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A completion of counterexamples to the classical central limit theorem for triplewise independent and identically distributed random variables

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ABSTRACT

By the Lindeberg–Lévy central limit theorem, standardized partial sums of a sequence of mutually independent and identically distributed random variables converge in law to the standard normal distribution. It is known that mutual independence cannot be relaxed to pairwise independence, nor even to triplewise independence. Counterexamples have been constructed for most marginal distributions: a recent construction works under a condition which excludes certain probability distributions with atomic parts, in particular almost all distributions on a fixed finite set. In the present paper, we show that this condition can be lifted: for any probability distribution F on the real line, which has finite variance and is not concentrated in a single point, there exists a sequence of triplewise independent random variables with distribution F, such that its standardized partial sums converge in law to a distribution which is not normal. There is also scope for extension to k-tuplewise independence.

1. Introduction

For a sequence of mutually independent and identically distributed random variables $X_1, X_2, \dots$ with $\mathbb{E}X_1 = \mu$ and $\operatorname{Var}(X_1) = \sigma^2$, where $0 < \sigma < \infty$, it is known that the standardized partial sums

$$S_n := \frac{1}{\sigma\sqrt{n}}\left(\sum_{k=1}^{n} X_k - n\mu\right) \tag{1}$$

converge in law to the standard normal distribution: this is known as the Lindeberg–Lévy central limit theorem: see Lindeberg (1922) and Lévy (1925). It is also known that mutual independence cannot in general be relaxed to the weaker pairwise independence, nor can it even be relaxed to triplewise independence. In general, K-tuplewise independence is defined as follows:

Definition 1. Let $K \in \{2, 3, 4, \dots\}$. An indexed family of random variables X_i, $i \in I$, is *K-tuplewise independent* if the random variables $X_{i_1}, X_{i_2}, \dots, X_{i_K}$ are mutually independent for any K-tuple of distinct indices $i_1, i_2, \dots, i_K$.

Counterexamples can be traced back to Révész and Wschebor (1965), who provide a sequence of pairwise independent and identically distributed random variables taking the values 1 and −1 with equal probabilities. By Theorem 1 ibidem, the absolute

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values of the (unstandardized) partial sums of that sequence can be bounded by a fixed random variable, so that their standardized counterparts S_n converge in distribution to zero. Avanzi et al. (2021) provide a survey of further constructions and construct a broad family of counterexamples for pairwise independence. Pruss (1998) succeeds to construct a counterexample to the central limit theorem which is a sequence of K-tuplewise independent and identically distributed random variables, where K can be arbitrary and the marginal distribution can be any symmetric distribution with finite variance. Bradley and Pruss (2009) construct a sequence of K-tuplewise independent and identically distributed random variables, which is strictly stationary.

Boglioni Beaulieu et al. (2021) modify the construction of Avanzi et al. (2021) to one which is based on a suitable sequence of graphs, each graph giving a family of K-tuplewise independent and identically distributed random variables. The random variables obtained from all graphs can be arranged into an array, each graph giving one row. They provide an increasing sequence of graphs giving triplewise independent rows and standardized row sums converging in law to a variance-gamma distribution, which is not normal: see Subsection 4.1 of Boglioni Beaulieu et al. (2021). From that array, a sequence can be extracted, such that its standardized partial sums do not converge to a normal distribution because it has a subsequence which converges to the variance-gamma distribution. Under some additional conditions, it can be shown that the entire sequence converges to the same distribution: see Lemma 8.

In the present paper, we focus on the constructions provided by Avanzi et al. (2021) and by Boglioni Beaulieu et al. (2021). They both allow for a broad choice for the (common) distribution of the summands $X_1, X_2, \dots$ Indeed, taking W to be a generic random variable with this distribution, they work under the following condition quoted below:

Condition 2. *There exists a Borel set $A \subseteq \mathbb{R}$, such that:*

- $\mathbb{P}(W \in A) = \ell^{-1}$ *for some integer* $\ell \geq 2$;
- $\mathbb{E}(W \mid W \in A) \neq \mathbb{E}(W \mid W \notin A)$.

Although this restriction is relatively mild, not all probability distributions on the real line fit it. In particular, discrete distributions in the "general position" are excluded, concretely any discrete distribution which is non-trivial and with point probabilities summing up only to 0, 1 or an irrational number; compare Remark 2 in Avanzi et al. (2021).

However, we show that Condition 2 can be lifted: the constructions provided by Avanzi et al. (2021) and Boglioni Beaulieu et al. (2021) can be adapted so that they allow for any distribution on the real line which makes sense. Indeed, instead of Condition 2, we only need that the distribution of W can be represented as a suitable mixture of two distributions. The following assertion states that this is true for all distributions which make sense (and we only need the case $\tau = \ell^{-1}$). We defer the proof to Section 3.

Proposition 3. *For each $\tau \in (0, 1)$ and any real-valued random variable W with finite expectation, which is not almost surely constant, there exist real-valued random variables U and V with different expectations, such that*

$$\mathbb{P}(W \in C) = (1-\tau)\mathbb{P}(U \in C) + \tau\mathbb{P}(V \in C) \tag{2}$$

for all Borel sets $C \subseteq \mathbb{R}$. Moreover, if W has finite variance, U and V can be chosen to have finite variances, too.

Based on the argument given by Avanzi et al. (2021) and Boglioni Beaulieu et al. (2021) extended by Proposition 3, we are able to complete the family of counterexamples to the central limit theorem for triplewise independent summands, as specified in the following result:

Theorem 4. *For any random variable W on the real line with finite variance, which is not almost surely constant, there exists a sequence $X_1, X_2, \dots$ of triplewise independent random variables, which follow the same distribution as W, such that the standardized partial sums S_n defined as in (1) converge in law to a probability distribution which is not normal.*

We defer the proof to the end of Section 2, where we give an outline of the arguments given by Avanzi et al. (2021) and Boglioni Beaulieu et al. (2021), exposing the point where Proposition 3 is applied. Notice that the latter is not related to the dependence structure of the summands, which depends on a sequence of graphs. So far, sequences leading to counterexamples for pair- and triplewise independence have been constructed. In future, it may turn out that higher degree of tuplewise independence can also be covered: see the discussion in Chapter 5 of Boglioni Beaulieu et al. (2021). As stated in Corollary 9, this would automatically extend Theorem 4, preserving the generality of the distribution of the summands.

2. Adaptation of construction

As mentioned in the Introduction, the construction provided by Boglioni Beaulieu et al. (2021) starts with a sequence of undirected graphs $G_1, G_2, \dots$ In order to provide K-tuplewise independence, all these graphs must be of girth at least $K+1$, that is, there must be no cycles of length K or less.

For a graph G, denote as usual by $V(G)$ its vertex set and by $E(G)$ its edge set. We work with abstract edges, assuming that each edge is assigned its two endpoints. In the sequence $G_1, G_2, \dots$, we assume that this assignment is consistent for all graphs in it, that is, any edge $k \in E(G_m) \cap E(G_n)$ has the same endpoints in both G_m and G_n.

Following Boglioni Beaulieu et al. (2021) (altering the notation to some extent), choose $\ell \in \{2, 3, 4, \dots\}$. Define $\mathcal{V} := \bigcup_{m=1}^{\infty} V(G_m)$ and $\mathcal{E} := \bigcup_{m=1}^{\infty} E(G_m)$. For each vertex $i \in \mathcal{V}$, consider a random variable M_i distributed uniformly over $\{1, 2, \dots, \ell\}$, letting all random variables M_i, $i \in \mathcal{V}$, be mutually independent. For each edge $k \in \mathcal{E}$ with endpoints i and j, define $D_k := 1$ if $M_i = M_j$ and $D_k := 0$ otherwise. Since each graph G_m has girth at least $K+1$, the family $D_k, k \in E(G_m)$, is K-tuplewise independent. Denoting by n_m the number of edges of G_m, let

$$\Xi_m^* := \sum_{k \in E(G_m)} D_k \,, \qquad \xi_m^* := \frac{\Xi_m^* - n_m \ell^{-1}}{\sqrt{n_m \ell^{-1}(1-\ell^{-1})}} \,.$$

Now choose two generic real-valued random variables U and V with finite variances. Let W be a random variable with distribution being a mixture of the distributions of U and V: more precisely,

$$\mathbb{P}(W \in C) = (1-\ell^{-1})\mathbb{P}(U \in C) + \ell^{-1}\mathbb{P}(V \in C) \tag{3}$$

for all Borel sets $C \subseteq \mathbb{R}$. Next, for each edge $k \in \mathcal{E}$, consider random variables U_k and V_k following the same distribution as U and V, respectively. Choose the random variables U_k and V_k, $k \in \mathcal{E}$, to be all mutually independent as well as independent of the random variables M_i, $i \in \mathcal{V}$. Letting

$$X_k := \begin{cases} U_k & ; D_k = 0 \\ V_k & ; D_k = 1 \end{cases} \tag{4}$$

and fixing m, observe that the random variables X_k, $k \in E(G_m)$, are K-tuplewise independent and follow the same distribution as W.

The constructions in the papers by Avanzi et al. (2021) and Boglioni Beaulieu et al. (2021) start with W and a Borel set $A \subseteq \mathbb{R}$, letting U and V follow the conditional distributions of W given A and A^c, respectively. This gives rise to Condition 2. However, there is no need to choose U and V this way: all that suffices for the continuation and desired properties of the construction, in particular Theorem 1 in Avanzi et al. (2021) and Theorem 3.1 in Boglioni Beaulieu et al. (2021), is the relationship (3): the latter (along with the observation that X_k follow the same distribution as W) corresponds to Formula (2.9) in Avanzi et al. (2021) and Formula (2.10) in Boglioni Beaulieu et al. (2021). Along with Formula (2.8) in Avanzi et al. (2021) and Formula (2.9) in Boglioni Beaulieu et al. (2021) (which both correspond to (4)) and the independence properties of the random variables U_k and V_k, $k \in E(G_m)$, this is all that is used in the proofs of Theorem 1 in Avanzi et al. (2021) and Theorem 3.1 in Boglioni Beaulieu et al. (2021). This proves the following modification of Theorem 3.1 in Boglioni Beaulieu et al. (2021):

Theorem 5. *With U, V, W, G_m, X_k and ξ_m^* as above, let $\mu = \mathbb{E}W$ and $\sigma^2 = \mathrm{Var}(W)$; assume that $0 < \sigma < \infty$. Provided that there exists a random variable Y, such that*

$$\xi_m^* \xrightarrow[n\to\infty]{\text{law}} Y \,,$$

the standardized sums

$$S_m^* := \frac{\sum_{k \in E(G_m)} X_k - n_m \mu}{\sigma \sqrt{n_m}}$$

converge in law to the random variable

$$S^{(\ell)} := \sqrt{1-r^2}\, Z + r Y \,,$$

where Z is a standard normal random variable, independent of Y, and where $r := \sqrt{\ell^{-1}(1-\ell^{-1})}\,(\mathbb{E}V - \mathbb{E}U)/\sigma$. □

Remark 6. If $r > 0$, then $S^{(\ell)}$ is normal if and only if Y is normal.

However, by Proposition 3, each real-valued random variable W with finite variance, which is not almost surely constant, admits random variables U and V with finite variances and different expectations, such that (3) is satisfied for all Borel sets $C \subseteq \mathbb{R}$. Recalling Remark 6, we have now proved the following assertion.

Corollary 7. *Let W be a real-valued random variable with expectation μ and variance σ^2; assume that $0 < \sigma < \infty$. Let G_m, n_m and ξ_m^*, $m = 1, 2, 3, \ldots$, be as above. Suppose that all graphs G_m have girth at least $K+1$ and that the random variables ξ_m^* converge in law to a probability distribution which is not normal. Then there exist random variables X_k, $k \in \mathcal{E}$, with the following properties:*

- *For each $k \in \mathcal{E}$, X_k has the same distribution as W.*
- *For each $m = 1, 2, 3, \ldots$, the family X_k, $k \in E(G_m)$, is K-tuplewise independent.*
- *The standardized sums $S_m^* = \frac{\sum_{k \in E(G_m)} X_k - n_m \mu}{\sigma \sqrt{n_m}}$ converge in law to a probability distribution which is not normal.* □

The preceding assertion allows us to construct counterexamples to the central limit theorem in terms of *arrays* of random variables, each graph giving one row. On the other hand, the central limit theorem is originally formulated in terms of *sequences*. The latter can also be constructed if the graphs G_m form an increasing sequence in the sense that $V(G_1) \subseteq V(G_2) \subseteq \cdots$, $E(G_1) \subseteq E(G_2) \subseteq \cdots$ and for each m, $E(G_m)$ is exactly the set of all edges in $E(G_{m+1})$ with both endpoints in $V(G_m)$. Notice that this allows us to define the endpoints of each edge $k \in \mathcal{E}$ consistently. All examples given by Boglioni Beaulieu et al. (2021) are of this kind.

Following Boglioni Beaulieu et al. (2021), arrange the edge set $\mathcal{E}$ into a sequence, so that the elements of $E(G_1)$ come first, followed by the elements of $E(G_2) \setminus E(G_1)$, then by $E(G_3) \setminus \big(E(G_1) \cup E(G_2)\big)$ and so on; otherwise, the order does not matter. Without loss of generality, we can just assume that $E(G_1) = \{1, 2, \dots, n_1\}$ and $E(G_m) \setminus \big(E(G_1) \cup \cdots \cup E(G_{m-1})\big) = \{n_{m-1} + 1, n_{m-1} + 2, \dots, n_m\}$ for $m = 2, 3, 4, \dots$ Thus, we have obtained a sequence of random variables $X_1, X_2, X_3, \dots$, which are K-tuplewise independent provided that each graph G_m has girth at least $K + 1$. Letting S_n be as in (1), notice that $S_m^* = S_{n_m}$. Therefore, under the conditions of Corollary 7, the sequence $S_1, S_2, \dots$ has a subsequence which converges in law to a non-normal distribution. Hence the sequence $S_1, S_2, \dots$ does not converge to a normal distribution. This is what is proved by Boglioni Beaulieu et al. (2021) (under Condition 2).

However, under some additional conditions, one can do a bit more, showing that the whole sequence of standardized partial sums actually converges in law.

Lemma 8. *Let $V_1, V_2, \dots$ be uncorrelated zero-mean random variables with the same variance σ^2, where $0 < \sigma < \infty$. Let*

$$T_n := \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{n} V_k$$

be their standardized partial sums. Take a sequence $n_1 < n_2 < \cdots$ of natural numbers with $\lim_{m\to\infty} n_{m+1}/n_m = 1$. If the subsequence $T_{n_1}, T_{n_2}, \dots$ converges in law to a random variable T, then the same is true for the whole sequence $T_1, T_2, \dots$

Proof. Letting $N_n := n_m$ for $n_m \le n < n_{m+1}$, we find that the sequence

$$\frac{1}{\sigma\sqrt{N_n}} \sum_{k=1}^{N_n} V_k\,; \quad n = 1, 2, 3, \dots$$

converges in law to T. Notice that the assumed condition implies that $\lim_{n\to\infty} n/N_n = 1$. Now consider the sequence

$$\frac{1}{\sigma\sqrt{N_n}} \sum_{k=N_n+1}^{n} V_k\,; \quad n = 1, 2, 3, \dots$$

and observe that $\mathrm{Var}\big(\frac{1}{\sigma\sqrt{N_n}} \sum_{k=N_n+1}^{n} V_k\big) = \frac{n-N_n}{N_n}$ tends to zero as $n \to \infty$. By Chebyshev's inequality, the random variables $\frac{1}{\sigma\sqrt{N_n}} \sum_{k=N_n+1}^{n} V_k$ then converge in law to zero as $n \to \infty$. By Slutsky's theorem, the sequence $\frac{1}{\sigma\sqrt{N_n}} \sum_{k=1}^{n} V_k$ then converges in law to T. The rest is completed by another part of Slutsky's theorem, recalling that $\lim_{n\to\infty} n/N_n = 1$. □

We can now summarize our observations into the following assertion:

Corollary 9. *Let W be a real-valued random variable with expectation μ and variance σ^2; assume that $0 < \sigma < \infty$. Let G_m, n_m and ξ_m^*, $m = 1, 2, 3, \dots$, be as above. Suppose that all graphs G_m have girth at least $K + 1$ and that the random variables ξ_m^* converge in law to a probability distribution which is not normal. Then there exist K-tuplewise independent random variables $X_1, X_2, X_3, \dots$, each of them following the same distribution as W, such that their standardized partial sums S_n defined as in (1) do not converge to a normal distribution. If, in addition, $n_1 < n_2 < \cdots$ and $\lim_{m\to\infty} n_{m+1}/n_m = 1$, then the random variables S_n converge in law to a probability distribution which is not normal.* □

With the preceding assertion, we are in a position to prove Theorem 4.

Proof of Theorem 4. Following Boglioni Beaulieu et al. (2021), choose $G_m := K_{m,m}$; as usual, $K_{m,m}$ denotes the bipartite graph with vertices divided into two groups of m vertices, where two vertices are adjacent if and only if they belong to different groups. Notice that $K_{m,m}$ has m^2 edges and girth 4 for $m \ge 2$. Choosing any $\ell \in \{2, 3, 4, \dots\}$, Theorem 4.1 in Boglioni Beaulieu et al. (2021) shows that the underlying random variables ξ_m^* defined as above converge in law to a variance gamma distribution, which is not normal. Thus, the conditions of Corollary 9 are fulfilled, proving the result. □

3. Construction of mixture

It remains to prove Proposition 3, which claims that any suitable probability distribution on the real line can be represented as a suitable mixture of two distributions.

Proof of Proposition 3. Let $a := \sup\{w \in \mathbb{R}\,;\, \mathbb{P}(W < w) < 1 - \tau\}$ and $b := \inf\{w \in \mathbb{R}\,;\, \mathbb{P}(W > w) < \tau\}$ be the lower and upper $(1 - \tau)$-quantile of the random variable W. Clearly, $a \le b$, $\mathbb{P}(W < a) \le 1 - \tau \le \mathbb{P}(W \le a)$ and $\mathbb{P}(W > b) \le \tau \le \mathbb{P}(W \ge b)$. We now distinguish two cases.

First, if $a < b$ or $\mathbb{P}(a \le W \le b) = 0$, then $\mathbb{P}(W \le a) = 1 - \tau$ and $\mathbb{P}(W \ge b) = \tau$. In this case, the construction is exactly the same as in Avanzi et al. (2021) and Boglioni Beaulieu et al. (2021): choosing U and V to follow the conditional distributions of W given $W \le a$ and $W \ge b$, respectively, (2) is immediate. Moreover, $\mathbb{E}U = \mathbb{E}(W \mid W \le a) \le a < b = \mathbb{E}(W \mid W \ge b) = \mathbb{E}V$. Finally, if W has finite variance, that is, if $\mathbb{E}(W^2) < \infty$, then $\mathbb{E}(U^2) = \mathbb{E}(W^2 \mid W \le a)$ and $\mathbb{E}(V^2) = \mathbb{E}(W^2 \mid W \ge b)$ are finite, too.

It remains to consider the case where $a = b$ and $\mathbb{P}(W = a) > 0$. Then define the distributions of U and V by

$$\mathbb{P}(U \in C) = \frac{1}{1-\tau}\left(\mathbb{P}(W \in C, W < a) + \frac{1 - \tau - \mathbb{P}(W < a)}{\mathbb{P}(W = a)}\,\mathbb{P}(W \in C, W = a)\right)$$

and

$$\mathbb{P}(V \in C) = \frac{1}{\tau}\left(\mathbb{P}(W \in C, W > a) + \frac{\tau - \mathbb{P}(W > a)}{\mathbb{P}(W = a)}\,\mathbb{P}(W \in C, W = a)\right).$$

A brief calculation shows that the latter two formulas indeed define probability distributions and that (2) is fulfilled. Next, we show that we again have $\mathbb{E}U < \mathbb{E}V$. First, observe that both expectations exist with

$$\mathbb{E}U = \frac{\mathbb{E}\big[W\mathbb{1}(W < a)\big] + a\big(1 - \tau - \mathbb{P}(W < a)\big)}{1 - \tau}$$

and

$$\mathbb{E}V = \frac{\mathbb{E}\big[W\mathbb{1}(W > a)\big] + a\big(\tau - \mathbb{P}(W > a)\big)}{\tau}.$$

Now if $\mathbb{P}(W < a) > 0$, then $\mathbb{E}(W \mid W < a) < a$. A brief calculation shows that $\mathbb{E}U < a$ in this case. Similarly, if $\mathbb{P}(W > a) > 0$, then $\mathbb{E}V > a$. Since W is not almost surely constant, at least one of these two cases occurs. Noting that $\mathbb{E}U \le a \le \mathbb{E}V$, we conclude that $\mathbb{E}U < \mathbb{E}V$.

Finally, observe that if W has finite variance, that is, if $\mathbb{E}(W^2) < \infty$, we also have

$$\mathbb{E}(U^2) = \frac{\mathbb{E}\big[W^2\mathbb{1}(W < a)\big] + a^2\big(1 - \tau - \mathbb{P}(W < a)\big)}{1 - \tau} < \infty$$

and

$$\mathbb{E}(V^2) = \frac{\mathbb{E}\big[W^2\mathbb{1}(W > a)\big] + a^2\big(\tau - \mathbb{P}(W > a)\big)}{\tau} < \infty$$

and the proof is complete. □

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Data availability

No data was used for the research described in the article.

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