Name and surname: ______________________

Identification number: | | | | | | | | |

University of Primorska

FAMNIT, Mathematics

Probability

Midterm 1

April 16th, 2024

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. Time allowed: 120 minutes. Good luck!

Question	a.	b.	c.	d.	Total
1.			•	•	
2.			•	•	
3.				•	
4.			•	•	
5.			•	•	
6.				•	
Total					

1. (20) A standard deck of 52 playing cards is shuffled well and cards are dealt from the top. The gamblers have to guess the colour (red or black) of the 26th card after 25 cards are dealt. Gambler Louis predicts a red card if there are fewer red cards than black ones among the first 25 cards, else he predicts a black card. Denote

$$S_m = \sum_{k=0}^{m} \frac{\binom{26}{k}\binom{26}{25-k}}{\binom{52}{25}} \quad \text{and} \quad T_m = \sum_{k=0}^{m} k \cdot \frac{\binom{26}{k}\binom{26}{25-k}}{\binom{52}{25}}$$

for $m = 0, 1, \ldots, 25$.

a. (5) Let B_k be the event that among the first 25 cards dealt there are k red cards for $k = 0, 1, \ldots, 25$. Compute $P(B_k)$.

Solution: the first 25 cards are a random sample from all of the cards. For

$$X = \textit{number of red cards among first 25 cards}$$

we have $X \sim \mathit{HiperGeom}(25, 26, 52)$, *and therefore*

$$P(B_k) = \frac{\binom{26}{k}\binom{26}{25-k}}{\binom{52}{25}}\,.$$

b. (5) Let A be the event that Louis's prediction is correct. Compute the conditional probability $P(A|B_k)$.

Solution: the 26*th card is any of the remaining 27 cards with the same (conditional) probability. It follows that*

$$P(A|B_k) = \begin{cases} \frac{26-k}{27} & \textit{for } k = 0, 1, \ldots, 12 \\ \frac{26-(25-k)}{27} & \textit{for } k = 13, 14, \ldots, 25. \end{cases}$$

c. (10) Express the probability that Louis correctly predicts the colour with S_m and T_m.

Solution: the events $B_0, B_1, \ldots, B_{25}$ *are a partition. Using the law of total probability, we get*

$$\begin{aligned} P(A) &= \sum_{k=0}^{25} P\left(A \mid B_k\right) P(B_k) \\ &= \sum_{k=0}^{12} \frac{26-k}{27} \cdot \frac{\binom{26}{k}\binom{26}{25-k}}{\binom{52}{25}} + \sum_{k=13}^{25} \frac{1+k}{27} \cdot \frac{\binom{26}{k}\binom{26}{25-k}}{\binom{52}{25}} \\ &= \frac{26}{27} \cdot S_{12} - \frac{1}{27} T_{12} + \frac{1}{27}\left(S_{25} - S_{12}\right) + \frac{1}{27}\left(T_{25} - T_{12}\right) \\ &= \frac{25}{27} S_{12} + \frac{1}{27} S_{25} - \frac{2}{27} T_{12} + \frac{1}{27} T_{25}\,. \end{aligned}$$

With the help of Mathematica we get $P(A) \doteq 0{,}5545$.

2. (20) We toss a fair coin n times, where $n \geq 3$. Let A be the event the pattern HTT appears in the n tosses. The tosses are independent.

a. (10) Define the events

$$A_i = \{\text{tosses } i,\, i+1,\, i+2 \text{ are HTT}\}$$

for $i = 1, 2, \ldots, n-2$. What are the possible values for probabilities

$$P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}\right)$$

for $i_1 < i_2 < \cdots < i_r$ and $r \leq n-2$?

Solution: if the sets $\{i, i+1, i+2\}$ and $\{j, j+1, j+2\}$ for $i \neq j$ have non-empty intersection, we have $P(A_i \cap A_j) = 0$. The probability of the intersection will be positive if the sets $\{i_k, i_k+1, i_k+2\}$ are disjoint for $k = 1, 2, \ldots, r$. This means $3r \leq n$. Due to independence

$$P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}\right) = \left(\frac{1}{2}\right)^{3r}.$$

b. (10) Express the probability of the event A. You do not need to simplify the sums.

Hint: to count the possible selections of non-overlapping sets $\{i, i+1, i+2\}$, $i = 1, 2, \ldots, r$, join the chosen sets into one element.

Solution: we have $A = \cup_{i=1}^{n-2} A_i$. For $3r \leq n$, we need to count in how many ways can we select r disjoint subsets of three consecutive numbers. We select r numbers among $n-2r$ numbers first, and add two additional numbers. There is $\binom{n-2r}{r}$ selections. Using the inclusion-exclusion formula we get

$$P(A) = \sum_{r;\, 3r \leq n} (-1)^{r-1} \binom{n-2r}{r} \left(\frac{1}{2}\right)^{3r}.$$

3. (20) We are tossing a fair coin until we get heads followed by tails. Assume the tosses are independent. Let X be the number of tosses needed, including the last tails.

a. (5) Let A_{jk} be event that the first tosses j are tails, the next $k-j-1$ tosses are heads, and the k-th toss is tails for $j = 0, 1, \ldots, k-2$. Express the event $\{X = n\}$ with events A_{jk}.

   Solution: we have $\{X = k\} = \cup_{j=0}^{k-2} A_{jk}$.

b. (10) Compute the distribution of the random variable X.

   Solution: the random variable X *takes values* $2, 3, 4, \ldots$ *The events* A_{jk} *are disjoint and have probability* 2^{-k}. *It follows:*

$$P(X = k) = \frac{k-1}{2^k}\,.$$

c. (5) For every pair (j, k), where j and k are natural numbers and $j < k$, compute the conditional probability of the event that the j-th toss is heads, given the event $\{X = k\}$.

   Solution: the question is to find

$$P\big(A_{0k} \cup A_{1k} \cup \cdots \cup A_{j-1,k} \bigm| X = k\big) = \frac{P(A_{0k} \cup A_{1k} \cup \cdots \cup A_{j-1,k})}{P(X = k)} = \frac{j}{k-1}\,.$$

4. (20) Let $U \sim U(0,1)$. Let $f\colon (0,1) \to \mathbb{R}$ be given by

$$f(u) = \frac{1}{2}\log\left(\frac{1+u}{1-u}\right)$$

and define

$$X = f(U)\,.$$

a. (10) Find the probability density function of the random variable X.

Solution: the function f is strictly increasing on $(0,1)$ and maps the interval $(0,1)$ onto $(0,\infty)$. We need the inverse of the function f, which means thet we need to solve the equation

$$x = \frac{1}{2}\log\left(\frac{1+u}{1-u}\right)\,.$$

Using the exponential function we get

$$e^{2x} = \frac{1+u}{1-u}\,,$$

or

$$e^{2x} - 1 = u(e^{2x}+1)\,.$$

It follows

$$u = f^{-1}(x) = \frac{e^{2x}-1}{e^{2x}+1} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \tanh(x)\,.$$

We compute for $x > 0$

$$\begin{aligned} F_X(x) &= P\left(X \le x\right) \\ &= P\left(f(U) \le x\right) \\ &= P\left(U \le f^{-1}(x)\right) \\ &= \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \end{aligned}$$

Taking the derivative, we get

$$f_X(x) = \frac{4}{(e^{x}+e^{-x})^2}$$

for $x > 0$, and $f_X(x) = 0$ else.

b. (10) For given $p \in (0,1)$, find x_p such that $P\left(X \le x_p\right) = p$.

Solution: we compute

$$\begin{aligned} P\left(X \le x\right) &= P\left(f(U) \le x\right) \\ &= P\left(U \le f^{-1}(x)\right) \\ &= f^{-1}(x)\,. \end{aligned}$$

We need to find x such that $f^{-1}(x) = p$, or

$$x_p = f(p) = \frac{1}{2}\log\left(\frac{1+p}{1-p}\right)\,.$$

5. (20) From an urn which initially contains a white and $b \geq 2$ black balls, balls are drawn uniformly at random. Each time a ball is drawn, replace it by a white ball regardless of its colour. The drawings are independent. Let X be the number of balls drawn until and including the first black ball, and let Y be the number of balls drawn between the first two black balls, including the second, but not the first black ball.

a. (10) Find the joint distribution of X and Y.

Solution: the possible values of the random vector (X, Y) are the pairs (k, l) where $k, l \geq 1$. The event $\{X = k, Y = l\}$ happens if we first draw $k-1$ white balls, then a black ball, than $l-1$ white balls, and finally a black ball. We have

$$P(X = k, Y = l) = \left(\frac{a}{a+b}\right)^{k-1} \cdot \frac{b}{a+b} \cdot \left(\frac{a+1}{a+b}\right)^{l-1} \cdot \frac{b-1}{a+b},$$

which simplifies to

$$P(X = k, Y = l) = \frac{a^{k-1}(a+1)^{l-1} b(b-1)}{(a+b)^{k+l}}.$$

In other words, X and Y are independent with $X \sim \mathrm{Geom}\left(\frac{b}{a+b}\right)$ and $Y \sim \mathrm{Geom}\left(\frac{b-1}{a+b}\right)$.

b. (10) Compute $P(X \geq Y)$.

Solution: we compute using independence

$$\begin{aligned}
P(X \geq Y) &= \sum_{k \geq l \geq 1} P(X = k, Y = l) \\
&= \sum_{k \geq l \geq 1} P(X = k) P(Y = l) \\
&= \sum_{l=1}^{\infty} P(Y = l) \sum_{k=l}^{\infty} P(X = k) \\
&= \sum_{l=1}^{\infty} P(Y = l) \left(\frac{a}{a+b}\right)^{l-1} \\
&= \sum_{l=1}^{\infty} \left(\frac{b-1}{a+b}\right) \left(\frac{a+1}{a+b}\right)^{l-1} \left(\frac{a}{a+b}\right)^{l-1} \\
&= \frac{(b-1)(a+b)}{(a+b)^2 - a(a+1)} \\
&= \frac{(b-1)(a+b)}{b^2 - a(2b-1)}.
\end{aligned}$$

6. (20) Assume that we have 52 cards numbered with numbers $1, 2, \ldots, 13$. Every number appears exactly four times. The player is dealt five cards from the well-shuffled deck of cards. Any seletion of five cards is equally likely.

a. (10) Compute the probability that the player gets cards with five consecutive numbers, e.g., $\{3, 4, 5, 6, 7\}$. The order in which cards are dealt is irrelevant.

Solution: For every $i = 1, 2, \ldots, 9$ let denote

$$A_i = \{\textit{player will get the numbers } i, i+1, i+2, i+3, i+4\}\,.$$

We are looking for the probability of the union $A = \cup_{i=1}^{9} A_i$. The events in the union are disjoint, therefore it is enough to find the probabilities of A_i. We have to count all subsets of five cards where there is one representative of the numbers $i, i+1, i+2, i+3, i+4$. While there are 4 options in every category, there are 4^5 fivetuples. The probability we are searchng for is equal to

$$P(A_i) = \frac{4^5}{\binom{52}{5}}\,,$$

and consequently

$$P(A) = \frac{9 \cdot 4^5}{\binom{52}{5}}\,.$$

b. (10) Assume that a *Joker* card is added to the deck. Assume it can replace any other card. There is 53 cards in the deck now. Example: If the Joker card is denoted by J, the sequence $\{3, \mathrm{J}, 5, 6, 7\}$ means five consecutive numbers because J replaces the number 4. Compute the probability that the player gets five consecutive numbers in this case.

Solution: we can get the consecutive quintuple in the following disjoint ways

- *five consecutive cards:* $9 \cdot 4^5$ *outcomes*
- *four consecutive cards plus joker:* $10 \cdot 4^4$ *outcomes*
- $i, i+2, i+3, i+4$ *plus joker:* $9 \cdot 4^4$ *outcomes*
- $i, i+1, i+3, i+4$ *plus joker:* $9 \cdot 4^4$ *outcomes*
- $i, i+1, i+2, i+4$ *plus joker:* $9 \cdot 4^4$ *outcomes*

Together $73 \cdot 4^4$ outcomes, where there is $\binom{53}{5}$ all outcomes. The final probability is

$$\frac{73 \cdot 4^4}{\binom{53}{5}} = \frac{18688}{2869685} \doteq 0{,}006512213\,.$$