Name and surname: ______________________

Identification number: [ ][ ][ ][ ][ ][ ][ ][ ]

University of Primorska

FAMNIT, Mathematics

Probability

Written examination

June 21st, 2021

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems, and a handbook of mathematics. Time allowed: 120 minutes.

Question	a.	b.	c.	d.	Total
1.			•	•	
2.			•	•	
3.			•	•	
4.			•	•	
5.				•	
6.			•	•	
Total					

SOLUTIONS

1. (20) Assume that four players are dealt five cards each from a well shuffled deck of standard 52 cards. One of the possible hands is the *Royal flush* which means that a player gets the top five cards of the same suit. Example: 10 of Hearts, Jack of Hearts, Queen of Hearts, King of Hearts and Ace of Hearts.

a. (10) Find the probability that none of the players get a *Royal flush*. You do not need to simplify binomial symbols.

Solution: let $A_i = \{\text{player } i \text{ gets a Royal flush}\}$ *for* $i = 1, 2, 3, 4$. *We have*

$$\begin{aligned} P(A_1) &= 4 \cdot \frac{1}{\binom{52}{5}}, \\ P(A_1 \cap A_2) &= 4 \cdot 3 \cdot \frac{1}{\binom{52}{5}\binom{47}{5}}, \\ P(A_1 \cap A_2 \cap A_3) &= 4 \cdot 3 \cdot 2 \cdot \frac{1}{\binom{52}{5}\binom{47}{5}\binom{42}{5}}, \\ P(A_1 \cap A_2 \cap A_3 \cap A_4) &= 4 \cdot 3 \cdot 2 \cdot 1 \cdot \frac{1}{\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}}. \end{aligned}$$

We are interested in the probability $1 - P\left(\bigcup_{i=1}^4 A_i\right)$. *The inclusion-exclusion formula and symmetry give*

$$P\left(\bigcup_{i=1}^4 A_i\right) = 4P(A_1) - 6P(A_1 \cap A_2) + 4P(A_1 \cap A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

and the desired probability is

$$1 - \frac{16}{\binom{52}{5}} + \frac{72}{\binom{52}{5}\binom{47}{5}} - \frac{96}{\binom{52}{5}\binom{47}{5}\binom{42}{5}} + \frac{24}{\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}}.$$

The numerical result is $1 - 6.16 \cdot 10^{-6}$.

b. (10) Sir Lancelot is one of the players. Find the conditional probability that Sir Lancelot gets a *Royal flush* given that at least one of the players gets a *Royal flush*.

Solution: let B *be the event that at least one of the players gets a Royal flush, and* A *the event that Sir Lancelot gets one. We have*

$$P(A) = 4 \cdot \frac{1}{\binom{52}{5}}$$

and $A \cap B = A$. *Thus, the conditional probability equals* $P(A)/P(B)$. *The numerical result is* 0.25000073.

2. (20) Let σ be a permutation of $\{1, 2, \ldots, n\}$. The element $k \geq 2$ is a point of increase if $\sigma(k-1) < \sigma(k)$. Assume that we choose a permutation randomly in such a way that all permutations are equally likely. Let X be the number of points of increase in the permutation.

a. (10) Find $E(X)$.

Solution: define for $k = 2, 3, \ldots, n$

$$I_k = \begin{cases} 1 & \text{if } k \text{ is a point of increase and} \\ 0 & \text{else.} \end{cases}$$

We have $X = I_2 + \cdots + I_n$. *By symmetry the two elements in positions* k *and* $k-1$ *are a random pair, such that all possible pairs of distinct elements are equally likely. Hence the probability of a point of increase is* $\frac{1}{2}$. *It follows that*

$$E(X) = \frac{n-1}{2}\,.$$

b. (10) Find $\operatorname{var}(X)$.

Solution: we need probabilities $P(I_k = 1, I_l = 1)$ *for* $k \neq l$. *There are three cases:*

(i) *If* $k = l$, *then* $\operatorname{cov}(I_k, I_l) = \operatorname{var}(I_k) = \frac{1}{4}$.

(ii) *If* $l = k + 1$, *then the three elements in positions* $k-1, k, k+1$ *must be in increasing order. By symmetry this probability is* $\frac{1}{6}$ *and so* $\operatorname{cov}(I_k, I_l) = -\frac{1}{12}$. *The same for* $k = l + 1$.

(iii) *If* $|k - l| > 1$, *then* I_k *and* I_l *are independent by symmetry, so that* $\operatorname{cov}(I_k, I_l) = 0$.

Summing up, we find that

$$\operatorname{var}(X) = \frac{n-1}{4} - \frac{2(n-2)}{12} = \frac{n+1}{12}\,.$$

3. (20) Let U_1, U_2 and U_3 be independent and uniformly distributed on the interval $(0,1)$. Define

$$X = U_1, \qquad Y = U_2(1-U_1) \qquad \text{and} \qquad Z = U_3(1-U_2)(1-U_1)\,.$$

a. (10) Find the density of (X,Y,Z). State explicitly where the density is positive.

Solution: define the map

$$\Phi(u_1,u_2,u_3) = \big(u_1,\, u_2(1-u_1),\, u_3(1-u_2)(1-u_1)\big)\,.$$

The map Φ maps the set $(0,1)^3$ bijectively onto the set $\Delta = \big\{(x,y,z)\colon\ x,y,z > 0,\ x+y+z<1\big\}$. We have

$$\Phi^{-1}(x,y,z) = \left(x,\ \frac{y}{1-x},\ \frac{z}{1-x-y}\right)\,.$$

The Jacobian matrix is lower triangular so we only need diagonal elements to compute $J_{\Phi^{-1}}$. We get

$$J_{\Phi^{-1}}(x,y,z) = \frac{1}{(1-x)(1-x-y)}\,.$$

It follows that

$$f_{X,Y,Z}(x,y,z) = \frac{1}{(1-x)(1-x-y)}$$

on the set Δ and 0 elsewhere.

b. (10) Find the distribution of $W = 1-X-Y-Z$.

Hint: one way is to express W with U_1, U_2 and U_3.

Solution: we find that

$$W = (1-U_1)(1-U_2)(1-U_3)\,.$$

Note that $-\log(1-U_i) \sim \exp(1)$ so $-\log W \sim \Gamma(3,1)$, that is

$$f_{-\log W}(x) = \frac{x^2}{2}\,e^{-x}$$

for $x>0$ and $f_{-\log W}(x) = 0$ elsewhere. The function $\Psi(x) = e^{-x}$ maps $(0,\infty)$ bijectively onto $(0,1)$ with $\Psi^{-1}(w) = -\log w$, $J_{\Psi^{-1}}(w) = \big(\psi^{-1}\big)'(w) = -\frac{1}{w}$. Again by trensformation formula,

$$f_W(w) = \frac{(\log w)^2}{2}$$

for $w \in (0,1)$ and $f_W(w) = 0$ elsewhere.

4. (20) Let $X_1, X_2, \ldots$ be independent with $X_i \sim \text{Geom}(p)$. Denote $S_k = X_1 + X_2 + \cdots + X_k$ for $k \geq 1$.

a. (10) Let $n \geq 1$ be fixed and let $B_k = \{S_k \leq n, S_{k+1} > n\}$. Compute $P(B_k)$.

Hints:

- $B_k = \cup_{l=k}^{n} \{S_k = l, S_{k+1} > n\}$;
- $\binom{l-1}{k-1} + \binom{l-1}{k} = \binom{l}{k}$ *(under the convention that* $\binom{l}{k} = 0$ *if* $k < 0$ *or* $k > l$*).*

Solution: we know that $S_k \sim \text{NegBin}(k, p)$*. Let* $q = 1 - p$*. By independence*

$$\begin{aligned} P(B_k) &= P\big(S_k \leq n, S_{k+1} > n\big) \\ &= \sum_{l=k}^{n} P\big(S_k = l, X_{k+1} > n - l\big) \\ &= \sum_{l=k}^{n} P(S_k = l) P(X_{k+1} > n - l) \\ &= \sum_{l=k}^{n} \binom{l-1}{k-1} p^k q^{l-k} \cdot q^{n-l} \\ &= p^k q^{n-k} \sum_{l=k}^{n} \binom{l-1}{k-1} . \end{aligned}$$

Now use the second hint to rewrite

$$\begin{aligned} P(B_k) &= p^k q^{n-k} \sum_{l=k}^{n} \left[\binom{l}{k} - \binom{l-1}{k} \right] \\ &= p^k q^{n-k} \left[\binom{n}{k} - \binom{k-1}{k} \right] \\ &= p^k q^{n-k} \binom{n}{k} . \end{aligned}$$

b. (10) Find the conditional distribution of $S_{k+1} - n$ given B_k for $k \leq n$.

Solution: Fix $m \geq 1$*. Compute*

$$\begin{aligned} P(\{S_{k+1} - n = m\} \cap B_k) &= \sum_{l=k}^{n} P\big(\{S_{k+1} - n = m\} \cap \{S_k = l\}\big) \\ &= \sum_{l=k}^{n} P\big(\{X_{k+1} = m + n - l\} \cap \{S_k = l\}\big) \\ &= \sum_{l=k}^{n} p q^{m+n-l-1} \binom{l-1}{k-1} p^k q^{l-k} \\ &= p^{k+1} q^{m+n-k-1} \sum_{l=k}^{n} \binom{l-1}{k-1} \\ &= p^{k+1} q^{m+n-k-1} \binom{n}{k} . \end{aligned}$$

We get

$$P(S_{k+1} - n = m | B_k) = pq^{m-1}\,,$$

so that the desired conditional distribution is $\mathrm{Geom}(p)$.

5. (20) Tweedledum has two identical octahedral dice with numbers from 1 to 8 written on the faces. Tweedledee also has two octahedral dice. The numbers on the faces of his dice are possibly repeated non-negative integers. If the two roll their two dice the distributions of the sum of numbers that come up are identical. Assume that the dice are rolled independently and all faces are equally likely.

a. (5) Let X be the sum Tweedledum gets. Argue that

$$G_X(s) = \frac{1}{64}\, s^2(1+s)^2(1+s^2)^2(1+s^4)^2 \,.$$

Solution: the random variable X is the sum of two independent random variables with generating functions

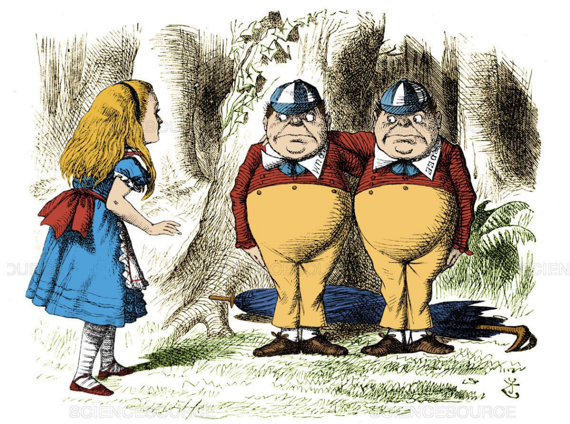
$$\frac{1}{8}\left(s+s^2+\cdots+s^8\right) = \frac{1}{8}\, s(1+s)(1+s^2)(1+s^4)\,.$$

The generating function of X is the square of the preceding expression, which is precisely the desired function.

b. (5) You know that one of Tweedledee's dice has number 0 on one of its faces. Argue that the other of his dice does not have 0 or 1 on the faces.

Solution: the sum is at least 2 with probability 1. If the first die has 0 on one of the faces, the smallest number on the other must be 2.

c. (10) Now you know that on the faces of one of Tweedledee's dice, the smallest number is 0 and the largest one is 8. What are the numbers written on the faces of Tweedledee's dice? Note that numbers on Tweedledee's dice are possibly repeated.



Tweedledum and Tweedledee from Lewis Carroll's Through the Looking Glass.

Solution: let Y be the random number we get rolling Tweedledee's die with 0 and let Z be the random number we get rolling the other Tweedledee's die. We need to have $G_X(s) = G_Y(s)\cdot G_Z(s)$. The two generating functions on the right *must be polynomials. This means that* $G_Y(s)$ *and* $G_Z(s)$ *must consist of factors of* $G_X(s)$*. One can assume that each of these factors equals* 1 *for* $s = 1$*, so that these factors are*

$$s\,, \quad \frac{1+s}{2}\,, \quad \frac{1+s^2}{2} \quad \textit{and} \quad \frac{1+s^4}{2}\,,$$

each of them appearing exactly twice in total. Since all probabilities must be multiples of $1/8$*, each one of the generating functions* G_Y *and* G_Z *can contain at most three of the factors* $\frac{1+s}{2}$*,* $\frac{1+s^2}{2}$ *and* $\frac{1+s^4}{2}$*. However, since there are six of them in total, each one must contain exactly three of these factors, possibly repeated.*

As the die with number 0 *has the largest number* 8*,* $G_Y(s)$ *is a polynomial of degree* 8 *containing no factor* s*, but exactly exactly three of the factors* $\frac{1+s}{2}$*,* $\frac{1+s^2}{2}$ *and* $\frac{1+s^4}{2}$*, possibly repeated. The only possibility is*

$$G_Y(s) = \frac{(1+s^2)^2(1+s^4)}{8} = \frac{1+2s^2+2s^4+2s^6+s^8}{8}\,,$$

making

$$G_Z(s) = \frac{s^2(1+s)^2(1+s^4)}{8} = \frac{s^2+2s^3+s^4+s^6+2s^7+s^8}{8}\,,$$

which means that we get $0, 2, 2, 4, 4, 6, 6, 8$ *for one die and* $2, 3, 3, 4, 6, 7, 7, 8$ *for the other.*

6. (20) Berti opens a stand with a game involving three dice. Every game costs 1 euro and the three dice are rolled. If no sixes show Berti keeps the stake. If exactly one six shows, Berti returns the stake to the player with additional 1 euro. If exactly two sixes show, Berti returns the stake to the player with additional 2 euros. If three sixes show, Berti returns the stake to the player with additional 14 euros. Assume the dice are fair and that all the rolls are independent.

a. (10) Compute the expected value and the variance of Berti's profit after n games.

Solution: Let X_i denote Berti's profit in i-th game. It holds:

$$X_i \sim \begin{pmatrix} -14 & -2 & -1 & 1 \\ \frac{1}{216} & \frac{15}{216} & \frac{75}{216} & \frac{125}{216} \end{pmatrix},$$

After short computation work we get $E(X_i) = 1/36$ and $\mathrm{var}(X_i) = 2735/1296$. Denoting Berti's profit after n games by S_n, it holds $E(S_n) = n/36$ and $\mathrm{var}(S_n) = 2735n/1296$.

b. (10) After approximately how many games will Berti have a positive profit with approximately 95% probability?

Solution: Let denote the number of games by n again. From the central limit theorem we get that, approximately.

$$1 - \Phi\left(\frac{-\frac{1}{36}n}{\sqrt{\frac{2735}{1296}n}}\right) = \Phi\left(\frac{\sqrt{n}}{\sqrt{2735}}\right) = 0{,}95$$

or

$$\frac{\sqrt{n}}{\sqrt{2735}} \doteq 1{,}645$$

which is true if n is approximately 7400.