Name and surname: ______________________

Identification number: | | | | | | | | |

University of Primorska

FAMNIT, MA & MF

Probability

Midterm 2

May 28$^{\text{th}}$, 2024

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems.

Question	a.	b.	c.	d.	Total
1.			•	•	
2.			•	•	
3.			•	•	
4.			•	•	
5.			•	•	
6.			•	•	
Total					

1. (20) We toss a coin until we get two heads in a row or two tails in a row. Denote the number of necessary tosses, including the last one, by X. Assume the tosses are independent and the probability of landing heads is $p \in (0,1)$. Assume as known that for $|x| < 1$ we have

$$\sum_{k=1}^{\infty} k x^k = \frac{x}{(1-x)^2}\,.$$

a. (10) Compute $P(X = n)$ for $n = 2, 3, \ldots$.

Hint: do the computation for odd and even n separately.

Solution: let $n = 2k$ first. The event $\{X = 2k\}$ happens in two disjoint ways: (i) the first toss is heads, the next $(2k-3)$ tosses are alternately tails and heads, and then we get two heads in a row (ii) the first toss is tails, the next $(2k-3)$ tosses are alternately heads and tails, and then we get two tails in a row. The probability of the first way is $p^{k-1}q^{k-1}p^2$, and the probability of the second is $p^{k-1}q^{k-1}q^2$. It follows that

$$P(X = 2k) = p^{k+1}q^{k-1} + p^{k-1}q^{k+1}\,.$$

If $n = 2k+1$, for $k \geq 1$ a similar argument gives

$$P(X = 2k+1) = p^{k-1}q^k p^2 + p^k q^{k-1} q^2 = p^k q^k\,.$$

In a joint expression

$$P(X = n) = (pq)^{[\frac{n}{2}-\frac{1}{2}]}(p^2+q^2)^{\frac{1}{2}(1+(-1)^n)},$$

where $[x]$ is the integer part of x.

b. (10) Compute $E(X)$.

Solution: we compute by definition.

$$\begin{aligned}
E(X) &= \\
&= \sum_{n=2}^{\infty} n\,P(X=n) = \\
&= \sum_{k=1}^{\infty} (2k)P(X=2k) + \sum_{k=1}^{\infty}(2k+1)P(X=2k+1) = \\
&= 2\sum_{k=1}^{\infty} k(p^2+q^2)(pq)^{k-1} + \sum_{k=1}^{\infty}(2k+1)(pq)^k = \\
&= 2(p^2+q^2)\frac{1}{(1-pq)^2} + 2\sum_{k=1}^{\infty} k(pq)^k + \sum_{k=1}^{\infty}(pq)^k = \\
&= 2(p^2+q^2)\frac{1}{(1-pq)^2} + \frac{2pq}{(1-pq)^2} + \frac{pq}{1-pq} = \\
&= \frac{2(p^2+2pq+q^2) - pq - p^2q^2}{(1-pq)^2} = \\
&= \frac{2-pq-p^2q^2}{(1-pq)^2} = \frac{1-pq+(1-p^2q^2)}{(1-pq)^2} = \\
&= \frac{2+pq}{1-pq}\,.
\end{aligned}$$

2. (20) Consider an urn initially containing a white and b black balls. Assume that $n = a + b$ is even, and denote $m = n/2$. We randomly draw pairs of balls from the urn uniformly without replacement, until we draw all the pairs. Thus, the balls have been arranged into pairs with all $n!/(m! \cdot 2^m)$ possible arrangements having the same probability. Denote by X the number of pairs with both balls white and by Y the number of pairs with both balls black.

a. (10) Compute $E(X)$.

Solution:

First method: *define*

$$I_k := \begin{cases} 1 & \textit{both balls in the } k\textit{-th pair drawn are white;} \\ 0 & \textit{otherwise.} \end{cases}$$

We have $X = \sum_{k=1}^{m} I_k$ *and consequently* $E(X) = \sum_{k=1}^{m} E(I_k)$*. Next, by symmetry, the* k*-th pair drawn is selected uniformly at random out of all possible pairs of balls. Therefore,*

$$E(I_k) = P(I_k = 1) = \frac{a(a-1)}{n(n-1)}$$

and

$$E(X) = m \cdot \frac{a(a-1)}{n(n-1)} = \frac{a(a-1)}{2(n-1)}\,.$$

Second method: *enumerating white balls by* $1, 2, \ldots, a$*, define*

$$I'_k := \begin{cases} 1 & k\textit{-th white ball has been paired with a white ball;} \\ 0 & \textit{otherwise.} \end{cases}$$

Then we have $X = \frac{1}{2}\sum_{k=1}^{a} I'_k$ *and consequently* $E(X) = \frac{1}{2}\sum_{k=1}^{a} E(I'_k)$*. By symmetry, we have*

$$E(I'_k) = P(I'_k = 1) = \frac{a-1}{n-1}\,,$$

giving

$$E(X) = \frac{a(a-1)}{2(n-1)}\,,$$

which is the same as before.

Third method: *enumerating all possible pairs of white balls by* $1, 2, \ldots, \binom{a}{2}$*, define*

$$I''_k := \begin{cases} 1 & \textit{the balls in the } k\textit{-th pair have been drawn together;} \\ 0 & \textit{otherwise.} \end{cases}$$

Similarly as in the first method, we have $X = \sum_{k=1}^{\binom{a}{2}} I''_k$ *and consequently* $E(X) = \sum_{k=1}^{\binom{a}{2}} E(I''_k)$*. By symmetry, we have*

$$E(I''_k) = P(I''_k = 1) = \frac{1}{n-1}\,.$$

giving

$$E(X) = \binom{a}{2} \cdot \frac{1}{n-1} = \frac{a(a-1)}{2(n-1)}\,,$$

which is again the same as before.

b. (10) Compute $E(XY)$.

Solution:

First method: *in addition to the indicators from the first method of the solution of part a., define*

$$J_l = \begin{cases} 1 & \text{both balls in the } k\text{-th pair drawn are black;} \\ 0 & \text{otherwise.} \end{cases}$$

Write

$$XY = \sum_{k=1}^{m} I_k J_k + \sum_{\substack{1 \le k,l \le m \\ k \ne l}} I_k J_l$$

and notice that $I_k J_k = 0$ *for all* k. *Similarly as before, we have by linearity*

$$E(XY) = \sum_{\substack{1 \le k,l \le m \\ k \ne l}} E(I_k J_l) .$$

Now recall that

$$E(I_k J_l) = P(I_k = 1, J_l = 1) .$$

Again, by symmetry, the balls in the k*-th and* l*-th pair can be regarded as a quadruple selected uniformly at random. The probability that the first two balls are white and the last two are black equals*

$$P(I_k = 1, J_l = 1) = \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)} .$$

As there are exactly $m(m-1)$ *pairs with* $k \ne l$, *we have*

$$E(XY) = m(m-1) \cdot \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)} .$$

Second method: *enumerating black balls by* $1, 2, \ldots, b$, *define*

$$J_l' := \begin{cases} 1 & l\text{-th black ball has been paired with a black ball;} \\ 0 & \text{otherwise.} \end{cases}$$

Now observe that

$$XY = \frac{1}{4} \sum_{k=1}^{a} \sum_{l=1}^{b} I_k' J_k' , \qquad \text{so that} \qquad E(XY) = \frac{1}{4} \sum_{k=1}^{a} \sum_{l=1}^{b} E(I_k' J_k')$$

and by symmetry,

$$E(I_k' J_l') = P(I_k' = 1, J_l' = 1) = \frac{(a-1)(b-1)}{(n-1)(n-3)} .$$

As a result, we find that

$$E(XY) = \frac{ab(a-1)(b-1)}{4(n-1)(n-3)} ,$$

which is the same as before.

Third method: *enumerating all possible pairs of black balls by* $1, 2, \ldots, \binom{a}{2}$*, define*

$$J_l'' := \begin{cases} 1 & \textit{the balls in the } l\textit{-th pair have been drawn together;} \\ 0 & \textit{otherwise.} \end{cases}$$

Now observe that

$$XY = \sum_{k=1}^{\binom{a}{2}} \sum_{l=1}^{\binom{b}{2}} I_k'' J_k'' , \qquad \textit{so that} \qquad E(XY) = \sum_{k=1}^{\binom{a}{2}} \sum_{l=1}^{\binom{b}{2}} E(I_k'' J_k'')$$

and by symmetry,

$$E(I_k'' J_l'') = P(I_k'' = 1, J_l'' = 1) = \frac{1}{(n-1)(n-3)} .$$

As a result, we find that

$$E(XY) = \binom{a}{2}\binom{b}{2} \cdot \frac{1}{(n-1)(n-3)} = \frac{a(a-1)b(b-1)}{4(n-1)(n-3)} ,$$

which is again the same as before.

3. (20) Let random variables Y and W be independent with $Y \sim \exp(1)$ and $W \sim \mathrm{N}(0,1)$. Let

$$X = \theta Y + \sigma\sqrt{Y}W\,,$$

where $\theta > 0$ and $\sigma > 0$ are given constants.

a. (10) Find the density of the random vector (Y, X).

Solution: define the mapping

$$\Phi(y,w) = (y, \theta y + \sigma\sqrt{y}w)\ .$$

On the open set $U = \{(y,w)\colon y > 0\}$ the mapping is bijective and maps U onto itself. We compute

$$\Phi^{-1}(y,x) = \left(y, \frac{x-\theta y}{\sigma\sqrt{y}}\right)\ .$$

The differentiability assumptions are satisfied and we compute

$$J_{\Phi^{-1}}(y,x) = \frac{1}{\sigma\sqrt{y}}\,.$$

It follows

$$f_{Y,X}(y,x) = f_{Y,W}\left(y, \frac{x-\theta y}{\sigma\sqrt{y}}\right)\cdot J_{\Phi^{-1}}(y,x)\,.$$

By independence we have $f_{Y,W}(y,w) = f_Y(y)f_W(w)$. For $(y,x) \in U$, it follows

$$f_{Y,X}(y,x) = \frac{1}{\sqrt{2\pi}}\exp\left(-y - \frac{1}{2}\left(\frac{x-\theta y}{\sigma\sqrt{y}}\right)^2\right)\cdot\frac{1}{\sigma\sqrt{y}}\,,$$

and $f_{Y,X}(y,x) = 0$ otherwise.

b. (10) Find the density of the random variable X. Assume as known that for $a > 0$ and $b \geq 0$

$$\int_0^\infty \frac{e^{-ay-\frac{b}{y}}}{\sqrt{y}}\,dy = \frac{\sqrt{\pi}}{\sqrt{a}}\,e^{-2\sqrt{ab}}\,.$$

Solution: we compute the density of X as the marginal density. Integrating we get

$$\begin{aligned}
f_X(x) &= \int_0^\infty f_{Y,X}(x,y)\,dy \\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{y}}\exp\left(-y - \frac{x^2}{2\sigma^2 y} + \frac{\theta x}{\sigma^2} - \frac{\theta^2 y}{2\sigma^2}\right)dy \\
&= \frac{1}{\sigma\sqrt{2\pi}}\,e^{\theta x/\sigma^2}\int_0^\infty \frac{1}{\sqrt{y}}\exp\left(-\frac{x^2}{2\sigma^2 y} - \frac{(\theta^2+2\sigma^2)y}{2\sigma^2}\right)dy \\
&= \frac{1}{\sigma\sqrt{2\pi}}\,e^{\theta x/\sigma^2}\,\frac{\sqrt{\pi}}{\sqrt{\frac{\theta^2+2\sigma^2}{2\sigma^2}}}\exp\left(-2\sqrt{\frac{x^2}{2\sigma^2}}\cdot\sqrt{\frac{\theta^2+2\sigma^2}{2\sigma^2}}\right) \\
&= \frac{1}{\sqrt{\theta^2+2\sigma^2}}\,e^{-(|x|\sqrt{\theta^2+2\sigma^2}-\theta x)/\sigma^2}\,.
\end{aligned}$$

4. (20) In a deck of $a+b$ cards there are a white and b red cards. We shuffle the deck so that every permutation is equally likely. We deal cards from the top of the deck. Let X be the number of white cards before the first red card and Y the number of white cards after the last red card. Number the white cards by $k = 1, 2, \ldots a$, and define

$$I_k = \begin{cases} 1 & \text{if the } k\text{-th white card is before the first red card;} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$J_k = \begin{cases} 1 & \text{if the } k\text{-th white card is after the last red card;} \\ 0 & \text{otherwise;} \end{cases}$$

a. (10) Justify that $X = \sum_{k=1}^{a} I_k$. Compute $\mathrm{var}(X)$.

Solution: if we consider just the first white card and the red cards, these $1+b$ cards are randomly permuted. The probability that the first white card is before all red cards is

$$P(I_1 = 1) = \frac{1}{b+1}$$

and

$$\mathrm{var}(I_1) = \frac{b}{(b+1)^2}\,.$$

We need

$$P(I_1 = 1, I_2 = 1) = \frac{2}{(b+1)(b+2)}\,,$$

which we can obtain if we consider just the permutation of two white cards and b red cards. It follows,

$$\mathrm{cov}(I_1, I_2) = \frac{b}{(b+1)^2(b+2)}\,.$$

By symmetry the variances I_k are equal and the covariances of pairs (I_k, I_l) are equal. It follows that

$$\mathrm{var}(X) = \frac{ab}{(b+1)^2} + \frac{a(a-1)b}{(b+1)^2(b+2)}\,.$$

b. (10) Compute $\mathrm{cov}(X, Y)$.

Solution: by symmetry we have

$$E(J_1) = \frac{1}{b+1}\,.$$

Furthermore

$$P(I_1 = 1, J_1 = 1) = 0 \quad \textit{and} \quad P(I_1 = 1, J_2 = 1) = \frac{1}{(b+1)(b+2)}\,.$$

Taking into account the bilinearity of the covariance and symmetry we get

$$\mathrm{cov}(X, Y) = \sum_{k,l=1}^{a} \mathrm{cov}(I_k, J_l)\,.$$

By symmetry, for $k \neq l$ all the covariances are equal. All the covariances $\mathrm{cov}(I_k, J_k)$ are equal too. It follows that

$$\mathrm{cov}(X, Y) = a\,\mathrm{cov}(I_1, J_1) + a(a-1)\,\mathrm{cov}(I_1, J_2)\,.$$

Since $I_1 J_1 = 0$, it follows

$$\mathrm{cov}(I_1, J_1) = -\frac{1}{(1+b)^2}\,.$$

Furthermore,

$$\mathrm{cov}(I_1, J_2) = \frac{1}{(b+1)(b+2)} - \frac{1}{(b+1)^2} = -\frac{1}{(b+1)^2(b+2)}\,.$$

We get

$$\mathrm{cov}(X, Y) = -\frac{a}{(b+1)^2} - \frac{a(a-1)}{(b+1)^2(b+2)} = -\frac{a(a+b+1)}{(b+1)^2(b+2)}\,.$$

5. (20) The natural numbers decide to go to a Chinese restaurant, where there are infinitely many round tables numbered by $1, 2, \ldots$. Every table can accommodate infinitely many guests. The natural numbers arrive to the restaurant one by one and are seated following the rules: 1 sits at table 1. When n arrives, it chooses to sit at the empty table with the lowest number available with probability $1/n$, or it chooses to sit to the left of i with probability $1/n$, independently of the previous numbers $i = 1, 2, 3, \ldots, n-1$. Let $X_{n,i}$ be the number of guests at the i-th table just after the arrival of number n to the restaurant.

a. (10) Compute the conditional distribution of $X_{n,1}$ given the event $\{X_{n-1,1} = l\}$.

   Solution: from the wording of the problem it follows that

$$P\left(X_{n,1} = l+1 | X_{n-1,1} = l\right) = \frac{l}{n} \quad \textit{in} \quad P\left(X_{n,1} = l | X_{n-1,1} = l\right) = \frac{n-l}{n}$$

   for $l = 1, 2, \ldots, n-1$.

b. (10) Compute $E(X_{n,1})$.

   Solution: From a. we compute

$$E\left(X_{n,1} | X_{n-1,1} = l\right) = l \cdot \frac{n-l}{n} + (l+1) \cdot \frac{l}{n} = \frac{l(n+1)}{n}.$$

   We know that $E(X_{1,1}) = 1$. *By the total expectation formula we get*

$$E\left(X_{n,1}\right) = \frac{(n+1)}{n} \cdot E(X_{n-1,1}),$$

   and it follows

$$E(X_{n,1}) = \frac{(n+1)n(n-1)\cdots 3}{n(n-1)\cdots 2} \cdot E(X_{1,1}) = \frac{n+1}{2}.$$

6. (20) In the Perla Casino in Nova Gorica, the guest Gregoroni won 144,000€ playing roulette. From the records we infer that he persistently played in the same way. In every game Gregoroni bet 500€. He always placed 200€ on a "straight up" bet on the single number 17, and a "split" bet of 300€ on numbers 16 and 17. If the number 17 came up, Gregoroni got the initial stake back and an additional 35 times the bet. Otherwise he lost the stake. In the "split" case, that is if 16 or 17 came up, Gregoroni got the initial stake back and in additional 17 times the bet. Otherwise he lost the stake.

There are 37 numbers on roulette, all are equally likely to be the winning numbers, and subsequent spins are independent.

a. (10) Denote by X the net profit of the guest in one game for Gregoroni's betting strategy. List the possible values of X and its distribution. Compute $E(X)$ and $\mathrm{var}(X)$.

Solution: the possible values for X are -500, if 16 and 17 do not come up 4.900, if 16 is the winning number and 12.100, if 17 is the winning number. The corresponding probabilities are $35/37$, $1/37$ in $1/37$. We compute

$$E(X) = -\frac{35 \cdot 500}{37} + \frac{4.900}{37} + \frac{12.100}{37} = -\frac{500}{37} \doteq -13,51\,.$$

We compute

$$E(X^2) = \frac{35 \cdot 500^2}{37} + \frac{4.900^2}{37} + \frac{12.100^2}{37} = \frac{179.170.000}{37}\,.$$

It follows

$$\mathrm{var}(X) = E(X^2) - E(X)^2 = \frac{6.629.040.000}{1.369} \doteq (2.200,51)^2\,.$$

b. (10) Gregoroni won his money in 582 spins. Compute, approximately, the probability that a guest using Gregroni's betting strategy wins 144.000€ or more in $n = 582$ spins of roulette.

Solution: we use the central limit theorem. From the first part of the exercise we get $\mu = -13,51$ and $\sigma = 2.200,51$. In the formula we take $n = 582$. We compute

$$\begin{aligned} P\left(S_{582} \geq 144.000\right) &= P\left(\frac{S_{582} - 582\mu}{\sqrt{582}\sigma} \geq \frac{144.000 - 582\mu}{\sqrt{582}\sigma}\right) \\ &\approx P\left(Z \geq 2,86\right) \\ &= 0,002\,. \end{aligned}$$