NAME AND SURNAME: ______________________

IDENTIFICATION NUMBER: | | | | | | | | |

UNIVERSITY OF PRIMORSKA

FAMNIT, MATHEMATICS

PROBABILITY

WRITTEN EXAMINATION

SEPTEMBER 4^{th}, 2024

INSTRUCTIONS

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have 120 minutes.

Question	a.	b.	c.	d.	Total
1.			•	•	
2.			•	•	
3.			•	•	
4.			•	•	
5.				•	
6.			•	•	
Total					

2. (20) Two players A and B have an ordinary deck of 52 cards each. Both players shuffle their decks well and independently of each other. They both start placing cards on the table one by one face up simultaneously from the top of their respective decks. Use the inclusion/exclusion formula to compute the probabilities below.

a. (10) Let C be the event that at least once the players simultaneously place an Ace on the table. There are 4 Aces among 52 cards. Compute the probability $P(C)$. You do not need to simplify the resulting expressions.

Solution: define the events

$$C_i = \{\textit{the } i\textit{-th card that the two players simultaneously place are both Aces}\}$$

for $i = 1, 2, \ldots, 52$. *We have* $C = \cup_{i=1}^{52} C_i$. *We use the inclusion-exclusion formula, where we notice that the intersections of* 5 *or more events out of* $C_1, C_2, \ldots, C_n$ *are empty. Due to symmetry all the intersections of* k *different events have the same probability. It follows*

$$\begin{aligned} P(C) &= \binom{52}{1} P(C_1) - \binom{52}{2} P(C_1 \cap C_2) \\ &\quad + \binom{52}{3} P(C_1 \cap c_2 \cap C_3) - \binom{52}{4} P(C_1 \cap C_2 \cap C_3 \cap C_4)\,. \end{aligned}$$

By independence we have

$$P(C_1) = \left(\frac{4}{52}\right)^2 ,$$

$$P(C_1 \cap C_2) = \left(\frac{4 \cdot 3}{52 \cdot 51}\right)^2 ,$$

$$P(C_1 \cap C_2 \cap C_3) = \left(\frac{4 \cdot 3 \cdot 2}{52 \cdot 51 \cdot 50}\right)^2$$

and

$$P(C_1 \cap \cdots \cap C_4) = \left(\frac{4 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49}\right)^2 .$$

All together we get

$$P(C) = \frac{15229}{54145} \doteq 0,281263\,.$$

b. (10) Let D be an event that the players at least once simultaneously place the same card on the table. Compute $P(D)$.

Solution: the problem is identical to the example from the lectures, where 52 *couples attend the ball, and upon leaving every woman chooses a man at random . From the lectures we have*

$$P(D) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{52!} ,$$

or

$$P(D) = \frac{3332398089094688906756940683186552650196823142416430337261808287 83}{52717761549636521942261854154512265996921245386198220800000000000000} \doteq 0{,}6321\,.$$

5. (20) An urn contains b white and 3 red balls. Balls are selected from the urn at random without replacement. Let X be the number of white balls before the first red ball is drawn and Y the number of white balls between the first and the second red ball.

a. (10) Find the joint distribution of the random variables X and Y.

Solution: the possible values for the random variables are pairs (k, l), for which $k \geq 0$, $l \geq 0$ and $k + l \leq b$. The event $\{X = k, Y = l\}$ happens, if first k white balls is chosen, then a red ball, then l white ones and finally a red ball. Denote $b + 3 = n$. We compute

$$
\begin{aligned}
&P(X = k, Y = l) = \\
&= \frac{b(b-1)\cdots(b-k+1)}{n(n-1)\cdots(n-k+1)} \cdot \frac{3}{n-k} \cdot \\
&\qquad \cdot \frac{(b-k)(b-k-1)\cdots(b-k-l+1)}{(n-k-1)(n-k-2)\cdots(n-k-l)} \cdot \frac{2}{(n-k-l-1)} \\
&= \frac{b(b-1)\cdots(b-k-l+1)\cdot 3 \cdot 2}{n(n-1)\cdots(n-k-l-1)} \\
&= \frac{b! \cdot (n-k-l-2)! \cdot 3 \cdot 2}{(b-k-l)! \cdot n!}\,.
\end{aligned}
$$

b. (10) Show that the random variables X and Y have the same distribution. Compute the distribution of Y.

Hint: the distribution of X should be computed separately, not as marginal distribution. Then use the symmetry of the joint distribution.

Solution: the random variable X counts the number of white balls before the first red ball is drawn. The event $\{X = k\}$ happens if k white balls are chosen first and then a red ball. Denote $n = b + 3$. We get

$$
P(X = k) = \frac{b(b-1)\cdots(b-k+1)\cdot 3}{n(n-1)\cdots(n-k+1)(n-k)}
$$

for $k = 0, 1, \ldots, b$. On the other hand, we can get the distributions of X and Y as marginal distributions of the joint distribution. Because it is a symmetric function of k and l, the distributions of X and Y are equal.

4. (20) Let the random variables X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{for } x, y > 0 \text{ and } x + y < 1 \\ 0 & \text{else.} \end{cases}$$

Assume as known that the mapping Φ given by

$$\Phi(x,y) = \left(\frac{x}{x+y}, 1 - x - y\right)$$

is a bijection of $G = \{(x,y) \colon x, y > 0 \text{ in } x + y < 1\}$ to the square $H = (0,1)^2$.

a. (10) Compute the density of the vector

$$(U,V) = \left(\frac{X}{X+Y}, 1 - X - Y\right).$$

Justify that U and V are independent.

Solution: first we need the inverse $\Phi^{-1}(u,v)$, which means that we need to solve the system of equations

$$\frac{x}{x+y} = u, \qquad 1 - x - y = v$$

for $(u,v) \in (0,1)^2$. We get

$$x = x = u(1-v) \quad \textit{and} \quad y = (1-u)(1-v)\,.$$

It follows

$$J_{\Phi^{-1}}(u,v) = \det\begin{pmatrix} 1-v & -u \\ -(1-v) & -(1-u) \end{pmatrix} = -(1-v)\,.$$

By the transformation formula we get

$$f_{U,V}(u,v) = \begin{cases} 2(1-v) & \textit{for } (u,v) \in (0,1)^2 \\ 0 & \textit{otherwise.} \end{cases}$$

The density is product of terms that are dependent of u (this is 1) and v respectively, therefore U and V are independent random variables.

b. (10) Show that the random variables X, Y and $1 - X - Y$ have the same density.

Solution: for X and Y we compute the marginal density to get

$$f_X(x) = 2(1-x) \quad \textit{and} \quad f_Y(y) = 2(1-y)\,.$$

From the first part it follows that the density of $V = 1 - X - Y$ is equal to

$$f_V(v) = 2(1-v)\,.$$

2. (20) Each of the two players A and B has an ordinary deck of 52 cards. Both players shuffle their decks well and independently of each other. They both start placing cards on the table one by one face up simultaneously from the top of their respective decks.

a. (10) Let X be the number of times when the two players simultaneously place an Ace on the table. Compute $E(X)$.

Solutions: Define the indicators

$$I_k = \begin{cases} 1 & \textit{in the } k\textit{-th round both players place an Ace on the table,} \\ 0 & \textit{otherwise.} \end{cases}$$

With this definition we have $X = I_1 + \cdots + I_{52}$. *By symmetry, all indicators have equal expectation so*

$$E(X) = 52 \cdot E(I_1) = 52 \cdot P(I_1 = 1) = 52 \cdot \left(\frac{4}{52}\right)^2 .$$

b. (10) Let Y be the number of Aces that are still in the deck of the player B immediately after the first time the player A places an Ace on the table. Let N be the number of cards placed on the table until the player A places an Ace on the table. Assume as known that $E(N) = \frac{53}{5}$. Compute $E(Y)$.

Hint: for a nonnegative integer valued random variable N *we have* $E(N) = \sum_{n \geq 1} P(N \geq n)$.

Solution: Define indicators

$$J_k = \begin{cases} 1 & \textit{the player B places an Ace on the table in the } k\textit{-th and } N \geq k \\ 0 & \textit{otherwise.} \end{cases}$$

The sum $J_1 + \cdots + J_{49}$ *is equal to the number of Aces that player B places on the table, including the moment when the player A places an Ace on the table. It follows that* $Y = 4 - J_1 + \cdots + J_{49}$. *By independence and symmetry it follows that*

$$P(J_k = 1) = \frac{4}{52} P(N \geq k) .$$

It follows

$$E\left(J_1 + \cdots + J_{49}\right) = \frac{4}{52} \sum_{k=1}^{49} P(N \geq k) .$$

For every nonnegative integer valued random variable N *we have*

$$E(N) = \sum_{n \geq 1} P(N \geq n) .$$

The random variable N *can take values* $1, 2, \ldots, 49$. *From the above formula it follows that*

$$\sum_{k=1}^{49} P(N \geq k) = E(N) = \frac{53}{5} .$$

We get

$$E\left(J_1+\cdots+J_{49}\right)=\frac{4}{52}\cdot\frac{53}{5}\,.$$

The final result is equal to

$$E(Y)=4-E\left(J_1+\cdots+J_{49}\right)=\frac{207}{65}\,.$$

5. (20) There are n urns each containing a red and b white balls. We choose a ball at random from the first urn and transfer it to the second urn. Then we randomly choose a ball from the second urn and transfer it to the third urn. We continue this process until we choose a ball at random from the last urn.

a. (5) For $j = 1, 2, 3, \ldots, n-1$ let X_j be the number of red balls chosen from the j-th urn. Compute $E(X_1)$, and for $j \geq 2$ the conditional expectation $E(X_j|X_{j-1} = k)$ where k is a possible value of X_{j-1}.

Solution: when we choose from the j-th box, there will be $a+k$ red and $b+(1-k)$ white balls in the box for $k = 0, 1$. It follows

$$E(X_j|X_{j-1} = k) = P(X_j = 1|X_{j-1} = k) = \frac{a+k}{a+b+1} \, .$$

We get

$$E(X_1) = P(X_1 = 1) = \frac{a}{a+b} \, .$$

b. (10) For all $j = 1, 2, 3, \ldots, n-1$ compute $E(X_j)$.

Solution: we get

$$E(X_j) = \sum_{k=0}^{1} E(X_j|X_{j-1} = k) P(X_{j-1} = k) \, ,$$

therefore

$$E(X_j) = \frac{a}{a+b+1} + \frac{E(X_{j-1})}{a+b+1} \, .$$

We compute

$$E(X_2) = \frac{a}{a+b+1} + \frac{a}{(a+b)(a+b+1)} = \frac{a}{a+b} \, .$$

With induction can be shown

$$E(X_j) = \frac{a}{a+b}$$

for all j.

c. (5) What is the probability the ball selected from the last urn is red?

Hint: express

$$P\left(\text{the ball chosen from the last box is red}|X_{n-1} = k\right)$$

for all possible values k of the random variable X_{n-1}.

Solution: denote $A =$ the ball chosen from the last box is red. We have

$$P(A|X_{n-1} = k) = \frac{a+k}{a+b+1} \, .$$

It follows

$$P(A) = P(A|X_{n-1} = 0)P(X_{n-1} = 0) + P(A|X_{n-1} = 1)P(X_{n_1} = 1)\,.$$

Compute to get

$$P(A) = \frac{a}{a+b}\,.$$

6. (20) The French roulette has 37 slots of which 18 are black, 18 are red and one is green. The House introduces a new game: a gambler who bets 1€ loses his bet if the ball stops on black. If the balls stops on red, the player gets his stake back. If the ball stops on the only green slot, the gambler gets his bet back and an additional x €. We assume that all slots are equally likely, and that subsequent games are independent.

a. (5) Find x such that the game is fair i.e. such that the expected return for the House is zero?

Solution: let us denote by X_1 the profit of the House after one game. From

$$X_1 \sim \begin{pmatrix} -1 & 0 & x \\ 18/37 & 18/37 & 1/37 \end{pmatrix}$$

we can compute $E(X_1) = \dfrac{x-18}{37}$*, therefore the game is fair for* $x = 18$.

b. (15) Find x such that after 10.000 games the probability of the House not losing money is approximately 95%?

Solution: let us denote by X_k the profit of the Hose in the k-th game, and by S the profit after 10.000 games. We have $S = X_1 + X_2 + \cdots + X_{10000}$. By the central limit theorem S is approximately normally distributed with appropriate expectation and variance. We have

$$\mathrm{var}(X_1) = \frac{x^2+18}{37} - \left(\frac{x-18}{37}\right)^2$$

and

$$E(S) = \frac{10000(x-18)}{37} \quad \textit{and} \quad \mathrm{var}(S) = 10000\left[\frac{(x^2+18)}{37} - \left(\frac{x-18}{37}\right)^2\right],$$

therefore

$$P(S > 0) \approx 1 - \Phi\left(-\frac{E(S)}{\sqrt{\mathrm{var}(S)}}\right) = \Phi\left(\frac{100(x-18)}{\sqrt{37(x^2+18)-(x-18)^2}}\right).$$

It follows that x must be chosen in such a way that

$$\Phi\left(\frac{100(18-x)}{\sqrt{37(x^2+18)-(x-18)^2}}\right) = 0{,}95\,,$$

or

$$\frac{100(18-x)}{\sqrt{37(x^2+18)-(x-18)^2}} \doteq 1{,}645\,.$$

We must have $x < 18$, otherwise $E(S) \leq 0$ and the House would have a loss with the probablity at least $1/2$. For $x < 18$ the above equation is equal to the equation

$$10000(18-x)^2 = 1{,}645^2\big(37(x^2+18)-(x-18)^2\big)$$

or

$$(10000 - 36 \cdot 1{,}645^2)x^2 - (360000 + 36 \cdot 1{,}645^2)x + 3240000 - 342 \cdot 1{,}645^2 = 0\,.$$

or

$$9902{,}6\, x^2 - 360097{,}4\, x + 3239075 = 0\,.$$

This quadratic equation has roots

$$x_1 \doteq 16{,}31\,, \qquad x_2 \doteq 20{,}05$$

and the correct solution is the first one. For the green slot, the House has to pay an additional 16 euros and 31 cents.