5. Generating functions

5.1. Definition and basic properties

The idea of generating functions comes from analysis and combinatorics. If $c_0, c_1, \ldots$ is a sequence of complex numbers then we can define the power series

$$G(s) = \sum_{k=0}^{\infty} c_k \cdot s^k \quad \text{for } s \in \mathbb{C}.$$

We know from analysis that such power series converge for $|s| < R$ where R is the radius of convergence. Analysis further gives that

$$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|c_n|}.$$

If $|c_n| \le 1$ for all n then

$\frac{1}{R} \le 1 \Rightarrow R \ge 1$.

In this chapter we will only look at non-negative integer valued random variables.

<u>Definition</u>: Let X be a random variable with values $0, 1, 2, \ldots$. We define the generating function of X, denoted by $G_X(s)$ as the power series

$$G_X(s) = \sum_{k=0}^{\infty} P(X=k) \cdot s^k$$

<u>Comments</u>:

(i) The idea is to "pack" up the distribution in a function.

(ii) Since $\sum_{k=0}^{\infty} P(X=k) = 1$.

the power series is dominated by $P(X=k)$ for $|s| \leq 1$ and converges uniformly to a continuous function.

Examples :

(i) if $X \sim Bin(n,p)$ we have

$$G_X(s) = \sum_{k=0}^{n} P(X=k) \cdot s^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k \cdot q^{n-k} \cdot s^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} (ps)^k q^{n-k}$$

$$= (ps+q)^n.$$

(ii) if $X \sim Po(\lambda)$ we have

$$G_X(s) = \sum_{k=0}^{\infty} P(X=k) s^k$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^k}{k!} \cdot s^k$$

$$= (*)$$

$$(*) = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda s}$$

$$= e^{-\lambda(1-s)}$$

(iii) Let $X \sim NegBin(m, p)$.

From analysis we have that for $|x| < 1$

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k \quad \text{where}$$

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}$$

The above formula is known as the Newton formula. Replace x by $-x$ and let $a = -r$ for some integer $r > 0$.

We get

$$(1-x)^{-r} = \sum_{k=0}^{\infty} \binom{-r}{k} \cdot (-x)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-r)(-r-1)\cdots(-r-k+1)}{k!} (-1)^k \cdot x^k$$

$$= \sum_{k=0}^{\infty} \frac{r(r+1)\cdots(r+k-1)}{k!} \cdot x^k$$

$$= \sum_{k=0}^{\infty} \frac{(r-1)!\, r(r+1)\cdots(r+k-1)}{(r-1)! \cdot k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{(r+k-1)!}{(r-1)! \cdot k!} x^k$$

$$= \sum_{k=0}^{\infty} \binom{r+k-1}{r-1} x^k.$$

We compute

$$G_X(s) = \sum_{k=m}^{\infty} P(X=k) \cdot s^k$$

$$= \sum_{k=m}^{\infty} \binom{k-1}{m-1} \cdot p^m \cdot q^{k-m} \cdot s^k$$

$$= (*)$$

$$(*) = \sum_{\ell=0}^{\infty} \binom{m+\ell-1}{m-1} p^m \cdot q^\ell s^{m+\ell}$$

$$= p^m \cdot s^m \cdot \sum_{\ell=0}^{\infty} \binom{m+\ell-1}{m-1} s^\ell \cdot q^\ell$$

$$= \frac{p^m \cdot s^m}{(1-qs)^m}$$

$$= \left(\frac{ps}{1-qs} \right)^m$$

(iv) The computation in previous example gives

$$(1-x)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \cdot x^k \qquad |x| < 1.$$

Let X have the Pólya distribution

$$P(X=k) = \frac{\beta^a (a)_k}{k! \, (1+\beta)^{a+k}}$$

We have

$$G_X(s) = \sum_{k=0}^{\infty} P(X=k) \cdot s^k$$

$$= \sum_{k=0}^{\infty} \frac{\beta^a (a)_k}{k!\,(1+\beta)^{a+k}} \cdot s^k$$

$$= \sum_{k=0}^{\infty} \frac{\beta^a (a)_k}{k!\,(1+\beta)^a} \left(\frac{s}{1+\beta}\right)^k$$

$$= \frac{\beta^a}{(1+\beta)^a} \cdot \left(1 - \frac{s}{1+\beta}\right)^{-a}$$

$$= \left(\frac{\beta}{1+\beta-s}\right)^a$$

<u>Theorem 5.1</u> : Let X be a nonnegative integer valued random variable and let $G_X(s)$ be its generating function. Then $G_X(s)$ uniquely determines the distribution of X.

Proof: Since $G_X(s)$ converges for $|s| < 1$ we have

$$G_X^{(n)}(0) = n!\, P(X=n).$$

Theorem 5.2: Let X be an integer valued random variable with generating function $G_X(s)$.

(i)

$$E(X) = \lim_{s \uparrow 1} G_X'(s)$$

(ii)

$$E[X(X-1)\cdots(X-m+1)]$$

$$= \lim_{s \uparrow 1} G_X^{(m)}(s)$$

Proof: Let $\varepsilon > 0$ and assume first that $E(X) < \infty$.

There is a N_ε such that for $n \geq N_\varepsilon$ we have $\sum_{k=n}^{\infty} k P(X=k) < \varepsilon$.

This means that

$$E(X) - \sum_{k=0}^{N_\varepsilon - 1} k P(X=k) < \varepsilon.$$

Since all the coefficients in the power series are non-negative we have that for $s \in (0,1)$

$$\sum_{k=1}^{N_\varepsilon - 1} k P(X=k) s^{k-1} \leq G'_X(s) \leq E(X).$$

As $s \uparrow 1$ we have

$$\sum_{k=1}^{N_\varepsilon - 1} k P(X=k) \leq \lim_{s \uparrow 1} G'_X(s) \leq E(X)$$

The limit exists because $G_X(s)$ is nondecreasing on $(0,1)$. But the above means that

$$E(X) - \varepsilon \leq \lim_{s \uparrow 1} G_X(s) \leq E(X)$$

for all $\varepsilon > 0$.

If $E(X) = \infty$ we have that

$$\lim_{s \uparrow 1} G_X'(s) \geq \sum_{k=1}^{N} k\,P(X=k)$$

for any finite N. This implies that $\lim_{s \uparrow 1} G_X'(s) = \infty$.

(ii) The proof is similar.

<u>Theorem 5.3</u>: Let X, Y be independent. Then

$$\boxed{G_{X+Y}(s) = G_X(s) \cdot G_Y(s)}$$

<u>Proof</u>: We can write

$$E(s^X) = \sum_{k=0}^{\infty} s^k \cdot P(X=k) = G_X(s)$$

Then

$$E(s^{X+Y}) = E(s^X \cdot s^Y) \overset{\text{indp}}{=} E(s^X) \cdot E(s^Y)$$

$$= G_X(s) \cdot G_Y(s)$$

Comment: This is the most important property of generating functions.

By extension we have for independent $X_1, X_2, \dots, X_r$

$$\boxed{G_{X_1+X_2+\dots+X_r}(s) = G_{X_1}(s) \cdot G_{X_2}(s) \cdots G_{X_r}(s)}$$

Examples:

(i) X, Y independent $X \sim Bin(m, p)$ and $Y \sim Bin(n, p)$. We have

$$G_{X+Y}(s) = G_X(s) \cdot G_Y(s)$$

$$= (ps+q)^m \cdot (ps+q)^n$$

$$= (ps+q)^{m+n}$$

This last function is the generating function of the Bin $(m+n, p)$ distribution. It follows, by uniqueness, $X+Y \sim$ Bin $(m+n, p)$.

(ii) Let X, Y be independent and

$$P(X=k) = \frac{\beta^a (a)_k}{k!\,(1+\beta)^{a+k}}, \quad k=0,1,\dots$$

$$P(Y=l) = \frac{\beta^b (b)_l}{l!\,(1+\beta)^{b+l}}, \quad l=0,1,\dots$$

We have

$$G_{X+Y}(s) = G_X(s)\cdot G_Y(s)$$

$$= \left(\frac{\beta}{1+\beta-s}\right)^a \cdot \left(\frac{\beta}{1+\beta-s}\right)^b$$

$$= \left(\frac{\beta}{1+\beta-s}\right)^{a+b}$$

Conclusion:

$$P(X+Y=k) = \frac{\beta^{a+b}\,(a+b)_k}{k!\,(1+\beta)^{a+b}}, \quad k=0,1,\dots$$

Comment: We have computed the distribution of $X+Y$ before but the above is much more elegant.

(iii) Suppose $X_1, X_2, \dots, X_r$ are independent and $X_k \sim Geom(p)$. We have

$$G_{X_k}(s) = \sum_{i=1}^{\infty} s^i \cdot q^{i-1} \cdot p$$

$$= ps \sum_{i=0}^{\infty} (qs)^i$$

$$= \frac{ps}{1-qs}$$

It follows

$$G_{X_1+\dots X_r}(s) = \left(\frac{ps}{1-ps}\right)^r$$

Conclusion: $X_1+\dots+X_r \sim NegBin(r,p)$.

5.2. Branching processes

In applications of probability we often calculate sums of a random number of random variables. Let $X_1, X_2, \dots$ be random variables and N an integer valued nonnegative random variables. We need to define $X_1 + X_2 + \dots + X_N$. Formally we define

$$\boxed{X = \sum_{k=1}^{\infty} X_k \cdot 1(N \geq k)}$$

Comment: For a fixed $\omega \in \Omega$ we have $N(\omega) < \infty$ and so the sum is finite because only a finitely many terms are $\neq 0$.

We will write

$$X = X_1 + X_2 + \dots + X_N.$$

Theorem 5.4: Let $N, X_1, X_2, \dots$ be independent, $X_1, X_2, \dots$ equally distributed non-negative integer valued random variables and N non-negative integer valued. Let $X = X_1 + X_2 + \dots + X_N$. Then

$$\boxed{G_X(s) = G_N(G_{X_1}(s)).}$$

Proof: We use the formula for the total expectation.

$$\begin{aligned}
G_X(s) &= E(s^X) \\
&= \sum_{k=0}^{\infty} E(s^X \mid N=k) P(N=k) \\
&= \sum_{k=0}^{\infty} E(s^{X_1+\dots+X_k} \mid N=k) P(N=k) \\
&\overset{\text{indep}}{=} \sum_{k=0}^{\infty} E(s^{X_1+\dots+X_k}) P(N=k) \\
&= (*)
\end{aligned}$$

$$(*) = \sum_{k=0}^{\infty} [G_{X_1}(s)]^k \cdot P(N=k)$$

$$= G_N(G_{X_1}(s)).$$

Example : A hen lays N eggs. A chick hatches from each egg with probability p independent of all other eggs. Suppose $N \sim Po(\lambda)$. What is the distribution of the number of chicks? In mathematical notation we are asking about the distribution of $I_1 + I_2 + \dots + I_N$ where $I_1, I_2, \dots$ are independent with $I_k \sim \text{Bernoulli}(p)$ and independent of N. We have

$$G_X(s) = G_N(G_{X_1}(s)).$$

$$G_{I_1}(s) = s^0 P(I_1 = 0) + s^1 \cdot P(I_1 = 1)$$
$$= q + ps$$

We have

$$G_X(s) = G_N(q + ps)$$

$$= e^{-\lambda(1-q-ps)}$$

$$= e^{-\lambda(p-ps)}$$

$$= e^{-\lambda p(1-s)}$$

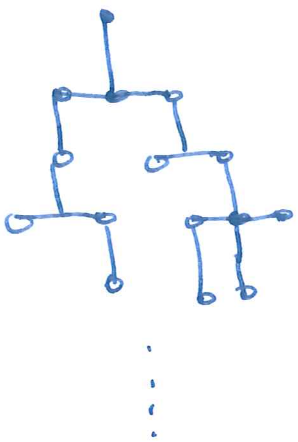
This last function is the generating function of the $P(\lambda p)$ distribution so $X \sim Po(\lambda p)$.

Branching processes

In 1874 Sir Francis Galton (1822-1911) asked the following question: suppose you take an English aristocrat. He will have a random number of sons. His sons will have a random number of sons, ...

The problem is to determine the probability that the family tree will die out.

Figure: A possible family tree



$Z_0 = 1$
$Z_1 = 3$
$Z_2 = 3$
$Z_3 = 5$
$Z_4 = 3$

The problem was solved by Galton and Watson in 1875 (F. Galton, H.W. Watson, [illegible] (1875) On the probability of the extinction of families, Journal of the Royal Anthropological Institute 4, 138 – 144) using generating functions.

To solve the problem mathematically we need a few additional assumptions:

(i) Generations are simultaneous.

(ii) Each individual has sons independently of all the others.

(iii) The random number of sons has the same distribution for all individuals.

The above assumptions imply the following mathematical formulation:

Let $\{\xi_{n,k}\}_{n\geq 1, k\geq 1}$ be independent, equally distributed non-negative integer valued random variables with generating function G.

We define

$$Z_0 = 1 \quad \text{and recursively}$$

$$Z_{n+1} = \xi_{n+1,1} + \xi_{n+1,2} + \cdots + \xi_{n+1,Z_n}$$

The sequence $Z_0, Z_1, \ldots$ of random variables is called the <u>branching process</u>.

The above means that Z_n individuals in the n-th generation have randomly many offspring.

The random variable Z_n depends on $\xi_{m,k}$ for $m \leq n$ so it is independent of

$$\xi_{n+1,1}, \xi_{n+1,2}, \ldots\ldots$$

Denote $G_n(s) = G_{Z_n}(s)$. By Theorem 5.4 we have

$$G_{n+1}(s) = G_n(G(s)).$$

By definition $G_1(s) = G(s)$ and by the above recursion

$$G_2(s) = G_1(G(s)) = (G \circ G)(s)$$

$$G_3(s) = G_2(G(s)) = (G \circ G \circ G)(s)$$

$$\vdots$$

$$G_n(s) = (G \circ G \circ \cdots \circ G)(s).$$

Since composition is associative we have

$$\boxed{G_{n+1}(s) = G(G_n(s))}$$

Let $A = \{$the family tree dies out$\}$.
The family tree dies out if one of the generations is empty so

$$A = \bigcup_{n=1}^{\infty} \{Z_n = 0\}.$$

But $\{Z_1 = 0\} \subseteq \{Z_2 = 0\} \subseteq \ldots\ldots$

In the first chapter we proved that for $A_1 \subseteq A_2 \subseteq \ldots$ we have

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

Denote $\eta = P(A)$. We have

$$\eta = P(A) = \lim_{n\to\infty} P(Z_n = 0).$$

But $P(Z_n = 0) = G_n(0)$.

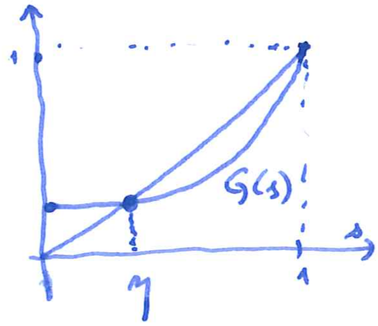
<u>Theorem 5.5</u> : The probability η satisfies the equation $\eta = G(\eta)$ and is the smallest solution of the above equation on $[0,1]$.

<u>Comments</u> :

(i) If $\eta = G(\eta)$ we say that η is a fixed point of G.

(ii) Since $G(1) = 1$ there is always at least one fixed point on $[0,1]$. The set of fixed points is compact so it contains a smallest point.

<u>Figure</u> :



<u>Proof</u> : $G(s)$ is continuous on $[0,1]$.

So we have

$$\eta = \lim_{n\to\infty} G_{n+1}(0) = \lim_{n\to\infty} G(G_n(0))$$

$$= G\left(\lim_{n\to\infty} G_n(0)\right) = G(\eta).$$

So η is a fixed point. To prove that η is the smallest fixed point let $\bar{\eta}$ be a fixed point on $[0,1]$. We have

$$0 \le \bar{\eta}$$

Since G is non decreasing on $[0,1]$ it follows

$$G(0) \le G(\bar{\eta}) = \bar{\eta}$$

$$G(G(0)) \le G(\bar{\eta}) = \bar{\eta}$$

$$\vdots$$

$$(G \circ \cdots \circ G)(0) = G_n(0) \le \bar{\eta}$$

So

$$\lim_{n\to\infty} G_n(0) = \eta \le \bar{\eta}.$$

This means that any fixed point $\bar{\eta}$ is $\geq \eta$ which proves the theorem.

Example: Suppose every individual has 0, 1, 2, 3 sons with probability 1/4 each. This means that

$$G(s) = \frac{1+s+s^2+s^3}{4}.$$

We need all solutions of

$$G(s) = s \iff 1 - 3s + s^2 + s^3 = 0$$

We know that $s = 1$ is a solution so we can factor

$$1 - 3s + s^2 + s^3 = (s-1)(s^2+2s-1)$$

The solutions are

$s = 1$
$s = -1+\sqrt{2}$
$s = -1-\sqrt{2}$

The smallest fixed point on $[0,1]$ is $-1+\sqrt{2}$ $\doteq 0.4142$.

Example: Suppose $G(s) = \frac{p}{1-qs}$.

What is $G_n(s)$?

$$G_2(s) = G(G(s))$$

$$= \frac{p}{1 - q \cdot \frac{p}{1-qs}}$$

$$= \frac{p(1-qs)}{1 - pq - qs}$$

$$G_3(s) = \frac{p}{1 - q \cdot \frac{p(1-qs)}{1-pq-qs}}$$

$$= \frac{p(1-pq-qs)}{1-pq-qs-pq(1-qs)}$$

$$= \frac{p(1-pq-qs)}{1-2pq-q(1-pqs)}$$

We see that

$$G_n(s) = \frac{a_n - b_n s}{c_n - d_n s}$$

$$G_{n+1}(s) = \frac{a_n - b_n \cdot \frac{p}{1-qs}}{c_n - d_n \cdot \frac{p}{1-qs}}$$

Multiplying out we get

$$a_{n+1} = a_n - p \cdot b_n$$
$$b_{n+1} = q a_n$$

We have $G_0(s) = s \Rightarrow a_0 = 0, \; b_0 = -1$

Write in matrix form

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 1, & -p \\ q, & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

Iteration gives

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1, & -p \\ q & 0 \end{pmatrix}^n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1, & -p \\ q & 0 \end{pmatrix}^n \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

We need to find the power of the matrix. Suppose $p \neq q$.

We can check by multiplication that

$$\underbrace{\begin{pmatrix} p & 1 \\ q & 1 \end{pmatrix}}_{A} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \underbrace{\begin{pmatrix} \frac{1}{p-q}, & -\frac{1}{p-q} \\ -\frac{q}{p-q} & \frac{p}{p-q} \end{pmatrix}}_{A^{-1}}$$

$$= \begin{pmatrix} 1, & -p \\ q, & 0 \end{pmatrix}$$

We have diagonalized the matrix $\begin{pmatrix} 1, & -p \\ q & 0 \end{pmatrix}$. This means

$$\begin{pmatrix} 1, & -p \\ q, & 0 \end{pmatrix}^n = \begin{pmatrix} p & 1 \\ q & 1 \end{pmatrix} \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix} \begin{pmatrix} 1, & -1 \\ -q & p \end{pmatrix} \cdot \frac{1}{p-q}$$

$$= \begin{pmatrix} p^{n+1} - q^{n+1}, & -p^{n+1} + pq^n \\ qp^n - q^{n+1}, & -qp^n + pq^n \end{pmatrix} \cdot \frac{1}{p-q}$$

We find:

$$\begin{pmatrix} 1, & -p \\ q & 0 \end{pmatrix}^n \begin{pmatrix} 0 \\ -1 \end{pmatrix} =$$

$$= \frac{1}{p-q} \begin{pmatrix} p(p^n - q^n) \\ pq(p^{n-1} - q^{n-1}) \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

For c_n, d_n the procedure is the same except that $\begin{pmatrix} c_0 \\ d_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$\begin{pmatrix} c_n \\ d_n \end{pmatrix} = \begin{pmatrix} p^{n+1} - q^{n+1} \\ q(p^n - q^n) \end{pmatrix} \cdot \frac{1}{p-q}$$

Finally,

$$G_n(s) = \frac{p\left(p^n - q^n - qs\left(p^{n-1} - q^{n-1}\right)\right)}{p^{n+1} - q^{n+1} - qs\left(p^n - q^n\right)}$$

We find

$$P(Z_n = 0) = G_n(0)$$

$$= \frac{p\left(p^n - q^n\right)}{p^{n+1} - q^{n+1}}$$

We get: if $p > q$, then

$$\lim_{n\to\infty} P(Z_n = 0) = 1$$

if $p < q$

$$\lim_{n\to\infty} P(Z_n = 0) = \frac{p}{q} < 1.$$

<u>Comment</u>: The fixed points satisfy $\frac{p}{1-qs} = s \Rightarrow$

$$qs^2 - s + p = (s-1)(qs-p) = 0 \Rightarrow$$

There is a fixed point in $[0,1]$ other than 1 if $p < q$.

Comment: For $p = q = 1/2$ we get

$$G_n(s) = \frac{n - (n-1)s}{n+1 - ns}$$

and

$$G_n(0) = \frac{n}{n+1} \to 1 \,, \quad \text{as } n \to \infty.$$

Theorem 5.6: Let $Z_0, Z_1, Z_2, \dots$ be a branching process. Let $\mu = E(Z_1)$.

(i) If $\mu < 1$ then $\eta = 1$.

(ii) If $\mu > 1$ then $\eta \in [0,1)$.

(iii) If $\mu = 1$ and $G(s) \neq s$ then $\eta = 1$.

Comment: The case $G(s) = s$ is uninteresting.

The proof of the theorem has more to do with analysis than probability.

Proof: Note that $\mu = \lim_{s \uparrow 1} G'(s)$.

The function $G'(s)$ is nondecreasing on $(0,1)$.

(i) if $\mu < 1$, then $G'(s) \leq \mu < 1$ for all $s \in (0,1)$. If $G(\bar{\eta}) = \bar{\eta}$ for $\bar{\eta} \in (0,1)$ then by Lagrange theorem

$$G(1) - G(\bar{\eta}) = G'(\xi)(1-\bar{\eta}) \quad \text{for } \xi \in (\bar{\eta}, 1).$$
$$\overset{\shortparallel}{1 - \bar{\eta}}$$

But $G'(\xi) < 1$. So we have a contradiction.

(ii) If $\mu > 1$ there is a $\delta > 0$ such that $G'(s) > 1$ for $s \in (1-\delta, 1)$. By Lagrange for $s \in (1-\delta, 1)$ we have

$$G(1) - G(s) = G'(\xi)(1-s) > (1-s)$$

for $\xi \in (s,1)$. This implies

$$1 - G(s) > 1 - s \Rightarrow G(s) < s.$$

On the other hand $G(0) \geq 0$.

For $s \in (1-\delta, 0)$ we have that

$$F(0) = G(0) - 0 \geq 0$$

$$F(s) = G(s) - s < 0.$$

There must be a zero of F on $(0, s) \subset (0,1) \Rightarrow \eta \in (0,1)$.

(iii) If $G(s) \neq s$ the either $G(s) = 1$ in which case $\eta = 1$ or G is strictly convex on $(0,1)$, or $G'(s)$ is strictly increasing on $(0,1)$.

If $G(\bar{\eta}) = \bar{\eta}$ for some $\bar{\eta} \in (0,1)$ this would imply

$$G(1) - G(\bar{\eta}) = 1 - \bar{\eta}$$

$$= G'(\xi)(1-\bar{\eta})$$

for some $\xi \in (\bar{\eta}, 1)$. This means $G'(\xi) = 1$. But $G'(s)$ is strictly increasing meaning $\lim_{s\uparrow 1} G'(s) > 1$.

A contradiction.

Panjer recursion

If $X = X_1 + X_2 + \dots + X_N$ we know that

$$G_X(s) = G_N(G_{X_1}(s)).$$

In principle we get $P(X=k)$ by expanding the right side into power series. But this is often difficult and recursive formulae are needed. This problem is often dealt with in insurance.

Definition: The random variable N is of Panjer class if

$$P(N=n) = \left(a + \frac{b}{n}\right) P(N=n-1)$$

for $n = 1, 2, \dots\dots$

Examples: (i) Take $a = 0$ and $b > 0$.

We get $P(N=n) = \frac{b}{n} P(N=1) \Rightarrow$

$P(N=n) = e^{-b} \cdot \frac{b^n}{n!} \Rightarrow N \sim Po(b)$.

(ii) Suppose $N \sim Bin(M, p)$. We have computed

$$\frac{P(N=n)}{P(N=n-1)} = \frac{M-n+1}{n} \cdot \frac{p}{q}$$

$$= \left(-\frac{p}{q} + \frac{(M+1)p}{q \cdot n} \right)$$

We take $a = -\frac{p}{q}$, $b = \frac{(M+1)p}{q}$

We see that $P(N = M+1) = 0$.

We compute

$$P(N=1) = \left(-p/q + \frac{(M+1)p}{q} \right) P(N=0)$$

$$= \frac{p}{q} M \cdot P(N=0)$$

$$P(N=2) = \left(-p/q + \frac{(M+1)p}{2q} \right) P(N=1)$$

$$= \frac{p}{q} \left(-1 + \frac{M+1}{2q} \right) P(N=1)$$

$$= \frac{p}{q} \cdot \frac{M-1}{2} \cdot \frac{p}{q} \cdot \frac{M}{1}$$

Continuing we get

$$P(N=n) = \left(\frac{p}{q}\right)^n \cdot \frac{M(M-1)\cdots(M-n+1)}{n!} P(N=0)$$

$$= \left(\frac{p}{q}\right)^n \binom{M}{n} \cdot P(N=0)$$

Because all probabilities must add to 1 we have

$$\left(1+\frac{p}{q}\right)^M P(N=0) = 1 \Rightarrow$$

$$P(N=0) = q^M$$

or

$$P(N=n) = \binom{M}{n} p^n q^{M-n}.$$

<u>Conclusion</u>: N is in the Panjer class.

Theorem 5.7: For $|s| < 1$ the generating function of N satisfies

$$(1 - as)\, G_N'(s) = (a+b)\, G_N(s).$$

Proof: From the recursion equation we have

$$P(N=n) \cdot s^n = \left(a + \frac{b}{n}\right) P(N=n-1) \cdot s^n.$$

Sum both sides over $n = 1, 2, \ldots$. We get

$$G_N(s) - G_N(0)$$

$$= a \sum_{n=1}^{\infty} P(N=n-1)\, s^n$$

$$+ b \cdot \sum_{n=1}^{\infty} \frac{s^n}{n} P(N=n-1)$$

$$= as\, G_N(s) + b \sum_{n=0}^{\infty} \left(\int_0^s u^n \, du \right) P(N=n)$$

$$= as\, G_N(s) + b \cdot \int_0^s \left(\sum_{n=0}^{\infty} u^n P(N=n) \right) du$$

$$= as\, G_N(s) + b \int_0^s G_N(u)\, du.$$

It is legitimate to interchange summation and integration because the sum converges uniformly on $[0,1]$.

Take derivatives to get

$$G_N'(s) = a\,G_N(s) + as\,G_N'(s)$$

$$+ b\,G_N(s).$$

Rearranging gives the equation.

If $X = X_1 + X_2 + \cdots + X_N$ where X_1, X_2 are independen equally distributed we get

$$G_X(s) = G_N(G_{X_1}(s))$$

Taking derivatives we get

$$G_X'(s) = G_N'(G_{X_1}(s)) \cdot G_{X_1}'(s)$$

Multipy both sides by

$$1 - aG_{X_1}(G_{X_1}(s))$$

and use Theorem 5.7. We get

$$\boxed{G'_X(s)\left(1 - aG_{X_1}(s)\right) = (a+b)\, G_X(s)\, G'_{X_1}(s)}$$

Denote $P(N=n) = p_n$ for $n = 0, 1, \ldots$.

Denote $P(X=r) = g_r$ and

$P(X_1 = k) = f_k$ for $k = 0, 1, \ldots$

We have that $X = 0$ if either

$N = 0$ or $N > 0$ and $X_1 + \cdots + X_N = 0$.

It follows

$$P(X=0) = P(N=0) + \sum_{n=1}^{\infty} P(N=n) \cdot f_0^n$$

$$= G_N(f_0)$$

In our notation

$$P(X=0) = g_0 = G_N(f_0).$$

From Analysis we know that

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{k=0}^{\infty} c_k x^k$$

with

$$c_k = \sum_{i=0}^{k} a_i b_{k-i} .$$

<u>Comment</u>: This is called the Cauchy product of power series.

In the formula connecting the generating functions

Equate coefficients for s^n.

$$h: \quad (n+1)\, g_{n+1} - a \sum_{k=0}^{n} f_k \cdot (n+1-k)\, g_{n+1-k}$$

$$R: \quad (a+b) \sum_{k=0}^{n} (k+1)\, f_{k+1} \cdot g_{n-k}$$

$$= (a+b) \sum_{k=1}^{n+1} k f_k \cdot g_{n-k+1}$$

Rearranging we get

$$(n+1) g_{n+1} - a f_0 (n+1) g_{n+1}$$

$$= a \sum_{k=1}^{n} f_k (n+1-k) g_{n+1-k} +$$

$$(a+b) \sum_{k=0}^{n} (k+1) f_{k+1} g_{n-k}$$

$$= a \sum_{k=1}^{n} f_k (n+1-k) g_{n+1-k}$$

$$+ (a+b) \sum_{k=1}^{n+1} k f_k \cdot g_{n+1-k}$$

$$= a \cdot \sum_{k=1}^{n+1} f_k (n+1-k) g_{n+1-k}$$

$$+ (a+b) \sum_{k=1}^{n+1} k f_k g_{n+1-k}$$

Divide by $(n+1)(1-af_0)$ to get

$$g_{n+1} = \frac{1}{1-af_0} \sum_{k=1} \left(a + \frac{bk}{n+1}\right) f_k g_{n+1-k}$$

<u>Theorem 5.8</u> (Panjer recursion)

We have

$$\boxed{g_{n+1} = \frac{1}{1-af_0} \sum_{k=1}^{n+1} \left(a + \frac{bk}{n+1}\right) f_k g_{n+1-k}}$$