Name and surname:

Identification number:

University of Primorska

FAMNIT, Mathematics

Probability

Written examination

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Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems, and a handbook of mathematics. Time allowed: 120 minutes.

Question	a.	b.	c.	d.	Total
1.			•	•	
2.			•	•	
3.			•	•	
4.				•	
5.			•	•	
6.			•	•	
Total					

1. (20) A player rolls a fair die. Assume the rolls are independent. Let A_k be the event that the rolls $k-5, k-4, \ldots, k$ equal $1, 2, 3, 4, 5, 6$, respectively. We call this a *run ending at k*. Fix n and let $A = \{$there is at least one completed run in the first n rolls$\}$.

a. (10) Let $6 \leq k_1 < k_2 < \ldots < k_r \leq n$. What are the possible values for $P(A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_r})$?

Solution: if any of the sets $\{k_i, k_i+1, \ldots, k_i+5\}$ overlap, the probability of the intersection is 0. Should this not happen, we must have $6r \leq n$. If the sets $\{k_i-5, k_i-4, \ldots, k_i\}$ are disjoint for $i = 1, 2, \ldots, r$, the events A_i are independent with probability $\left(\frac{1}{6}\right)^6$. In this case

$$P(A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_r}) = \left(\frac{1}{6}\right)^{6r} .$$

b. (10) Find the probability $P(A)$. You do not need to compute the sums explicitly.

Hint: to count in how many ways we can choose non-overlapping sets $\{k_i-5, k_i-4, \ldots, k_i\}$ for $i = 1, 2, \ldots, r$, collapse the chosen sets into one element.

Solution: we use the inclusion-exclusion formula. For a given r with $6r \leq n$, we need to count in how many ways we can choose non-overlapping sets $\{k_i-5, k_i-4, \ldots, k_i\}$ for $i = 1, 2, \ldots, r$. The hint implies that we need to choose r elements out of $n-5r$ elements. Finally,

$$P(A) = \sum_{r;\, 6r \leq n} (-1)^{r-1} \binom{n-5r}{r} \left(\frac{1}{6}\right)^{6r} .$$

2. (20) A standard deck of cards contains 52 cards. There are four aces. The cards are shuffled well. Let X be the position of the first ace from the top, and Y the position of the last ace from the top.

a. (10) Find the probabilities $P(X = k)$ for $1 \le k \le 49$.

Solution:

First method: *by symmetry the first k cards are a random sample of the 52 cards. The distribution of the number of aces among these k cards is* HiperGeom$(k, 4, 52)$. *The event $\{X > k\}$ happens if there are no aces in a random sample of k cards. It follows,*

$$P(X > k) = \frac{\binom{48}{k}}{\binom{52}{k}},$$

and

$$P(X = k) = P(X > k - 1) - P(X > k) = \frac{\binom{48}{k-1}}{\binom{52}{k-1}} - \frac{\binom{48}{k}}{\binom{52}{k}} = \frac{4}{49 - k}\,\frac{\binom{48}{k}}{\binom{52}{k}},$$

where we interpret $\binom{a}{b} = 0$ if $b > a$.

Second method: *consider the positions of the aces in the deck, not distinguishing between the aces. Out of $\binom{52}{4}$ equiprobable possibilities, there are $\binom{52-k}{3}$ such that the most upper ace appear in the k-th place. Therefore,*

$$P(X = k) = \frac{\binom{52-k}{3}}{\binom{52}{4}}$$

and it can be verified that this is the same value as the one obtained by the first method.

b. (10) Find the probabilities $P(Y \le l | X = k)$ for $1 \le k \le 48$ and $k - l \ge 3$. Derive the distribution of the pair (X, Y).

Solution: conditionally on $\{X = k\}$, the remaining $52 - k$ cards contain three aces, and are well shuffled. The conditional probability $P(Y \le l | X = k)$ is equal to the probability that the cards in positions $k + 1, k + 2, \ldots, l$ contain three aces. Hence,

$$P(Y \le l | X = k) = \frac{\binom{49-k}{l-k-3}}{\binom{52-k}{l-k}} = \frac{\binom{l-k}{3}}{\binom{52-k}{3}},$$

where we interpret $\binom{a}{b} = 0$ if $b < 0$; again, one can verify that both forms yield the same value. Finally,

$$\begin{aligned} P(X = k, Y = l) &= P(X = k)\left(P(Y \le l | X = k) - P(Y \le l - 1 | X = k)\right) \\ &= \frac{12}{(49 - k)(l - k)}\,\frac{\binom{48}{k}\binom{49-k}{l-k-3}}{\binom{52}{k}\binom{52-k}{l-k}} = \frac{\binom{l-k-1}{2}}{\binom{52}{4}}. \end{aligned}$$

The result in the latter form can also be obtained directly: considering the positions of the four aces in the deck, there are $\binom{52}{4}$ possibilities, among which there are $\binom{l-k-1}{2}$ with $X = k$ in $Y = l$.

3. (20) Let X and Z be independent with $X \sim \exp(1)$ and $Z \sim \mathrm{N}(0,1)$.

a. (10) Find the density of the vector

$$\left(Z, \sqrt{2XZ^2}\right) .$$

Solution: the map

$$\Phi(x,z) = \left(z, \sqrt{2xz^2}\right)$$

takes $(0,\infty) \times \mathbb{R} \setminus \{0\}$ *bijectively onto* $\mathbb{R} \setminus \{0\} \times (0,\infty)$. *We have*

$$\Phi^{-1}(z,w) = \left(\frac{w^2}{2z^2}, z\right) ,$$

and

$$J_{\Phi^{-1}}(z,w) = -\frac{w}{z^2} .$$

The transformation formula gives with $W = \sqrt{2XZ^2}$

$$f_{Z,W}(z,w) = e^{-\frac{w^2}{2z^2}} \cdot \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}} \cdot \frac{w}{z^2} .$$

b. (10) Find the density of $W = \sqrt{2XZ^2}$. Assume as known that for $a, b > 0$

$$\int_0^\infty \frac{1}{\sqrt{u^3}}\, e^{-\frac{a}{2u}} e^{-bu}\, du = \sqrt{\frac{2\pi}{a}}\, e^{-\sqrt{2ab}} .$$

Solution: we need to integrate over z. *Noting that the density is even in* z *for fixed* w, *we integrate over* $(0,\infty)$ *instead. Introducing the new variable* $z^2 = y$, *we get*

$$\begin{aligned}
f_W(w) &= 2\int_0^\infty f(z,w)dz \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{w^2}{2z^2}} e^{-\frac{z^2}{2}}\, \frac{w}{z^2} dz \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{w^2}{2y}} e^{-\frac{y}{2}}\, \frac{w}{2\sqrt{y^3}} dy \\
&= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2\pi}{w^2}}\, e^{-w}\, w \\
&= e^{-w} .
\end{aligned}$$

4. (20) A player rolls a fair die. Assume the rolls are independent. We say that the player gets a *run* of length 6 on roll k, if the numbers of dots on rolls $k-5, k-4, \ldots, k$ are strictly increasing, i. e., $1, 2, 3, 4, 5, 6$. Let X be the number of rolls until the player gets a run of length 6 for the first time.

a. (5) Let B be the event that the first roll produces one dot. Denote $\alpha = E(X|B)$. Express $E(X)$ using the quantity α.

Solution:

First method: *to get a run, the player first has to get one dot. If Y is the number of rolls until the first one dot we have $Y \sim \text{Geom}(\frac{1}{6})$. By independence, we have*

$$E(X|Y=k) = k - 1 + \alpha\,.$$

By the formula for total expectation

$$E(X) = \sum_{k=1}^{\infty}(k-1+\alpha)P(Y=k) = E(Y) - 1 + \alpha\,.$$

Knowing that $E(Y) = 6$, we get

$$E(X) = 5 + \alpha\,.$$

Second method: *again, we apply the formula of total expectation, but we take the partition $\{B, B^c\}$ instead, leading to*

$$E(X) = E(X|B)\,P(B) + E(X|B^c)\,P(B^c) = \tfrac{1}{6}\,\alpha + \tfrac{5}{6}\,E(X|B^c)\,.$$

If the first roll does not produce one dot, the game "resets" itself, so that $E(X|B^c) = E(X) + 1$. Therefore,

$$E(X) = \tfrac{1}{6}\,\alpha + \tfrac{5}{6}\big(E(X) + 1\big)\,.$$

Solving for $E(X)$, we find that $E(X) = 5 + \alpha$, which is the same as before.

b. (5) Let

$$C_l = \{\text{no run on the first 6 rolls, and the sixth roll produces } l\}$$

for $l = 1, 2, \ldots, 6$. Compute $P(C_l)$.

Solution: we start from the end. The event C_6 happens if the last roll produces a 6 and the first 5 rolls are not 1,2,3,4,5. By independence,

$$P(C_6) = \frac{1}{6}\cdot\left(1 - \left(\frac{1}{6}\right)^5\right)\,.$$

For the other C_l the event {sixth roll produces l} implies that the player does not get a run on the first 6 rolls. So $P(C_l) = \frac{1}{6}$.

c. (10) Compute $E(X)$.

Solution: let B_j be the event that in the first j rolls we get the outcomes $1, 2, \ldots, j$: we have $B_1 = B$. Further, let $B_{j,k}$ be the event that in the first j rolls we get the outcomes $1, 2, \ldots, j$, and on the $(j+1)$-th roll we get k. We have:

$$\begin{aligned} E(X \mid B_{j,1}) &= j + E(X \mid B) = j - 5 + E(X)\,, \\ E(X \mid B_{j,j+1}) &= E(X \mid B_{j+1})\,, \end{aligned}$$

and for $k \in \{1, 2, 3, 4, 5, 6\} \setminus \{1, j+1\}$ we get:

$$E(X \mid B_{j,k}) = j + 1 + E(X)\,.$$

Next we need to compute the conditional expectations $E(X \mid B_j)$. We have $E(X \mid B_6) = 6$, and for $j = 1, 2, 3, 4, 5$:

$$E\big[X \mathbf{1}_{B_j}\big] = \sum_{k=1}^{6} E\big[X \mathbf{1}_{B_{j,k}}\big]\,,$$

hence

$$E(X \mid B_j)\, P(B_j) = \sum_{k=1}^{6} E(X \mid B_{j,k})\, P(B_{j,k})\,,$$

yielding

$$\begin{aligned} E(X \mid B_j) &= \sum_{k=1}^{6} E(X \mid B_{j,k})\, \frac{P(B_{j,k})}{P(B_j)} \\ &= \sum_{k=1}^{6} E(X \mid B_{j,k})\, P(B_{j,k} \mid B_j) \\ &= \tfrac{1}{6}\big(j - 5 + E(X)\big) + \tfrac{1}{6}\, E(X \mid B_{j+1}) + \tfrac{4}{6}\big(j + 1 + E(X)\big) \\ &= \tfrac{5}{6} j - \tfrac{1}{6} + \tfrac{5}{6}\, E(X) + \tfrac{1}{6}\, E(X \mid B_{j+1})\,. \end{aligned}$$

This is a recursion relation. By induction we get

$$E(X \mid B_j) = j + \big(1 - 6^{j-6}\big) E(X)\,.$$

For $j = 1$ we have

$$E(X \mid B_1) = 1 + \big(1 - 6^{-5}\big) E(X)\,,$$

and at the same time

$$E(X \mid B_1) = E(X \mid B) = E(X) - 5\,.$$

Equating the two gives $E(X) = 6^6$.

5. (20) Let X and Y be independent, non-negative, integer valued random variables with the same distribution. Assume that

$$P(X = k) = \frac{1}{4}\, P(X + Y = k - 1)$$

for all $k \geq 1$. Let $G(s)$ be the generating function of X and Y.

a. (10) Find an equation that is satisfied by $G(s)$.

Solution: multiply both sides of the above relation by s^k and sum over $k \geq 1$. Denoting $P(X = 0) = p$, we get

$$\sum_{k=1}^{\infty} P(X = k)s^k = G_X(s) - p$$

and

$$\sum_{k=1}^{\infty} \frac{1}{4}\, P(X + Y = k - 1)s^k = \frac{s}{4}\, G_{X+Y}(s)\,.$$

Since X and Y have the same distribution we have $G_{X+Y}(s) = G(s)^2$. The desired equation is

$$G(s) - p = \frac{s}{4}\, G(s)^2\,.$$

b. (10) Find the distribution of X.

Hint: first $G(1) = 1$, and by Newton's expansion we have that for $|x| < 1$

$$\sqrt{1 - x} = \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} x^k\,.$$

Solution: since $G(1) = 1$, the equation from the first part implies

$$1 - p = \frac{1}{4}\,.$$

Solving for $G(s)$ we get

$$G(s) = \frac{2\left(1 \pm \sqrt{1 - \frac{3s}{4}}\right)}{s}\,.$$

The coefficients of a generating function must be non-negative. Since $(-1)^k \binom{1/2}{k} < 0$ for all $k = 1, 2, 3, \ldots$, we have to choose the negative sign for the root. Expanding into a power series we get

$$G(s) = \sum_{k=1}^{\infty} 2\binom{1/2}{k}(-1)^{k-1} \frac{3^k s^{k-1}}{4^k}\,.$$

Finally,

$$P(X = k) = 2\binom{1/2}{k+1}(-1)^k \left(\frac{3}{4}\right)^{k+1}\,.$$

6. (20) Berti opens a stand with a game involving three dice. Every game costs 1 euro and the three dice are rolled. If no sixes show Berti keeps the stake. If exactly one six shows, Berti returns the stake to the player with additional 1 euro. If exactly two sixes show, Berti returns the stake to the player with additional 2 euros. If three sixes show, Berti returns the stake to the player with additional 14 euros. Assume the dice are fair and that all the rolls are independent.

a. (10) Compute the expected value and the variance of Berti's profit after n games.

Solution: Let X_i denote Berti's profit in i-th game. We have:

$$X_i \sim \begin{pmatrix} -14 & -2 & -1 & 1 \\ \frac{1}{216} & \frac{15}{216} & \frac{75}{216} & \frac{125}{216} \end{pmatrix},$$

We get $E(X_i) = 1/36$ and $\mathrm{var}(X_i) = 2735/1296$. Denoting Berti's profit after n games by S_n we have $E(S_n) = n/36$ and $\mathrm{var}(S_n) = 2735n/1296$.

b. (10) After approximately how many games will Berti have a positive profit with approximately 95% probability?

Solution: Denote the unknown number of games by nn. From the central limit theorem we get that, approximately,

$$1 - \Phi\left(\frac{-\frac{1}{36}n}{\sqrt{\frac{2735}{1296}n}}\right) = \Phi\left(\frac{\sqrt{n}}{\sqrt{2735}}\right) = 0.95$$

or

$$\frac{\sqrt{n}}{\sqrt{2735}} \doteq 1.645\,.$$

It follows that n is approximately 7400.