UNIVERSITY OF PRIMORSKA

FAMNIT, MATHEMATICS

# PROBABILITY

## EXAM

JUNE 11$^{\text{th}}$, 2019

NAME AND SURNAME: ______________________ IDENTIFICATION NUMBER: | | | | | | | | |

### INSTRUCTIONS

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have two hours.

| Problem | a. | b. | c. | d. | |
|---|---|---|---|---|---|
| 1. | | | • | • | |
| 2. | | | • | • | |
| 3. | | | • | • | |
| 4. | | | • | • | |
| 5. | | | • | • | |
| 6. | | | • | • | |
| Total | | | | | |

**1.** (20) There are $2n$ chairs around a round table. For dinner $n$ couples arrive and the host seats them around the table at random. All possible $(2n)!$ arrangements are equally likely. Let $X_n$ be the number of couples who sit diametrically to each other. Denote $p(n,k) = P(X_n = k)$ for $0 \leq k \leq n$.

a. (10) Compute $P(X_n = 0)$. You do not need to simplify the sums.

*Hint: compute the probability of the opposite event* $\{X_n > 0\}$.

*Solution: Let* $A = \{X_n > 0\}$ *and let define*

$$A_i = \{\text{on } i\text{-th and } (n+i)\text{-st chair a couple sit}\}$$

*for* $i = 1, 2, \ldots, n$. *It holds* $A = \cup_{i=1}^n A_i$. *We need probability* $P(A_1 \cap A_2 \cap \cdots \cap A_i)$. *We need to count the permutations of* $2n$ *elements that fit the above intersection. Among n couples we pick i couples, in each couple one of them and sit him/her in chairs* $1, 2, \ldots, i$. *We can do that on*

$$\binom{n}{i} 2^i i!$$

*ways. The rest* $2n - 2i$ *people can be seated on the rest chairs arbitrary on* $(2n-2i)!$ *ways. It follows*

$$P(A_1 \cap A_2 \cap \cdots \cap A_i) = \frac{\binom{n}{i} 2^i i! (2n-2i)!}{(2n)!} .$$

*We use inclusion exclusion formula and symmetry and get*

$$P(A) = \sum_{i=1}^{n} (-1)^{i+1} \frac{\binom{n}{i}^2 2^i i! (2n-2i)!}{2n)!} .$$

b. (10) Compute $P(X_n = k)$ for $k = 1, 2, \ldots, n$. You do not need to simplify the sums.

*Solution: We choose* $2k$ *chairs, where the couples sit diametral and we sit the rest in such way that they do not sit diametral. There arena* $\binom{n}{k}$ *choises for diametral seats. For different choises of k pairs of diametral chairs the above events are disjoint, their union is excactly the event we are looking for. The counting of the favorable permutations gives us*

$$\binom{n}{k}^2 2^k k! \cdot p(n-k, 0)(2n-2k)! .$$

*It follows*

$$P(X = k) = \frac{\binom{n}{k}^2 2^k k! \cdot b(n-k, 0)(2n-2k)!}{(2n)!}.$$

*We have considered the case when we sit the $k$ couples on diamteral, the others should not sit in diametral. We can think as the chosen $k$ couples are removed and the rest $n-k$ couples are seated in a way that none of them sits in diametral. This question is equivalent to the question from the first part of the exercise, but for $n-k$ couples.*

**2.** (20) Let $\xi_1, \xi_2, \ldots$ be independent random variables, uniformly distributed on the set $\{1, 2, \cdots, m\}$, where $m > 1$ is a given number, i.e.

$$P(\xi_k = i) = \frac{1}{m}$$

for $i = 1, 2 \ldots, m$.

a. (10) Define $B_r = \{\xi_1 < \xi_2 < \cdots < \xi_r\}$ for $r \leq m$. Compute the probability of the event $B_r$.

*Solution: We notice that*

$$B_r = \cup_{1 \leq k_1 < k_2 < \cdots < k_r \leq m} \{\xi_1 = k_1, \ldots, \xi_r = k_r\} \; .$$

*Events in the union are disjoint and have the probability $m^{-r}$. There are $\binom{m}{r}$ choises for increasing $r$-tuples. It follows that*

$$P(B_r) = \binom{m}{r} \left(\frac{1}{m}\right)^r \; .$$

b. (10) Define $A_r = \{\xi_1 < \xi_2 < \cdots < \xi_r\} \cap \{\xi_{r+1} \leq \xi_r\}$. Compute the probability of the event $A_r$.

*Solution: It holds*

$$P(A_{r,s}) = P(A_{1,s-r+1}) \; .$$

*It holds*

$$A_{1,r} = B_r \backslash B_{r+1} \, ,$$

*where $B_{m+1} = \emptyset$. While $B_{r+1} \subset B_r$, is*

$$P(A_r) = \binom{m}{r} \left(\frac{1}{m}\right)^r - \binom{m}{r+1} \left(\frac{1}{m}\right)^{r+1} \; .$$

**3.** (20) Let the random vector $(X, Y)$ have the density

$$f_{X,Y}(x,y) = \frac{1}{4\pi\sqrt{1-\rho^2}}\, e^{-\frac{x^2+y^2}{2(1-\rho^2)}} \left( e^{\frac{\rho xy}{1-\rho^2}} + e^{-\frac{\rho xy}{1-\rho^2}} \right)$$

for $|\rho| < 1$ and $\rho \neq 0$.

a. (10) Compute the marginal distributions of $X$ and $Y$ and determine whether $X$ and $Y$ are independent.

*Solution: The marginal distribution of random variable $X$ is an integral of bivariant:*

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y \\ &= \frac{1}{4\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2(1-\rho^2)}} \left( e^{\frac{\rho xy}{1-\rho^2}} + e^{-\frac{\rho xy}{1-\rho^2}} \right) \mathrm{d}y \\ &= \frac{1}{4\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}}\,\mathrm{d}y + \frac{1}{4\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2+2\rho xy+y^2}{2(1-\rho^2)}}\,\mathrm{d}y\,. \end{aligned}$$

*Now we notice that in both integrals there is a density of bivariant normal distribution, where both marginal distributions in both cases are standard normal. We get*

$$\begin{aligned} f_X(x) &= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\,. \end{aligned}$$

*Similarly we would get that also $Y$ is standardn normal. For $\rho \neq 0$ holds*

$$f_{X,Y}(x,y) \neq f_X(x)\, f_Y(y)\,,$$

*and the variables $X$ and $Y$ are not independent.*

b. (10) Define $(U, V) = (X+Y, X-Y)$. Compute the densitiy of the random vector $(U, V)$ and of the random variable $U$.

*Solution: The mapping*

$$\Phi(x,y) = (x+y, x-y)$$

*satisfies all the conditions for the usage of transformation formula and its inverse mapping is*

$$\Phi^{-1}(u,v) = \left( \tfrac{1}{2}(u+v),\ \tfrac{1}{2}(u-v) \right)\,,$$

*the Jacobi determinant is*

$$J_{\Phi^{-1}}(u,v) = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}\,.$$

*It follows*

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{8\pi\sqrt{1-\rho^2}}\, e^{-\frac{u^2+v^2}{4(1-\rho^2)}} \left( e^{\frac{\rho(u^2-v^2)}{4(1-\rho^2)}} + e^{-\frac{\rho(u^2-v^2)}{4(1-\rho^2)}} \right) \\ &= \frac{1}{8\pi\sqrt{1-\rho^2}} \left( e^{-\frac{(1-\rho)u^2+(1+\rho)v^2}{4(1-\rho^2)}} + e^{-\frac{(1+\rho)u^2+(1-\rho)v^2}{4(1-\rho^2)}} \right). \end{aligned}$$

*The marginal distribution can be obtained by integration. The computations contains two integrals. Without constants the first is equal to*

$$\int_{-\infty}^{\infty} e^{-\frac{(1-\rho)u^2+(1+\rho)v^2}{4(1-\rho^2)}}\, dv = e^{-\frac{(1-\rho)u^2}{4(1-\rho^2)}} \cdot \sqrt{2\pi}\sqrt{2-2\rho}\,,$$

*similarly for the second integral. With cancellation summation we get*

$$f_U(u) = \frac{1}{4\sqrt{\pi}\sqrt{1+\rho}}\, e^{-\frac{u^2}{4(1+\rho)}} + \frac{1}{4\sqrt{\pi}\sqrt{1-\rho}}\, e^{-\frac{u^2}{4(1-\rho)}}\,.$$

**4.** (20) A group of $n \geq 3$ gamblers are sitting around a round table. All of them roll their own die once; all dice are standard (1 to 6 dots), fair (every number of dots has equal probability) and the rolls are independent. Denote by $W$ the number of pairs of gamblers sitting next to each other at the table who roll a neigbouring number of dots. The numbers from the set $\{1, 2, 3, 4, 5, 6\}$ are neighbouring numbers if their difference is 1 in absolute value (4 and 3 are neighbouring numbers, but 6 and 1 are not, and 3 and 3 are not either).

a. (10) Compute $E(W)$.

*Solution: We can write* $W = I_1 + I_2 + \cdots + I_n$, *where* $I_i$ *is indicator for the event, where* $i$*-th gambler and his right neighbour toss neighbouring numbers. The probability of this event equals to*

$$E(I_i) = \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{6} = \frac{5}{18},$$

*therefore*

$$E(W) = \frac{5n}{18}.$$

*For computation of the variance there are two classical ways. We can start with*

$$\mathrm{var}(W) = E(W^2) - \big(E(W)\big)^2$$

*and*

$$E(W^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} E(I_i I_j).$$

*Random variable* $I_i I_j$ *is indicator of the event, that for* $i$*-th and* $j$*-th gambler holds that with their right neighbours toss a neighbouring number. For* $i = j$ *is the probability equal to* $5/18$*; there are* $n$ *such term in the abovve sum. If* $i$*-th and* $j$*-th gambler are neighbours, this means that three tosses of neighbouring gamblers are neighbouring numbers. The probability for this is*

$$E(I_i I_j) = \frac{2}{3} \cdot \frac{1}{9} + \frac{1}{3} \cdot \frac{1}{36} = \frac{1}{12};$$

*there are* $2n$ *such terms in above sum. If* $i$ *and* $j$ *are neither equal neither neighbours, the probability is* $(5/18)^2 = 25/324$*; there are* $n^2 - 3n$ *such terms in the above sum (here we need the assumption* $n \geq 3$*). We sum up and get*

$$E(W^2) = n \cdot \frac{5}{18} + 2n \cdot \frac{1}{12} + (n^2 - 3n) \cdot \frac{25}{324} = \frac{25n^2 + 69n}{324}.$$

*We get:*

$$\mathrm{var}(W) = \frac{23n}{108}.$$

*We can achieve this also wirh covariances:*

$$\mathrm{var}(W) = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}(I_i, I_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} E(I_iI_j) - E(I_i)E(I_j)\,.$$

*For* $i = j$ *is* $\mathrm{cov}(I_i, I_j) = 5/18 - (5/18)^2 = 65/324$. *If* $i$*-th and* $j$*-th gambler are neighbours* $\mathrm{cov}(I_i, I_j) = 1/12 - 25/324 = 1/162$. *In every other case is* $\mathrm{cov}(I_i, I_j) = 0$*: event that neighbours of* $i$*-th and neighbours of* $j$*-th gambler toss neighbouring numbers are independent. We sum up and get:*

$$\mathrm{var}(W) = n\cdot\frac{65}{324} + 2n\cdot\frac{1}{162} = \frac{23n}{108}\,,$$

*which is equal as before.*

b. (10) Compute $\mathrm{var}(W)$.

*Solution: We write* $S = J_1 + J_2 + \cdots + J_n$*, where* $J_j$ *is indicator of the event, that* $j$*-th gambler tosses six dots. For computing covariance of* $W$ *and* $S$ *there are again two standard ways. We can write*

$$\mathrm{cov}(W, S) = E(WS) - E(W)\,E(S)$$

*in*

$$E(WS) = \sum_{i=1}^{n}\sum_{j=1}^{n} E(I_iJ_j)\,.$$

*Random variables* $I_iI_j$ *is indicator for the event where* $i$*-th gambler and his right neighbour toss neighbouring numbers,* $j$*-th gamblers tosses six dots. For* $i = j$ *is this an event where* $i$*-th gambler tosses six dots, his right neighbour tosses five dots; probability of this event equals to* $1/36$ *and there are* $n$ *such terms in the above double sum. If* $j$*-th gambler is the right neighbour of the* $i$*-th gambler, is this an event where* $j$*-th gambler tosses six and* $i$*-th five dots; the probability of this event is* $1/36$ *and there are* $n$ *such terms in the sum. If* $j$*-th gambler is neither equal to* $i$*-th neither is his right neighbour, the probability is* $(1/6)\cdot(5/18) = 5/108$*; there are* $n^2 - 2n$ *such terms in the sum. We sum up and get*

$$E(WS) = 2n\cdot\frac{1}{36} + (n^2 - 2n)\cdot\frac{5}{108} = \frac{5n^2 - 4n}{108}\,.$$

*It holds* $E(J_j) = 1/6$ *and consequently* $E(S) = n/6$*. We get*

$$\mathrm{var}(W) = -\frac{n}{27}\,.$$

*We can achieve this also with covariances*

$$\mathrm{cov}(W) = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}(I_i, J_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} E(I_i J_j) - E(I_i)E(J_j)\,.$$

*If* $i = j$ *or* $j$*-th gambler is right neighbour of* $i$*-th, it holds* $\mathrm{cov}(I_i, J_j) = 1/36 - 5/108 = -1/54$, *otherwise* $\mathrm{cov}(I_i, J_j) = 0$. *We sum up and get*

$$\mathrm{cov}(W, S) = -\frac{n}{27}\,,$$

*which is the same as before.*

**5.** (20) Let $Z_0, Z_1, \ldots$ be a branching process. Denote $G(s) = E(s^{Z_1})$.

a. (5) Let $G_n(s)$ be the generating function of the random variable $Z_n$. Show that

$$G_{m+n}(s) = G_n(G_m(s))\,.$$

*Solution: We know from lectures that*

$$G_n = G \circ G \circ \cdots \circ G\,.$$

*The claim follows.*

b. (15) Denote $\mu_n = E(Z_n)$ and $\sigma_n^2 = \mathrm{var}(Z_n)$. Show that

$$\mu_{m+n} = \mu_n \mu_m$$

and

$$\sigma_{m+n}^2 = \mu_m \sigma_n^2 + \mu_n^2 \sigma_m^2\,.$$

*Hint: Differentiate the generating functions.*

*Solution: With differentiation we get*

$$G'_{m+n}(s) = G'_n(G_m(s))\, G'_m(s)\,.$$

*We consider*

$$\lim_{s \uparrow 1} G_X(s) = 1 \qquad \textit{in} \qquad \lim_{s \uparrow 1} G'_X(s) = E(X)$$

*and get*

$$\lim_{s \uparrow 1} G_{m+n}(s) = G'_n(1)\, G'_m(1)\,.$$

*The first claim follows. For the second claim we need*

$$\lim_{s \uparrow 1} G''_X(s) = E\left(X(X-1)\right)\,.$$

*With double differentiation we get*

$$G''_{m+n}(s) = G''_n(G_m(s))(G'_m(s))^2 + G'_n(G_m(s))\, G''_m(s)\,.$$

*When $s \uparrow 1$, we get*

$$\sigma_{m+n}^2 + \mu_{m+n}^2 - \mu_{m+n} = (\sigma_n^2 + \mu_n^2 - \mu_n)\mu_m^2 + \mu_n(\sigma_m^2 + \mu_m^2 - \mu_m)\,.$$

*We order the terms and use $\mu_{m+n} = \mu_m \mu_m$ in and we get what we wanted.*

**6.** (20) A coin is tossed $2n$ times. The tosses are independent, the probability of heads showing is $p = 1/2$. Denote by $S_{2n}$ the number of heads in $2n$ tosses.

a. (10) Determine as accurately as possible such $n$ that

$$P(S_{2n} = n) = 0,01\,?$$

Use $\Phi(0,0125) = 0,505$.

*Solution: Tossing coins is equal to choosing slips from the box, where there are only numbers 0 and 1 available. We know that $\mu = 1/2$ and $\sigma = 1/2$. We compute*

$$\begin{aligned}
P(S_{2n} = n) &= P\big(n - \frac{1}{2} \leq S_{2n} \leq n + \frac{1}{2}\big) \\
&= P\big(-\frac{1}{2} \leq S_{2n} - n \leq \frac{1}{2}\big) \\
&= P\big(-\frac{1}{\sqrt{2n}} \leq \frac{S_{2n} - n}{\sqrt{2n}/2} \leq \frac{1}{\sqrt{2n}}\big) \\
&\approx P\big(-\frac{1}{\sqrt{2n}} \leq Z \leq \frac{1}{\sqrt{2n}}\big) \\
&= \Phi\big(\frac{1}{\sqrt{2n}}\big) - \Phi\big(-\frac{1}{\sqrt{2n}}\big) \\
&= 2\Phi\big(\frac{1}{\sqrt{2n}}\big) - 1 \\
&= 0,01\,.
\end{aligned}$$

*It follows*

$$\Phi\big(\frac{1}{\sqrt{2n}}\big) = 0,505\,,$$

*therefore*

$$\frac{1}{\sqrt{2n}} = 0.0125\,.$$

*We get $n = 3183$.*

b. (10) Let $n = 5.000$. What approximately is the probability that the difference between the number of heads and the number of tails in $2n = 10.000$ tosses is less than 100?

*Hint: What should be the number of heads so that the difference between the number of heads and the number of tails is 100 or less?*

*Solution: We should translate the exercise a little bit. The numbers will differ for less than* 100*, if the number of heads will from* 4950 *to* 5050*. We compute*

$$
\begin{aligned}
P(4950 \leq S_{2n} \leq 5050) &= P(-50 \leq S_{2n} - 5000 \leq 50) \\
&= P\Big( - \frac{50}{\sqrt{2n}/2} \leq \frac{S_{2n} - 5000}{\sqrt{2n}/2} \leq \frac{50}{\sqrt{2n}/2}\Big) \\
&= P\Big( - 1 \leq \frac{S_{2n} - 5000}{\sqrt{2n}/2} \leq\leq 1\Big) \\
&\approx P(-1 \leq Z \leq\leq 1\Big) \\
&= \Phi(1) - \Phi(-1) \\
&= 0,68\,.
\end{aligned}
$$