University of Primorska

FAMNIT

Probability

Written examination

July 3^{rd}, 2018

Name and surname: ______________________ Identification number: | | | | | | | | |

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have two hours.

Naloga	a.	b.	c.	d.	
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2.				•	
3.			•	•	
4.			•	•	
5.				•	
6.			•	•	
Skupaj					

1. (20) There are n seats in the theatre hall. The ticket office sold n tickets numbered 1 to n. Patrons are entering the theatre in the same order as shown on their tickets. The first patron chooses a seat at random from the n seats. Subsequent patrons take the seat with the number shown in their ticket if it is available, and otherwise choose a seat at random from the remaining available seats.

a. (10) What is the probability that the seat of the patron 3 will be available when she enters the theatre hall?

b. (10) Let $n = 100$. What is the probability that the seat of the patron 50 will be available when she enters the hall, conditionally on the event that the first 49 patrons are occupying 30 out of the seats numbered $1, 2, \ldots, 49$?

2. (20) Let the vector of indicators $(I_1, I_2, \ldots, I_N)$ have the joint distribution

$$P\left(I_1 = i_1, \ldots, I_N = i_N\right) = \frac{1}{\binom{N}{n}}$$

for all the N-tuples $\{i_1, \ldots, i_N\}$ with $i_k \in \{0, 1\}$ and $i_1 + i_2 + \cdots + i_N = n$ for some fixed $n > 0$. For $b < N$ let $X_b = I_1 + I_2 + \cdots + I_b$.

a. (5) Compute the distribution of the random variable X_b.

b. (5) Compute $P(I_k = 1)$ and $P(I_k = 1, I_l = 1)$ for $k \neq l$.

c. (10) Compute $\mathrm{var}(X_b)$.

3. (20) Suppose the random vector (X, Y) has the density:

$$f_{X,Y}(x,y) = \begin{cases} x\,e^{-x} & ;\ x > 0,\ -ax^2 < y < ax^2 \\ 0 & ;\ \text{otherwise} \end{cases},$$

where $a > 0$.

a. (10) Determine the constant a.

b. (10) Compute the density of the random variable $Z = XY$.

4. (20) A deck of m red and m black cards is shuffled well so that every possible order of the cards is equally likely. The cards will be dealt from the top of the deck one by one. Denote by $n = 2m$ the number of all cards. For fixed k with $1 \leq k \leq n$ define

$$I_k = \begin{cases} 1 & \text{if the } k\text{-th card dealt from the top is red} \\ 0 & \text{otherwise.} \end{cases}$$

and let X_k be the number of red cards among the k cards dealt from the top.

a. (10) For $2 \leq k \leq n$ compute $P(I_k = 1, X_{k-1} = j)$ where $\max(0, k-m-1) \leq j \leq \min(k-1, m)$.

   Hint: $P(I_k = 1, X_{k-1} = j) = P(X_{k-1} = j)P(I_k = 1 | X_{k-1} = j)$.

b. (10) A gambler can bet that the next card will be red after $k-1$ cards have been dealt. She decides that she will place the bet only if there are strictly more red than black cards among the cards remaining in the deck. Compute the probability that the gambler will place the bet at some stage and win. The probability should be expressed as a sum. You do not need to simplify the sum.

5. (20) Let Π_n be a random permutation of n elements. Assume that all permutations are equally likely. Let X_n denote the number of fixed points in Π_n and let $G_n(s)$ be the generating function of the random variable X_n.

a. (10) Show that for $j \geq 0$ we have

$$P(X_n = j) = (j+1)P(X_{n+1} = j+1)\,.$$

b. (5) Show that $G'_{n+1}(s) = G_n(s)$ and express $G_n(s)$ as a polynomial in the variable $s-1$.

c. (5) For every $k = 0, 1, 2, \ldots, n$ compute $P(X_n = k)$. You do not need to simplify the sums.

6. (20) A Casino decides that a promotion game will be offered to the next $n = 30,000$ guests. Every guest rolls a fair die. If 6 shows the guest gets free entrance and additional two euros. If 5 shows the guest gets free entrance. In every other case the guest pays the usual entrance fee of one euro.

a. (10) Let X_1 denote the casino's profit after the first guest. Compute $E(X_1)$ and $\text{var}(X_1)$.

b. (10) Compute the approximate probability that the total profit after 30,000 guests will be 10,490 euros or less.