Name and surname: ______________________ Identification number: [| | | | | | |]

University of Primorska

FAMNIT, Mathematics

Probability

Midterm 1

April 20th, 2023

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems, and a handbook of mathematics. Time allowed: 120 minutes.

Question	a.	b.	c.	d.	Total
1.			•	•	
2.			•	•	
3.				•	
4.			•	•	
5.			•	•	
6.				•	
Total					

1. (20) Guinevere and Lancelot have a deck of 8 cards, 4 of which are red and 4 are white. They take turns drawing cards at random from the deck without replacing them until all the cards are drawn. Guinevere is the first to draw a card.

a. (10) What is the probability that Guinevere will be the first to draw a red card?

Solution: Guinevere is the first to draw a red card if the first $i-1$ draws produce white cards, and the i-th draw produces a red card for $i = 1, 3, 5$. Denote the described events by A_i. They are disjoint with

$$P(A_1) = \frac{4}{8}, \qquad P(A_3) = \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{4}{6}$$

and

$$P(A_5) = \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} \cdot \frac{4}{4}.$$

Adding the probabilities we get $\frac{23}{35}$.

b. (10) What is the conditional probability that Lancelot draws a red card just after Guinevere draws the first red card, given that Guinevere is the first to draw a red card?

Solution: let A be the event that Guinevere is the first to draw a red card, and B the event that Lancelot draws a red card just after Guinevere draws the first red card. The event $A \cap B$ happens if the first red card appears on the i-th draw for $i = 1, 3, 5$ and Lancelot draws a red card on the $(i+1)$-st draw. The ways are disjoint and we have

$$P(A_1 \cap B) = \frac{4}{8} \cdot \frac{3}{7}, \qquad P(A_3 \cap B) = \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{3}{5}$$

and

$$P(A_5 \cap B) = \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} \cdot \frac{4}{4} \cdot \frac{3}{3}.$$

Adding fractions gives $11/35$. *We have*

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{11}{23}.$$

2. (20) An urn contains a white and b black balls. We draw balls at random from the urn. We return the ball drawn on the k-th step to the urn just after the $(k+1)$-st draw. Let A_n be the event that the n-th draw results in a white ball.

a. (10) Compute $p_n = P(A_n)$.

Hint: compute p_2 and p_3.

Solution: for $n=1$ we have

$$p_1 = \frac{a}{a+b}\,.$$

The formula for total probabilities gives

$$P(A_n) = P(A_n \mid A_{n-1})\,P(A_{n-1}) + P(A_n \mid A_{n-1}^c)\,P(A_{n-1}^c)\,.$$

We rewrite

$$p_n = \frac{a-1}{a+b-1}\cdot p_{n-1} + \frac{a}{a+b-1}\cdot(1-p_{n-1})\,.$$

The recursion uniquely determines the probabilities p_n. However, the constant

$$p_n = \frac{a}{a+b}$$

satisfies the recursion equation, so $p_n = p_1$ for all n.

b. (10) Denote $p_{k,n} = P(A_n \mid A_k)$ for $k \le n$. Show that

$$p_{n,k} = \frac{a}{a+b} + \frac{(-1)^{n-k} b}{(a+b)(a+b-1)^{n-k}}\,.$$

Hint: what is

$$P(A_n \mid A_{n-1}\cap A_k)\,P(A_{n-1}\mid A_k) + P(A_n \mid A_{n-1}^c \cap A_k)\,P(A_{n-1}^c \mid A_k)\ ?$$

Note that $p_{k,k} = 1$.

Solution: by definition $p_{k,k}=1$. We check

$$P(A_n \mid A_k) = P(A_n \mid A_{n-1}\cap A_k)\,P(A_{n-1}\mid A_k) + P(A_n \mid A_{n-1}^c \cap A_k)\,P(A_{n-1}^c \mid A_k)\,.$$

It follows that

$$p_{n,k} = \frac{a-1}{a+b-1}p_{n-1,k} + \frac{a}{a+b-1}(1-p_{n-1,k})\,.$$

The recursion equation uniquely determines $p_{k,n}$ for $n \ge k$ with $p_{k,k}=1$. We need to check that the given expression for $p_{n,k}$ satisfies the recursion. We compute

$$\begin{aligned}
&\frac{a-1}{a+b-1}\cdot p_{n-1,k} + \frac{a}{a+b}\cdot(1-p_{n-1,k})\\
&\quad= \frac{a-1}{a+b-1}\cdot\left(\frac{a}{a+b} + \frac{(-1)^{n-k-1}b}{(a+b)(a+b-1)^{n-k-1}}\right)\\
&\qquad+ \frac{a}{a+b-1}\cdot\left(\frac{b}{a+b} - \frac{(-1)^{n-k-1}b}{(a+b)(a+b-1)^{n-k-1}}\right)\\
&\quad= \frac{a}{a+b} + \frac{(-1)^{n-k}b}{(a+b)(a+b-1)^{n-k}}\,(-a+1+a)\\
&\quad= p_{k,n}\,.
\end{aligned}$$

3. (20) A fair die is rolled until all six possible faces appear. Let X be the number of rolls. Assume that the rolls are independent.

a. (5) Denote by $A_{n,i}$ the event that the face i does not appear in the first n rolls for $i = 1, 2, 3, 4, 5, 6$. Compute $P(A_{n,1} \cap A_{n,2} \cap \cdots \cap A_{n,i})$.

   Solution: the intersection is the event that the first n rolls show the outcomes $i+1, i+2, \ldots, 6$. By independence, we have that

$$P(A_{n,1} \cap A_{n,2} \cap \cdots \cap A_{n,i}) = \left(\frac{6-i}{6}\right)^n .$$

b. (5) Express the event $\{X > n\}$ with the events $A_{n,i}$.

   Solution: the event $\{X > n\}$ happens if there is at least one of the six faces that we do not see in the first n rolls. It follows that

$$\{X > n\} = \bigcup_{i=1}^{6} A_{n,i} .$$

c. (10) Find the probability $P(X > n)$ and deduce the distribution of the random variable X. Write the result in closed form.

   Solution: the possible values of X are $n = 6, 7, \ldots$. We have

$$P(X = n) = P(X > n-1) - P(X > n) .$$

   Using symmetry, we find that the probability of any intersection of k distinct $A_{n,i}$ has the same probability. The inclusion-exclusion formula gives for $n = 6, 7, \ldots$ that

$$\begin{aligned} P(X > n) &= \sum_{i=1}^{5} (-1)^{i-1} \binom{6}{i} P(A_{n,1} \cap \cdots \cap A_{n,i}) \\ &= \sum_{i=1}^{5} (-1)^{i-1} \binom{6}{i} \left(\frac{6-i}{6}\right)^n \\ &= 6\left(\frac{5}{6}\right)^n - 15\left(\frac{4}{6}\right)^n + 20\left(\frac{3}{6}\right)^n - 15\left(\frac{2}{6}\right)^n + 6\left(\frac{1}{6}\right)^n . \end{aligned}$$

   In the above expression we take into account that

$$P\left(A_{n,1} \cap A_{n,2} \cap A_{n,3} \cap A_{n,4} \cap A_{n,5} \cap A_{n,6}\right) = 0 .$$

For $n = 5$ we obviously have $P(X > 5) = 1$ which turns out to be the result if we plug $n = 5$ into the inclusion-exclusion formula above. We have

$$
\begin{aligned}
P(X = n) &= \sum_{i=1}^{5} (-1)^{i-1} \binom{6}{i} \left[\left(\frac{6-i}{6} \right)^{n-1} - \left(\frac{6-i}{6} \right)^{n} \right] \\
&= \sum_{i=1}^{5} (-1)^{i-1} \binom{6}{i} \frac{i}{6} \left(\frac{6-i}{6} \right)^{n-1} \\
&= \sum_{i=1}^{5} (-1)^{i-1} \binom{5}{i-1} \left(\frac{6-i}{6} \right)^{n-1} .
\end{aligned}
$$

Remark: a direct computation gives

$$
P(X = 6) = \frac{6!}{6^6} = \frac{5!}{6^5}
$$

which coincides with the result above for $n = 6$.

4. (20) There are $B \geq 2$ white and R red balls in the urn. We select the balls from the urn randomly one by one without replacement. Let X be the number of selections up to and including the first white ball, and Y be the number of selections up to and including the second white ball.

a. (10) Compute *joint* distribution of random variables X and Y.

Solution: the possible pairs of values of random variables X and Y are all integer valued pairs (k, l), for which $1 \leq k < l \leq R + 2$. For the event $\{X = k, Y = l\}$ to happen, we need to get $k - 1$ red balls first, a white ball followed by $l - k - 1$ red balls, and again a white ball. Let us denote $N = B + R$. The probability of the event $\{X = k, Y = l\}$ is computed as

$$\begin{aligned} P(X = k, Y = l) &= \frac{R}{N} \cdot \frac{R-1}{N-1} \cdots \frac{R-k+2}{N-k+2} \cdot \frac{B}{N-k+1} \cdot \\ &\qquad \cdot \frac{R-k+1}{N-k} \cdots \frac{R-l+3}{N-l+2} \cdot \frac{B-1}{N-l+1} \\ &= \frac{B(B-1)\, R!\, (N-l)!}{(R-l+2)!\, N!}\,, \end{aligned}$$

or as

$$P(X = k, Y = l) = \frac{\binom{N-l}{B-2}}{\binom{N}{B}} = \frac{B(B-1)\, R!\, (N-l)!}{(R-l+2)!\, N!}\,.$$

b. (10) Show that for every $l = 2, 3, \ldots, R + 2$ and $k = 1, 2, \ldots, l - 1$ we have

$$P(X = k, Y = l) = \frac{1}{l-1}\, P(Y = l)\,.$$

Solution: we use the formula for marginal distributions to get

$$P(Y = l) = \sum_{k=1}^{l-1} P(X = k, Y = l)\,,$$

and notice that all the probabilities in the sum are equal (independent of k). The claim follows.

5. (20) Assume the random variable X has the distribution given by

$$P(X=k) = \binom{2k}{k} \frac{\beta^k}{4^k\,(1+\beta)^{k+\frac{1}{2}}}$$

for $k = 0, 1, \ldots$ and $\beta > 0$.

a. (10) Compute $E(X)$.

Hint: check that

$$kP(X=k) = \frac{2\beta}{4(1+\beta)}[2(k-1)+1]P(X=k-1)$$

for $k \geq 1$.

Solution: the equality can be checked by a straightforward computation. Adding both sides of the equation over $k = 1, 2, \ldots$ *gives*

$$E(X) = \frac{\beta}{1+\beta}E(X) + \frac{\beta}{2(1+\beta)}$$

or

$$E(X) = \frac{\beta}{2}\,.$$

b. (10) Compute $E(X^2)$.

Hint: check that

$$k^2 P(X=k) = \frac{\beta}{4(1+\beta)}\left[4(k-1)^2 + 6(k-1) + 2\right] P(X=k-1)$$

for $k = 1, 2, \ldots$.

Solution: the equality follows by a straightforward calculation. Adding both sides of the equation over $k = 1, 2, \ldots$ *gives*

$$E(X^2)(1+\beta) = \beta E(X^2) + \frac{3\beta}{2}E(X) + \frac{\beta}{2}$$

or

$$E(X^2) = \frac{\beta(2+3\beta)}{4}\,.$$

6. (20) Djoković and Nadal are playing tennis. The outcomes of individual games are independent, and one or the other player wins with probability $\frac{1}{2}$. The player who is the first to win six games and has a two game advantage over the opponent wins the set. Let X be the number of games until one of the players wins.

a. (5) Find $P(X = n)$ for $n = 6, 7, 8, 9, 10$.

Solution: the event $\{X = n\}$ can happen in two disjoint ways: (i) Djoković wins in the n-th game and has 5 victories in the first $n-1$ games. (ii) same as (i) except that the name is Nadal instead of Djoković. Using the independence assumption and the binomial distribution we get

$$P(X = n) = 2 \cdot \binom{n-1}{5} \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2}.$$

The expression simplifies into

$$P(X = n) = \binom{n-1}{5} \left(\frac{1}{2}\right)^{n-1}.$$

In particular we have:

$$P(X = 6) = \frac{1}{32}, \quad P(X = 7) = \frac{3}{32}, \quad P(X = 8) = \frac{21}{128},$$
$$P(X = 9) = \frac{7}{32}, \quad P(X = 10) = \frac{63}{256}.$$

b. (5) Find the probability $P(X > 10)$.

Solution: we have

$$P(X > 10) = 1 - P(X = 6) - P(X = 7) - P(X = 8) - P(X = 9) - P(X = 10)$$

which is $\frac{63}{256}$.

c. (10) Find $P(X = n)$ for $n > 10$. The probabilities are positive for even n only.

Solution: for n even we have $\{X = n\}$ if the players are even after the first 10 games, in the pairs of games $(11, 12)$,$(13, 14)$, ...,$(n-3, n-2)$ we have different winners, and in the games $(n-1, n)$ the winner is the same. The pairs of games are independent, and the desired events happen with probability $\frac{1}{2}$. Therefore, for $m = 6, 7, \ldots$ we have

$$P(X = 2m) = P(X > 10) \cdot \left(\frac{1}{2}\right)^{\frac{2m-10}{2}}.$$

The expression simplifies into

$$P(X = 2m) = \binom{10}{5} \left(\frac{1}{2}\right)^{m+5} = \frac{63}{2^{m+3}}.$$