Name and surname: ______________________

Identification number: | | | | | | | | |

University of Primorska

FAMNIT, Mathematics

Probability

Written examination

June 26th, 2024

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have 120 minutes.

Question	a.	b.	c.	d.	Total
1.			●	●	
2.			●	●	
3.			●	●	
4.			●	●	
5.			●	●	
6.			●	●	
Total					

1. (20) Four players are dealt 13 cards each from a well-shuffled deck of 52 cards. There are 4 Aces among the 52 cards.

a. (10) What is the probability at least one player has at least 2 Aces?

Solution: denote the event we are interested in by A. The opposite event is that every player has exactly one Ace. Let A_i be event that i-th player has exactly one Ace. We compute

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)P(A_4 \mid A_1 \cap A_2 \cap A_3)\,.$$

We have

$$\begin{aligned} P(A_1) &= \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} \\ P(A_2 \mid A_2) &= \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}} \\ P(A_3 \mid A_1 \cap A_2) &= \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}} \\ P(A_4 \mid A_1 \cap A_2 \cap A_3) &= 1 \end{aligned}$$

Combining the results in the above expression gives

$$P(A^c) = \frac{4! \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49}\,.$$

The final answer equals to $1 - P(A^c)$.

b. (10) What is the conditional probability that at least one player has at least two Aces given the event that the first player has exactly one Ace.

Solution: conditionally we deal cards to three players from a deck of 39 cards, 3 of which are Aces. Denote by B event that at least one player out of the remaining three players has at least two Aces. Similarly as in the first part we get

$$P(B^c \mid A_1) = \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}} \cdot \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}}\,,$$

which can be simplified to

$$P(B^c \mid A_1) = \frac{3! \cdot 13^3}{39 \cdot 38 \cdot 37}\,.$$

2. (20) Four players are dealt 13 cards each from a well-shuffled deck of 52 cards. There are 4 Aces among 52 cards. Let X_i be the number of Aces of the i-th player for $i = 1, 2, 3, 4$.

a. (10) Find the distribution of the random vector (X_1, X_2, X_3, X_4).

Solution: the possible values of the random vector are quadruples (k_1, k_2, k_3, k_4) where $k_i \geq 0$ and $\sum_{i=1}^4 k_i = 4$. We can assume that the first player gets the first 13 cards, the second player the next 13 cards, ... Among all 52! permutations we need to count those having k_1 Aces among the first 13 cards, k_2 Aces among the next 13 cards ... Let us choose the positions for Aces first. We can do that in

$$\binom{13}{k_1}\binom{13}{k_2}\binom{13}{k_3}\binom{13}{k_4}$$

ways. We can arrange the Aces in these positions in 4! ways. The remaining 48 cards can be arbitrary permuted. It follows

$$P(X_1 = k_1, X_2 = k_2, X_3 = k_3, X_4 = k_4) = \frac{4! \cdot 48! \cdot \binom{13}{k_1} \cdot \binom{13}{k_2} \cdot \binom{13}{k_3} \cdot \binom{13}{k_4}}{52!} .$$

b. (10) Compute $\mathrm{cov}(X_i, X_j)$ for $i \neq j$.

Solution:

First way:

Define

$$I_n = \begin{cases} 1 & \text{if the } i\text{-th player has the } n\text{-th Ace;} \\ 0 & \text{otherwise;} \end{cases}$$

and

$$J_n = \begin{cases} 1 & \text{if the } j\text{-th player has the } n\text{-th Ace;} \\ 0 & \text{otherwise.} \end{cases}$$

By symmetry

$$P(I_n = 1) = P(J_n = 1) = \frac{1}{4} .$$

Since $I_n J_n = 0$, we have

$$\mathrm{cov}(I_n, J_n) = -\frac{1}{16} .$$

For $n \neq m$ we have

$$P(I_m = 1, J_n = 1) = \frac{13^2}{52 \cdot 51} ,$$

and therefore

$$\mathrm{cov}(I_m, J_n) = \frac{1}{816} .$$

It follows

$$\begin{aligned}
\mathrm{cov}(X_i, X_j) &= \mathrm{cov}\left(\sum_{m=1}^{4} I_m, \sum_{n=1}^{4} J_n\right) \\
&= 4\mathrm{cov}(I_1, J_1) + 12\mathrm{cov}(I_1, J_2) \\
&= -\frac{1}{4} + \frac{1}{68} \\
&= -\frac{4}{17}\,.
\end{aligned}$$

Second way:

The cards of the i-th and the j-th player are a radnom sample of 26 cards among all cards, therefore $X_i + X_j \sim \mathrm{HiperGeom}(26, 4, 52)$. It follows

$$\mathrm{var}(X_1 + X_2) = 26 \cdot \frac{4}{52} \cdot \frac{48}{52} \cdot \frac{52-26}{52-1}\,.$$

Similarly $X_i \sim \mathrm{HiperGeom}(13, 4, 52)$, which means

$$\mathrm{var}(X_i) = 13 \cdot \frac{4}{52} \cdot \frac{48}{52} \cdot \frac{52-13}{52-1}\,.$$

We have

$$\begin{aligned}
\mathrm{cov}(X_i, X_j) &= \frac{1}{2}\left(\mathrm{var}(X_i + X_j) - \mathrm{var}(X_i) - \mathrm{var}(X_j)\right) \\
&= \frac{1}{2}\left(\frac{16}{17} - \frac{24}{17}\right) \\
&= -\frac{4}{17}\,.
\end{aligned}$$

Third way:

Since $X_1 + X_2 + X_3 + X_4 = 4$, we have

$$\mathrm{cov}(X_1, X_1 + X_2 + X_3 + X_4) = 0\,.$$

By symmetry all the covariances are equal, and using bilinearity we have

$$\mathrm{cov}(X_i, X_j) = -\mathrm{var}(X_1)\,.$$

We know that $X_1 \sim \mathrm{HiperGeom}(13, 4, 52)$, and therefore

$$\mathrm{cov}(X_i, X_j) = -13 \cdot \frac{4}{52} \cdot \frac{48}{52} \cdot \frac{52-13}{52-1} = -\frac{4}{17}\,.$$

3. (20) Let X and Y be independent with $X \sim \Gamma(a+b,\lambda)$ and $Y \sim \text{Beta}(a,b)$. Define

$$Z = XY \qquad \text{and} \qquad W = X(1-Y)\,.$$

a. (10) Find the density of the random variable Z.

Solution: let us find the density of the random vector (X, XY). The mapping

$$\Phi(x,y) = (x, xy)$$

maps $(0,\infty)\times(0,1)$ bijectively onto $\{(x,z)\colon x>0, 0<z<x\}$ and is differentiable. We have

$$\Phi^{-1}(x,z) = \left(x, \frac{z}{x}\right)$$

and $J_{\Phi^{-1}}(x,z) = 1/x$. We get

$$f_{X,Z}(x,z) = \frac{\lambda^{a+b}}{\Gamma(a+b)} x^{a+b-1}e^{-\lambda x} \cdot \frac{1}{\mathrm{B}(a,b)}(z/x)^{a-1}(1-z/x)^{b-1} \cdot \frac{1}{x}\,.$$

for $x>0$ and $0<y<x$. The density of Z is the marginal density of $f_{X,Z}(x,z)$. We compute

$$\begin{aligned} f_Z(z) &= \frac{\lambda^{a+b}}{\Gamma(a+b)\mathrm{B}(a,b)} z^{a-1}\int_z^\infty (x-z)^{b-1}e^{-\lambda x}dx \\ &= \frac{\lambda^{a+b}}{\Gamma(a+b)\mathrm{B}(a,b)} z^{a-1}\int_0^\infty u^{b-1}e^{-\lambda(z+u)}du \\ &= \frac{\lambda^{a+b}}{\Gamma(a+b)\mathrm{B}(a,b)} z^{a-1}e^{-\lambda z}\int_0^\infty u^{b-1}e^{-\lambda u}du \end{aligned}$$

We notice that the density is proportional to $z^{a-1}e^{-\lambda z}$, therefore Z has distribution $\Gamma(a,\lambda)$. All of the constants multiply to $\lambda^a/\Gamma(a)$.

b. (10) Are random variables XY and $X(1-Y)$ independent?

Solution: let us define

$$\Phi(x,y) = (xy, x(1-y))\,.$$

The mapping maps the domain $(0,\infty)\times(0,1)$ bijectively onto $(0,\infty)^2$ and is continously partially differentiable. We have

$$\Phi^{-1}(z,w) = \left(z+w, \frac{z}{z+w}\right)$$

and

$$J_{\Phi^{-1}}(y,w) = -\frac{1}{z+w}\,.$$

The transformation formula gives

$$\begin{aligned} f_{Z,W}(z,w) &= f_X(z+w) f_Y\left(\frac{z}{z+w}\right) \cdot \frac{1}{z+w} \\ &= \frac{\lambda^{a+b}}{\Gamma(a+b)}(z+w)^{a+b-1} e^{-\lambda(y+w)} \cdot \\ &\qquad \frac{1}{\mathrm{B}(a,b)}\left(\frac{z}{z+w}\right)^{a-1}\left(1-\frac{z}{z+w}\right)^{b-1}\frac{1}{z+w} \\ &= \frac{\lambda^{a+b}}{\Gamma(a+b)\mathrm{B}(a,b)} z^{a-1} e^{-\lambda z} w^{b-1} e^{-\lambda w} . \end{aligned}$$

The density of (Y,W) *is a product of a function depending on* z *only, and a function dependent on* w *only on* $(0,\infty)^2$*, therefore independence follows. We see that* $W \sim \Gamma(b,\lambda)$*.*

4. (20) Let Π be a random permutation of n elements. For a given permutation π we have $P(\Pi = \pi) = \frac{1}{n!}$. A pair (i, j) with $1 \leq i < j \leq n$ is called an inversion of the permutation π, if $\pi(i) > \pi(j)$. Let S_n be the number of all inversions of random permuation Π.

a. (10) For fixed $2 \leq j \leq n$ define random variables

$$X_j = \sum_{i=1}^{j-1} I_{ij} \,,$$

where the indicators I_{ij} are defined as

$$I_{ij} = \begin{cases} 1 & \text{if } \pi(i) > \pi(j) \\ 0 & \text{otherwise}\,. \end{cases}$$

Show that random variables $X_2, \ldots, X_n$ are independent and find their distributions. Show that $S_n = X_2 + \cdots + X_n$.

Solution: let $k_2, \ldots, k_n$ be nonnegative integers with $k_j < j$. From he last number k_n we infer that $\pi(n) = n - k_n$. Once we know $\pi(n)$, from k_{n-1} we can infer the position of $\pi(n-1)$. We continue and conclude that the numbers $k_2, \ldots, k_n$ uniquely determine the permutation. It follows

$$P\left(X_2 = k_2, \ldots, X_n = k_n\right) = \frac{1}{n!}\,.$$

For every j, the number $\Pi(j)$ is distributed uniformly on the set $\{1, 2, \ldots, n\}$. It follows

$$\Pi(X_n = k) = P(\Pi(n) = n - k) = \frac{1}{n}$$

for $k = 0, 1, \ldots, n$. We sum up to get

$$\sum_{k_n=0}^{n-1} P\left(X_2 = k_2, \ldots, X_n = k_n\right) = \frac{1}{(n-1)!}\,.$$

If n is eliminated from the permutation of n elements, and the order of the remaining elements is not changed, we get the random permutation of $n-1$ elements. By induction it follows

$$P(X_j = k) = \frac{1}{j}$$

for $0 \leq k < j$. The equality $S_n = X_2 + \cdots + X_n$ is just counting inversions in a slightly different order.

b. (10) Compute $E(S_n)$ and $\mathrm{var}(S_n)$.

Solution: from the first part it follows

$$E(X_j) = \frac{j-1}{2}$$

and

$$\mathrm{var}(X_j) = \frac{j^2 - 1}{12}\,.$$

It follows

$$E(S_n) = \frac{n(n-1)}{4}$$

and

$$\mathrm{var}(S_n) = \frac{n\,\left(-5 + 3\,n + 2\,n^2\right)}{72}\,.$$

5. (20) In a simple model of an epidemic one assumes that in the first wave of infections every individual in the population of size n gets infected with probability p independently of all the others. In the second wave of infections, every non-infected individual is infected with probability equal to the proportion of infected individuals in the first wave, independently of all the other non-infected individuals. Let X be the number of infected individuals after the first wave of infections and by Y the number of all the infected individuals after the second wave of infection. More mathematically, we have:

$$P(Y = l|X = k) = \binom{n-k}{l-k}\left(\frac{k}{n}\right)^{l-k}\left(1-\frac{k}{n}\right)^{n-l}$$

for $0 \leq k \leq l \leq n$.

a. (10) Compute $E(Y|X = k)$.

Solution: we compute

$$\begin{aligned} E(Y|X = k) &= \sum_{l=k}^{n} lP(Y = l|X = k) \\ &= \sum_{l=k}^{n} l\binom{n-k}{l-k}\left(\frac{k}{n}\right)^{l-k}\left(1-\frac{k}{n}\right)^{n-l} \\ &= \sum_{i=0}^{n-k}(k+i)\binom{n-k}{i}\left(\frac{k}{n}\right)^{i}\left(1-\frac{k}{n}\right)^{n-k-i} \\ &= k + \frac{(n-k)k}{n}\,. \end{aligned}$$

We have used the known expected value for the binomial distribution.

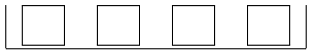
b. (10) Compute $E(Y)$.

Solution: since $X \sim \mathrm{Bin}(p)$*, using the formula for total expectation it follows*

$$\begin{aligned} E(Y) &= \sum_{k=0}^{n} E(Y|X = k)P(X = k) \\ &= \sum_{k=0}^{n}\left(k + \frac{(n-k)k}{n}\right)P(X = k) \\ &= E(X) + \frac{1}{n}E(X(n-X)) \\ &= np - \frac{1}{n}\left(nE(X) - E(X^2)\right) \\ &= np - \frac{1}{n}\left(nE(X) - \mathrm{var}(X) - E(X)^2\right) \\ &= np - \frac{1}{n}\left(n^2p - npq - n^2p^2\right) \\ &= np + \frac{1}{n}\cdot n(n-1)pq \\ &= np + (n-1)pq\,. \end{aligned}$$

6. (20) Consider the following game: Players A and B will each toss a fair coin 1000 times. Denote by X the number of heads after 1000 tosses of the player A, and by Y the number of heads after 1000 tosses of the player B. If $|X - Y| \leq 15$, A wins, otherwise B wins.

a. (10) There are 4 possible outcomes if two fair coins are tossed independently: HH, HT, TH, TT. Every outcome happens with probability 1/4. Fill the missing part in the following sentence: The difference $X - Y$ equals the sum of _____ random numbers generated by selecting tickets at random with replacement from the box



Hint: What happens to the difference of the number of heads after a toss of two coins?

Solution: for every toss of the two coins, the difference between the number of heads tossed does not change with probability $1/2$ *(they both get tails or both get heads), the difference increases with probability* $1/4$ *and decreases with probability* $1/4$*. The claim follows.*

b. (10) Compute, approximately, the probability that player A wins.

Solution: the mean of the box is $\mu = 0$ *and standard deviation is* $\sigma = \sqrt{1/2}$*. Let us denote the sum of random numbers that we get with random selection of* $n = 1000$ *tickets from the box by* S_{1000}*. The central limit theorem gives*

$$
\begin{aligned}
P(|X - Y| \leq 15) &= P(|S_{1000}| \leq 15) \\
&= P(-15 \leq S_{1000} \leq 15) \\
&= P\left(-\frac{15}{\sqrt{1000/2}} \leq \frac{S_{1000}}{\sqrt{1000/2}} \leq \frac{15}{\sqrt{1000/2}}\right) \\
&\approx P(-0,67 \leq Z \leq 0,67) \\
&= \Phi(0,67) - \Phi(-0,67) \\
&= 0,5\,.
\end{aligned}
$$