Financial Mathematics 2022

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Lecturer

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Course Program

1. Introduction
    - simple market model,
    - basic concepts.
2. Risk-free assets
    - time value of money,
    - interest,
    - money market,
    - loans.
3. Risky assets
    - dynamics of prices,
    - binomial model,
    - other models.

Course Program

4. Discrete time market models.
5. Portfolio management
   - expected return and risk of portfolio,
   - efficient frontier,
   - CAPM.

Literature

1. (OBLIGATORY) Marek Capinski and Tomasz Zastwaniak, *Mathematics for Finance: An Introduction to Financial Engineering*, Springer, 2003.
2. (SUPPLEMENTARY) Stanley R. Pliska, *Introduction to Mathematical Finance*, Blackwell, 1997.
3. (OBLIGATORY) Marcin Anholcer, *Financial Mathematics, Course Materials*, UP FAMNIT Moodle, 2021.

Passing Conditions

1. Homeworks.
2. Final quiz (theory).

Basic assumptions and notation

- Two assets: risk-free (e.g. bond) ans risky (e.g. stock).
- Today: $t = 0$, one period from now: $t = 1$.
- Price of a share: $S(t)$. $S(0)$ known to all, $S(1)$ - random variable.
- Price of a bond: $A(t)$. $A(0)$ and $A(1)$ known to all.
- Rates of return (or return):

$$K_S = \frac{S(1) - S(0)}{S(0)}$$

$$K_A = \frac{A(1) - A(0)}{A(0)}$$

Assumptions (1)

Assumption (Randomness)

$S(1)$ is random variable with at least two possible values. $S(0)$, $A(0)$ and $A(1)$ are deterministic.

Assumption (Positivity of Prices)

All prices are positive: $S(t) > 0$, $A(t) > 0$.

Portfolio

Assume investor holds x stock shares and y bonds. Then the pair

$$(x, y)$$

is called the portfolio. Total wealth of investor (value of portfolio) at time t equals to:

$$V(t) = xS(t) + yA(t)$$

Return equals to:

$$K_V = \frac{V(1) - V(0)}{V(0)}$$

Special cases: $x = 0$, $y = 0$.

Example

Let $A(0) = 10€$, $A(1) = 12€$, $S(0) = 20€$ and

$$S(1) = \begin{cases} 22€ & \text{with probability } p \\ 18€ & \text{with probability } 1-p \end{cases}$$

Calculate the rates of return: on stock, on bonds and on portfolio consisting of 10 shares and 30 bonds.

Example

Let $A(0) = 20€$, $A(1) = 21€$, $S(0) = 14€$ and

$$S(1) = \begin{cases} 19€ & \text{with probability } p \\ 15€ & \text{with probability } 1-p \end{cases}$$

Calculate the rates of return: on stock, on bonds and on portfolio consisting of 20 shares and 25 bonds.

Assumptions (2)

Assumption (Divisibility)

The numbers x and y of shares and bonds may be integer or fractional.

Assumption (Short Selling)

The numbers x and y of shares and bonds may be negative.

Assumption (Liquidity)

The numbers x and y of shares and bonds are not bounded.

Types of positions

- Positive number of shares/bonds: investor has **long position**.
- Negative number of shares/bonds: investor has **short position**. Examples: issuing and selling bonds or borrowing cash (risk-free), short selling of shares (risky).

Assumptions (3)

Assumption (Solvency)

The investor's wealth is always non-negative:

$$V(t) \geq 0$$

Assumption (Discrete Prices)

The future price $S(1)$ can take only finitely many values.

Arbitrage

Arbitrage is a possibility of gaining risk-free profit with no initial investment. Examples:

- Sport bets.
- Currency market.
- Financial market (in general).

Example

Before the football match, bookmaker B_1 offers $2,5$ to 1 for the victory of team X and $1,5$ to 1 for Y. Bookmaker B_2 offers $1,9$ to 1 for the victory of team X and 2 to 1 for Y. Spot a chance of arbitrage.

Example (Capinski, Zastawniak (2003), p.6)

Dealer A in New York offers following prices (buy/sell) for Euro and Pounds:

- 1€ - 1.0202\$ / 1.0284\$.
- 1£ - 1.5718\$ / 1.5844\$.

Dealer B in London offers following prices (buy/sell) for Euro and US Dollars:

- 1€ - 0.6324£ / 0.6401£.
- 1\$ - 0.6299£ / 0.6375£.

Spot a chance of arbitrage.

Assumptions (4)

Assumption (No-Arbitrage Principle)

The arbitrage is impossible. In other words, there is no admissible portfolio with $V(0) = 0$, for which $V(1) > 0$ with probability $p > 0$.

In practice, the arbitrage opportunity exists very rarely and is inaccessible for small investors.

Binomial Model

$$S(1) = \begin{cases} S^u & \text{with probability } p \\ S^d & \text{with probability } 1-p \end{cases}$$

S^u - stock goes up, S^d - stock goes down, $S^u > S^d$.

Bounds on the bonds prices

Proposition

If $S(0) = A(0)$*, then*

$$S^d < A(1) < S^u$$

Proof (sketch). If the value of $A(1)$ exceeds the bounds, the arbitrage opportunity arises.

- If $A(1) \leq S^d$, then consider the portfolio $(x, y) = (1, -1)$.
- If $A(1) \geq S^u$, then consider the portfolio $(x, y) = (-1, 1)$.

Expected Return and Risk

Expectation of Return

$$E(K_V) = pK_V^u + (1-p)K_V^d$$

Standard Deviation:

$$\sigma_V = [p(K_V^u - E(K_V))^2 + (1-p)(K_V^d - E(K_V))^2]^{1/2}$$

Example

Let $A(0) = 10€$, $A(1) = 12€$, $S(0) = 20€$ and

$$S(1) = \begin{cases} 22€ & \text{with probability } p \\ 18€ & \text{with probability } 1-p \end{cases}$$

Calculate the expected rate of return and risk of portfolio consisting of 10 shares and 30 bonds. Compare it to the portfolio consisting of 20 shares and 25 bonds.

Example

Let $A(0) = 20€$, $A(1) = 21€$, $S(0) = 14€$ and

$$S(1) = \begin{cases} 19€ & \text{with probability } p \\ 15€ & \text{with probability } 1-p \end{cases}$$

Calculate the expected rate of return and risk of portfolio consisting of 10 shares and 30 bonds. Compare it to the portfolio consisting of 20 shares and 25 bonds.

Definition

Definition (Forward Contract)

Forward contract is an agreement to buy or sell a risky asset at a specified future time (delivery date), for a price F fixed at the present moment (forward price).

- Investor who agrees to buy *enters into a long forward contract* or *takes a long forward position*.
- Investor who agrees to sell *enters into a short forward contract* or *takes a short forward position*.

Example

Suppose that $F = 19€$ and

$$S(1) = \begin{cases} 22€ & \text{with probability } p \\ 18€ & \text{with probability } 1 - p \end{cases}$$

What are the possible gains of the investor taking a long and short position? What would they be if $F = 17€$? And if $F = 25€$?

Value of the portfolio

- Payoff for the long forward is $S(1) - F$, for short forward $F - S(1)$.
- Portfolio: (x, y, z), x-shares, y-bonds, z-forward ($z > 0$ for long and $z < 0$ for short position).
- Values of portfolio:

$$V(0) = xS(0) + yA(0)$$

$$V(1) = xS(1) + yA(1) + z(S(1) - F)$$

Uniqueness of the forward price

Proposition

The forward price must be equal to

$$F^{\star} = \frac{A(1)}{A(0)} S(0)$$

Otherwise the arbitrage opportunity will occur.

Uniqueness of the forward price

Proof (sketch). Consider the portfolio:

$$(x, y, z) = (x, -x\frac{S(0)}{A(0)}, -x)$$

- If $F > F^{\star}$, then $V(0) = 0$ and $V(1) > 0$ for $x > 0$ (buy asset, borrow cash, enter short forward).
- If $F < F^{\star}$, then $V(0) = 0$ and $V(1) > 0$ for $x < 0$ (short-sell asset, invest cash risk-free, enter long forward).

Example

Let $A(0) = 30$€, $A(1) = 33$€ and $S(0) = 20$€. What should be the forward price to avoid the arbitrage? What should do the investor if $F = 20$€? What if $F = 23$€?

Example (Capinski, Zastawniak (2003), p.13)

Suppose that $A(0) = 100€$, $A(1) = 105€$, the present price of pound sterling is $S(0) = 1.60€$ and the forward price with delivery date 1 is $F = 1.50€$. How much should a sterling bond cost today if it promises to pay 100£ at time 1? Solution: consider the prices of *bond* in €.

Definitions

Definition (Call Option)

Call option is a contract giving the holder the right to buy risky asset at a specified future time (exercise time), for a price C fixed at the present moment (strike price, exercise price).

Definition (Put Option)

Put option is a contract giving the holder the right to sell risky asset at a specified future time (exercise time), for a price P fixed at the present moment (strike price, exercise price).

Options vs Forwards

- You don't pay for a forward, you have to pay for an option.
- The holder of a forward **has to** buy/sell the asset. The holder of an option **doesn't have to** buy/sell the asset.

Payoffs of the options

Call:

$$C(1) = \begin{cases} S(1) - C & \text{if } S(1) > C \\ 0 & \text{if } S(1) \leq C \end{cases}$$

Put:

$$P(1) = \begin{cases} P - S(1) & \text{if } S(1) < P \\ 0 & \text{if } S(1) \geq P \end{cases}$$

Example

Suppose that

$$S(1) = \begin{cases} 22€ & \text{with probability } p \\ 18€ & \text{with probability } 1 - p \end{cases}$$

What are the possible payoffs for the investor holding an call option with $C = 17€$? What if $C = 21€$ or $C = 24€$? What are the possible payoffs for the investor holding a put option with $P = 18€$, $P = 19€$ and $P = 23€$?

Call Option Pricing

1. Replicating the option - construct an investment in $x^\star$ stocks and $y^\star$ options s.t.

$$x^\star S(1) + y^\star A(1) = C(1)$$

   no matter what will be the value of $S(1)$
2. Compute option price $C(0)$ from the equation

$$C(0) = x^\star S(0) + y^\star A(0)$$

Replicating the call option

$$\begin{cases} x^\star S^u + y^\star A(1) = S^u - C \\ x^\star S^d + y^\star A(1) = 0 \end{cases}$$

$$\begin{cases} x^\star = \frac{S^u - C}{S^u - S^d} > 0 \\ y^\star = -x^\star \frac{S^d}{A(1)} = -\frac{(S^u - C)S^d}{A(1)(S^u - S^d)} < 0 \end{cases}$$

Example (Capinski, Zastawniak (2003), p.13)

Let $A(0) = 100€$, $A(1) = 110€$, $S(0) = 100€$ and

$$S(1) = \begin{cases} 120€ & \text{with probability } p \\ 80€ & \text{with probability } 1 - p \end{cases}$$

Calculate the price of the call option with $C = 100€$.

Uniqueness of the call option price

Proposition

The price of the call option must be equal to

$$C^\star = x^\star S(0) + y^\star A(0)$$

Otherwise the arbitrage opportunity will occur.

Uniqueness of the call option price

Proof (sketch).

- If $C(0) > C^\star$, then consider the portfolio $(x, y, z) = (x^\star, y^\star, -1)$ (buy assets, borrow cash, issue and sell option). $V(0) < 0$, so the cash balance is positive (equal to $-V(0)$). Invest this amount risk-free. As $V(1) = 0$, risk-free investment plus interest always generates the positive income.
- If $C(0) < C^\star$, then consider the portfolio $(x, y, z) = (-x^\star, -y^\star, 1)$ (short sell shares, buy bonds, buy option). $V(0) < 0$, so the cash balance is positive (equal to $-V(0)$). Invest this amount risk-free. As $V(1) = 0$, risk-free investment plus interest always generates the positive income.

Example (Capinski, Zastawniak (2003), p.16)

Let $A(0) = 100€$, $A(1) = 110€$, $S(0) = 100€$, $C = 100€$ and

$$S(1) = \begin{cases} 120€ & \text{with probability } p \\ 80€ & \text{with probability } 1-p \end{cases}$$

Analyse the cases when $C(0) = 13€$ and $C(0) = 15€$.

Put Option Pricing

1. Replicating the option - construct an investment in $x^\star$ stocks and $y^\star$ options s.t.

$$x^\star S(1) + y^\star A(1) = P(1)$$

   no matter what will be the value of $S(1)$

2. Compute option price $P(0)$ from the equation

$$P(0) = x^\star S(0) + y^\star A(0)$$

Replicating the put option

$$\begin{cases} x^\star S^u + y^\star A(1) = 0 \\ x^\star S^d + y^\star A(1) = P - S^d \end{cases}$$

$$\begin{cases} x^\star = \frac{P - S^d}{S^d - S^u} < 0 \\ y^\star = -x^\star \frac{S^u}{A(1)} = -\frac{(P - S^d)S^u}{A(1)(S^d - S^u)} > 0 \end{cases}$$

Example (Capinski, Zastawniak (2003), p.13)

Let $A(0) = 100€$, $A(1) = 110€$, $S(0) = 100€$, $P = 100€$ and

$$S(1) = \begin{cases} 120€ & \text{with probability } p \\ 80€ & \text{with probability } 1 - p \end{cases}$$

Calculate the price of the put option.

Uniqueness of the put option price

Proposition

The price of the put option must be equal to

$$P^\star = x^\star S(0) + y^\star A(0)$$

Otherwise the arbitrage opportunity will occur.

Uniqueness of the put option price

Proof (sketch).

- If $P(0) > P^\star$, then consider the portfolio $(x, y, z) = (x^\star, y^\star, -1)$ (short sell shares, buy bonds, issue and sell option). $V(0) < 0$, so the cash balance is positive (equal to $-V(0)$). Invest this amount risk-free. As $V(1) = 0$, risk-free investment plus interest always generates the positive income.
- If $P(0) < P^\star$, then consider the portfolio $(x, y, z) = (-x^\star, -y^\star, 1)$ (buy assets, borrow cash, buy option). $V(0) < 0$, so the cash balance is positive (equal to $-V(0)$). Invest this amount risk-free. As $V(1) = 0$, risk-free investment plus interest always generates the positive income.

Example (Capinski, Zastawniak (2003), p.16)

Let $A(0) = 100€$, $A(1) = 110€$, $S(0) = 100€$, $P = 100€$ and

$$S(1) = \begin{cases} 120€ & \text{with probability } p \\ 80€ & \text{with probability } 1 - p \end{cases}$$

Analyse the cases when $P(0) = 4€$ and $P(0) = 6€$.

Derivative securities

- One may invest in more than one kind of option and in addition to open a forward position - we will use $z_1, z_2, \ldots$.
- The payoffs of these securities depend on the stock prices. Thus we call them **derivative securities** or **derivatives**.

Example (Capinski, Zastawniak (2003), p.19)

Let $A(0) = 100€$, $A(1) = 110€$, $S(0) = 100€$, $C = 100€$ and

$$S(1) = \begin{cases} 120€ & \text{with probability } p \\ 80€ & \text{with probability } 1 - p \end{cases}$$

Consider investing 1000€ in either only shares or only call options. Compare the risk and expected return. Does exist such p that one of the strategies is better than the other one?

Example (Capinski, Zastawniak (2003), p.19)

Let $A(0) = 100€$, $A(1) = 110€$, $S(0) = 100€$ and

$$S(1) = \begin{cases} 160€ & \text{with probability } p \\ 40€ & \text{with probability } 1 - p \end{cases}$$

Consider the following strategies (calculate the expected return and risk):

1. Wait until time 1 and purchase the stock for $S(1)$.
2. Borrow money to buy one call option with strike price $C = 100€$. At time 1 repay the loan with interest and purchase the stock. Exercise the option if it is profitable.
3. Borrow money to buy two call options with strike price $C = 100€$. At time 1 repay the loan with interest and purchase the stock. Exercise the options if it is profitable.

Reasons

- Compensation for postponed consumption.
- Increase of prices in the same period.
- Risk of loosing the money.

Questions

- What is the future value of money invested today?
- What amount of money should be invested today in order to receive predefined amount in the future?

Basic Concepts

- Principal $V(0)$
- Interest Rate r
- Future Value $V(t)$

Applications

- Interest always paid in cash and not re-invested.
- Interest credited to 0%-account.
- Interest credited to the original account after some longer period.

Future Value of the investment

$$V(t) = (1 + tr)V(0)$$

Here t is arbitrary positive real number (not in practice!) and $1 + tr$ is called *growth factor*. More generally, for positive s (start) and t (end) we have

$$V(s, t) = (1 + (t - s)r)V(s)$$

Return on an Investment

In general:

$$K(s,t) = \frac{V(t) - V(s)}{V(s)}$$

In case of simple interest:

$$K(s,t) = (t-s)r$$

Return over one year:

$$K(t,t+1) = r$$

Example

Consider the deposit of 200€, attracting simple interest at a rate of 10%. Calculate the value of the deposit and the return on the investment:

- in 50 days, when the investment starts today,
- in 3 years, when the investment starts in 1 year.

Present Value of the investment

$$V(0) = (1 + tr)^{-1} V(t)$$

Here $(1 + tr)^{-1}$ is called *discount factor*. More generally, for positive s (start) and t (end) we have

$$V(s) = (1 + (t - s)r)^{-1} V(s, t)$$

Example

What should be the initial value of the deposit, attracting simple interest at a rate of 10%, if it should become 2000€:

- in 50 days, when the investment starts today,
- in 3 years, when the investment starts in 1 year?

Perpetuity

Sequence of payments of a fixed amount of money to be made at equal time intervals and continuing to infinity. If the payment is R and interest rate r, the principle must be equal to

$$V(0) = \frac{R}{r}$$

in order to keep the value of the deposit at the constant level.

Future Value

- Interest added to the deposit earns the interest after some period, most frequently quarter or month.
- The value after one year (m compounding periods):

$$V(1) = \left(1 + \frac{r}{m}\right)^{m} V(0)$$

The value after t years (t being integer multiple of $\frac{1}{m}$):

$$V(t) = \left(1 + \frac{r}{m}\right)^{mt} V(0)$$

More generally:

$$V(s,t) = \left(1 + \frac{r}{m}\right)^{m(t-s)} V(s)$$

Example

Consider the deposit of 200€, attracting p.a. interest at a rate of 10%. Calculate the value of the deposit in 2 years subject to:

- semi-annual compounding,
- quarterly compounding,
- monthly compounding.

Present Value

- The value one year before:

$$V(0) = \left(1 + \frac{r}{m}\right)^{-m} V(1)$$

The value t years before:

$$V(0) = \left(1 + \frac{r}{m}\right)^{-mt} V(t)$$

More generally:

$$V(s) = \left(1 + \frac{r}{m}\right)^{-m(t-s)} V(s, t)$$

Example

What should be the initial value of the deposit, attracting p.a. interest at a rate of 7%, if it should bring 2500€ in 3 years subject to:

- semi-annual compounding,
- quarterly compounding,
- monthly compounding?

Return on an Investment

In general:

$$K(s,t) = \frac{V(t) - V(s)}{V(s)}$$

In case of periodic compounding:

$$K(s,t) = \left(1 + \frac{r}{m}\right)^{m(t-s)} - 1$$

Return over one compounding period:

$$K\left(t, t + \frac{1}{m}\right) = \frac{r}{m}$$

The Monotonicity of $V(t)$

Proposition

The value of $V(t)$ increases if any of the values m, t, r or $V(0)$ increases, the others remaining unchanged.

Proof (sketch).

- t, r or $V(0)$ - trivial.
- m - binomial formula.

Future Value

- Infinitely many compounding periods. Good approximation of the situation where m is large,
- The value after one year:

$$V(1) = \lim_{m \to \infty} \left\{ \left(1 + \frac{r}{m} \right)^m \right\} V(0) = e^r V(0)$$

- The value after t years:

$$V(t) = e^{rt} V(0)$$

- More generally:

$$V(s,t) = e^{r(t-s)} V(s)$$

Example

Consider the deposit of 200€, attracting p.a. interest at a rate of 10%. Calculate the value of the deposit in 2 years subject to continuous compounding.

Present Value

- The value before one year:

$$V(0) = e^{-r} V(1)$$

- The value before t years:

$$V(0) = e^{-rt} V(t)$$

- More generally:

$$V(s) = e^{-r(t-s)} V(s,t)$$

Example

What should be the initial value of the deposit, attracting p.a. interest at a rate of 7%, if it should bring 2500€ in 3 years subject to continuous compounding?

Return on an Investment

In general:

$$K(s,t) = \frac{V(t) - V(s)}{V(s)}$$

In case of continuous compounding:

$$K(s,t) = e^{r(t-s)} - 1$$

Logarithmic returns

In general:

$$k(s,t) = \ln \frac{V(t)}{V(s)}$$

In case of continuous compounding we have:

$$k(s,t) = r(t-s)$$

and thus

$$r = \frac{k(s,t)}{t-s}$$

Additivity vs. Multiplicativity

Fact

- *The returns on a deposit subject to simple interest are additive.*
- *The growth factors (not the returns!) subject to periodic compounding are multiplicative.*
- *The growth factors subject to continuous compounding are multiplicative.*
- *The logarithmic returns are additive.*

Formulae

- Future value:

$$V(n) = \sum_{i=1}^{n} (1+r)^{i} R(i)$$

- Present value:

$$V(0) = \sum_{i=1}^{n} (1+r)^{-i} R(i)$$

- Infinite payments:

$$V(0) = \sum_{i=1}^{\infty} (1+r)^{-i} R(i)$$

Present Value Factor for an Annuity

We assume that the payments are equal, i.e. $R(i) = R$ for every i.

$$PA(r,n) = \sum_{i=1}^{n} (1+r)^{-i} = \frac{1-(1+r)^{-n}}{r}$$

then

$$V(0) = PA(r,n) \times R$$

Special case - perpetuity:

$$PA(r,\infty) = \frac{1}{r}$$

$$V(0) = \frac{R}{r}$$

Example

Consider the stream of 10 annual payments of 1000€ at the interest rate 3%. What will be the balance of the account in 10 years?

Example

Consider the stream of 10 annual payments of 1000€ at the interest rate 3%. What is the present value of the stream?

Example

Consider the stream of infinitely many annual payments of 1000€ at the interest rate 3%. What is the present value of the stream?

Loans

- Constant instalments: derived from the equation

$$R = \frac{V(0)}{PA(r, n)}$$

- Decreasing instalments: constant fraction of the loan plus decreasing interests.
- Repayment schedule.

Example

Consider a loan of 1700€ to be paid back in 3 equal yearly instalments subject to an interest rate of 4%. Calculate the value of each instalment and present the repayment schedule.

Example

Consider a loan of 1700€ to be paid back in 3 decreasing yearly instalments subject to an interest rate of 4%. Calculate the value of each instalment and present the repayment schedule.

Example

Consider a loan of 250000€ to be paid back in 360 decreasing or equal monthly instalments subject to an interest rate of 3%. In both cases calculate the value of each instalment and present both repayment schedules.

Effective Interest Rate

- Allows the comparison of the investments with more than one different parameters.
- The formulae:

$$r_e = \left(1 + \frac{r}{m}\right)^m - 1$$

$$r_e = e^r - 1$$

Example

Compare the following compounding methods:

- yearly compounding, $r = 0.1$,
- quarterly compounding, $r = 0.097$,
- monthly compounding, $r = 0.094$,
- continuous compounding $r = 0.09$.

Internal Interest Rate

- Allows the comparison of the investments in the form of streams.
- In order to find *IRR* one has to solve the equation

$$V(0) = \sum_{i=1}^{n} (1+r)^{-i} R(i)$$

in r.

Example

Compare the following investments:

- cost: 1000€ today, income: 600€ in one year and 600€ in two years,
- cost: 1200€ today, income: 750€ in one year and 700€ in two years.

Elements

Money Market - risk-free (default-free) assets

- bonds,
- treasury bills and notes,
- mortgage,
- commercial papers,
- e.t.c.

Zero-Coupon Bonds

- The issuing institution (*bond writer*) promises to exchange bond for money F_T (face value) at given day T (maturity date). It is like lending money to the bond writer.
- Present value:

$$V(0) = F_T \left(1 + \frac{r}{m}\right)^{-mT} = F_T(1 + r_e)^{-T}$$

- Usually, prices are set. The effective interest rate equals to:

$$r_e = \left(\frac{F_T}{V(0)}\right)^{\frac{1}{T}} - 1$$

Zero-Coupon Bonds

- Bond price at time t:

$$B(t,T) = F_T\left(1+\frac{r}{m}\right)^{-m(T-t)} = F_T(1+r_e)^{-(T-t)}$$

- Continuous compounding:

$$B(t,T) = F_T e^{-r(T-t)}$$

Example

An investor paid 90€ for zero-coupon bond with face value 110€ maturing in a year. What is the effective rate of return if the compounding is:

- annual,
- monthly,
- continuous?

Coupon Bonds

- The issuing institution (*bond writer*) promises to exchange bond for money F_T (face value) at given day T (maturity date) and to pay regularly (usually annually, semi-annually or quarterly) the *coupons* $C_T(i)$. We assume the constant interest rates.
- Present value:

$$V(0) = F_T(1 + r_e)^{-T} + \sum_{i=1}^{T} (1 + r_e)^{-i} C_T(i)$$

- If the price is set, the effective rate may be derived from the above equation.

Example

An investor bought the bond with face value 110€ and annual 10€ - coupons maturing in 5 years. What is price of the bond if the compounding is ($r = 0.06$):

- annual,
- monthly,
- continuous?

Coupon Bonds

Proposition

Assume the coupons are paid annually. The coupon rate is equal to the interest rate for annual compounding if and only if the price of the bond is equal to its face value.

In such a case we say that the bond *sells at par* (or *trades at par*).

Proof (sketch).

- Start with the price $V(0)$, use $r_C = r_e$, obtain F_T.
- Start with $V(0) = F_T$, reduce F_T, obtain $r_C = r_e$. Another way: use the fact that $V(0)$ is monotonous in r thus only once has value F_T.

Example

A bond with face $F_T = 120$€ and annual 5€ - coupons, maturing after 5 years is trading at par. What is the effective rate of return if the compounding is:

- annual,
- monthly,
- continuous?

Money-Market Account

- Account with a financial intermediary. Long position - buying asset, short position - borrowing money.
- Assuming some type (e.g. continuous) compounding, the value (balance) does not depend on the type of the asset:

$$A(t) = e^{rt}A(0)$$

Notation and assumptions

- Price at moment t denoted by $S(t)$.
- Random variable:

$$S(t) : \Omega \to (0, \infty)$$

- Current value $S(0)$ known, but may be considered as constant random variable.
- Time is discrete, $t = n\tau$ (τ - time unit), for simplicity we use $S(t)$ instead of $S(\tau t)$.

Example (Capinski, Zastawniak (2003), p.48)

There are two possible scenarios: ω_1 and ω_2, and the possible prices of an asset are:

Scenario	$S(0)$	$S(1)$
ω_1	10	12
ω_2	10	7

Present the tree of price movements.

Example (Capinski, Zastawniak (2003), p.48)

There are three possible scenarios: ω_1, ω_2 and ω_3 and the possible prices of an asset are:

Scenario	$S(0)$	$S(1)$	$S(2)$
ω_1	55	58	60
ω_2	55	58	52
ω_3	55	52	53

Present the tree of price movements.

Return without dividend

Rate of return:

$$K(s,t) = \frac{S(t) - S(s)}{S(s)}$$

One-period rate of return:

$$K(t) = K(t-1,t) = \frac{S(t) - S(t-1)}{S(t-1)} = \frac{S(t)}{S(t-1)} - 1$$

Thus:

$$S(t) = S(t-1)(1 + K(t))$$

Return with dividend

One-period rate of return:

$$K(t) = \frac{S(t) - S(t-1) + div(t)}{S(t-1)} = \frac{S(t) + div(t)}{S(t-1)} - 1$$

Thus:

$$S(t) = S(t-1)(1 + K(t)) - div(t)$$

Example (Capinski, Zastawniak (2003), p.50)

There are three possible scenarios: ω_1, ω_2 and ω_3 and the possible prices of an asset are:

Scenario	$S(0)$	$S(1)$	$S(2)$
ω_1	55	58	60
ω_2	55	58	52
ω_3	55	52	53

Calculate the one-period returns $K(1)$ and $K(2)$.

Example (Capinski, Zastawniak (2003), p.50)

There are three possible scenarios: ω_1, ω_2 and ω_3, $S(0) = 45€$ and the possible returns are:

Scenario	$K(1)$	$K(2)$	$K(3)$
ω_1	10%	5%	−10%
ω_2	5%	10%	10%
ω_3	5%	−10%	10%

Present the tree of price movements.

Example (Capinski, Zastawniak (2003), p.50)

There are three possible scenarios: ω_1, ω_2 and ω_3, $S(0) = 45€$ and the possible returns are:

Scenario	$K(1)$	$K(2)$	$K(3)$
ω_1	10%	5%	-10%
ω_2	5%	10%	10%
ω_3	5%	-10%	10%

Present the tree of price movements, assuming that the divident of 1€ is paid at the end of every period.

Example (Capinski, Zastawniak (2003), p.51)

There are three possible scenarios: ω_1, ω_2 and ω_3, $S(0) = 45€$ and the possible returns are:

Scenario	$K(1)$	$K(2)$	$K(3)$
ω_1	10%	5%	−10%
ω_2	5%	10%	10%
ω_3	5%	−10%	10%

Calculate $K(0,2)$ and $K(0,3)$ and compare it to the sums of respective 1-period returns.

Growth factors are multiplicative

Proposition

The relationship between one-period and multi-period rates of return is as follows:

$$1 + K(s,t) = \prod_{i=s+1}^{t} (1 + K(i))$$

Proof (sketch). Compare two formulae:

$$K(s,t) = \frac{S(t)}{S(s)} - 1$$

$$K(t) = \frac{S(t)}{S(t-1)} - 1$$

Example (Capinski, Zastawniak (2003), p.52)

There are three possible scenarios: ω_1, ω_2 and ω_3 and the possible prices of an asset are:

Scenario	$S(0)$	$S(2)$
ω_1	55	60
ω_2	55	65
ω_3	55	70

Calculate the one-period returns assuming that they are equal.

Example (Capinski, Zastawniak (2003), p.52)

Given that $K(1) \in \{-10\%, 10\%\}$ and $K(0,2) \in \{-1\%, 10\%, 21\%\}$ find a possible structure of scenarios such that $K(2)$ takes at most two possible values.

Logarithmic Returns

Rate of return:

$$k(s,t) = \ln \frac{S(t)}{S(s)}$$

One-period rate of return:

$$k(t) = k(t-1,t) = \ln \frac{S(t)}{S(t-1)}$$

Thus

$$S(t) = S(s)e^{k(s,t)}$$

and

$$S(t) = S(t-1)e^{k(t)}$$

Example (Capinski, Zastawniak (2003), p.53)

There are three possible scenarios: ω_1, ω_2 and ω_3 and the possible prices of an asset are:

Scenario	$S(0)$	$S(1)$	$S(2)$
ω_1	55	58	60
ω_2	55	58	52
ω_3	55	52	53

Calculate the one-period logarithmic returns $k(1)$ and $k(2)$.
Compare them to the logarithmic returns $k(0, 2)$.

Logarithmic rates are additive

Proposition

The relationship between one-period and multi-period logarithmic rates of return is as follows:

$$k(s,t) = \sum_{i=s+1}^{t} k(i)$$

Proof (sketch). Compare two formulae:

$$k(s,t) = \ln \frac{S(t)}{S(s)}$$

$$k(t) = \ln \frac{S(t)}{S(t-1)}$$

Ordinary vs. Logarithmic Rates of Return

Proposition

The relationship between ordinary and logarithmic rates of return is as follows:

$$1 + K(s,t) = e^{k(s,t)}$$

Proof (sketch). Compare both definitions.

Expected Return and Risk

Expectation of Return

$$E(K(s,t)) = \sum_{\omega \in \Omega} K_\omega(s,t) Pr(\omega)$$

Standard Deviation:

$$\sigma_{K(s,t)} = [\sum_{\omega \in \Omega} (K_\omega(s,t) - E(K(s,t)))^2 Pr(\omega)]^{1/2}$$

Expectation of Logarithmic Return

$$E(k(s,t)) = \sum_{\omega \in \Omega} k_\omega(s,t) Pr(\omega)$$

Standard Deviation:

$$\sigma_{k(s,t)} = [\sum_{\omega \in \Omega} (k_\omega(s,t) - E(k(s,t)))^2 Pr(\omega)]^{1/2}$$

Example (Capinski, Zastawniak (2003), p.54)

There are three possible scenarios: $Pr(\omega_1) = 0.3$, $Pr(\omega_2) = 0.4$ and $Pr(\omega_3) = 0.3$ and the possible prices of an asset are:

Scenario	$S(0)$	$S(1)$	$S(2)$
ω_1	55	58	60
ω_2	55	58	52
ω_3	55	52	53

Calculate the expected one-period returns and compare it to the multi-period ones. Do the same with the logarithmic expected returns.

Expected growth factors are multiplicative

Proposition

If the one-period rates of return are independent, then the relationship between one-period and multi-period expected rates of return is as follows:

$$1 + E(K(s,t)) = \prod_{i=s+1}^{t} (1 + E(K(i)))$$

Proof (sketch). Use the relation between returns and the fact that the product of expectations of independent random variables is the same as expectation of their product.

Expected logarithmic rates are additive

Proposition

The relationship between one-period and multi-period expected logarithmic rates of return is as follows:

$$E(k(s,t)) = \sum_{i=s+1}^{t} E(k(i))$$

Proof (sketch). Use the relation between the logarithmic returns and the additivity of expectation.

Introduction

- Very simple model.
- On one hand tractable.
- On the other hand covers many real-world situations.

Conditions

- One-period returns are identically distributed:

$$K(t) = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1-p \end{cases}$$

  where $-1 < d < u$ and $0 < p < 1$.

- One-period return on a risk-free investment satisfies the condition:

$$d < r < u$$

Example

Assume the initial value of an asset is $S(0)$.

- What are possible values of $S(1)$, $S(2)$, $S(3)$? What are the corresponding probabilities? How the tree of price movements looks like if $S(0) = 1$€?
- What are possible values of $S(t)$? What are the corresponding probabilities?

Expected Stock Prices

Proposition

The expected stock prices are given by:

$$E(S(t)) = S(0)(1 + E(K(1)))^t$$

Proof (sketch).

$$E(S(t)) = E(S(0)\prod_{i=1}^{t}(1 + E(K(i))))$$

Risk-free investment

$$S(0)(1 + E(K(1)))^n = S(0)(1 + r)^n$$

and so

$$E(K(1)) = r$$

Risk-averse investor: $E(K(1)) > r$. Risk-seeker: $E(K(1)) < r$, but maximum possible income is huge (see e.g. lottery).

Risk-Neutral Probability

$$r = E_\star(K(1)) = p_\star u + (1 - p_\star)d$$

implies

$$p_\star = \frac{r - d}{u - d}$$

Example: Risk-Neutral Probability - properties

- Investigate the properties of $p_\star$ as a function of d.
- Investigate the properties of $p_\star$ as a function of u.
- Investigate the properties of $p_\star$ as a function of r.

Risk-Neutral Probability - geometric interpretation

$$r = p_\star u + (1 - p_\star)d$$

implies

$$p_\star(u - r) + (1 - p_\star)(d - r) = 0$$

It means that vectors $(p_\star, 1 - p_\star)$ and $(u - r, d - r)$ are orthogonal.

Risk-Neutral Probability - barycentric interpretation

$$r = p_\star u + (1 - p_\star)d$$

implies

$$p_\star(u - r) = (1 - p_\star)(r - d)$$

It means that vectors masses $p_\star$ and $1 - p_\star$ should hang at points u and d if r is supposed to be the center.

Example (Capinski, Zastawniak (2003), p.61)

Assume $S(0) = 100€$, $u = 0.2$, $d = -0.1$, $r = 0.1$.

- Present the two-period tree of price movements.
- Calculate the risk-neutral probability and corresponding expected stock price after two periods.
- Assume that in the first period the stock goes up. What is the expected price after two periods compared to the price after first period. What is the situation if the stock goes down?

Risk-Neutral Conditional Expectation

Proposition

Given $S(t)$ known at time t, the risk-neutral conditional expectation of $S(t+1)$ is:

$$E_\star(S(t+1)|S(t)) = S(t)(1+r)$$

Proof (sketch). Let $S(t) = x$ for some x. Then

$$E_\star(S(t+1)|S(t) = x) = [p_\star(1+u) + (1-p_\star)(1+d)]x = (1+r)x$$

Thus

$$E_\star(S(t+1)|S(t) = x) = x(1+r)$$

for every x.

Martingale Property

Proposition

Let $\tilde{S}(t) = (1+r)^{-t}S(t)$ be discounted stock price. Then

$$E_\star(\tilde{S}(t+1)|S(t)) = \tilde{S}(t)$$

Proof (sketch). Multiply both sides of

$$E_\star(S(t+1)|S(t)) = S(t)(1+r)$$

by

$$(1+r)^{-t-1}$$

and obtain

$$E_\star((1+r)^{-t-1}S(t+1)|S(t)) = S(t)(1+r)^{-t}$$

Example

- (Capinski, Zastawniak (2003), p.63) Let $r = 0.2$ and $S(2) = 110$€. Find $E(S(3)|S(2))$.
- Let $r = 0.2$ and $S(2) = 110$€. Find $E(\tilde{S}(3)|S(2))$.

Introduction

- Generalization of previously discussed models.
- m risky assets, prices $S_1(t), S_2(t), \dots, S_m(t)$, positions $x_1, x_2, \dots, x_m$.
- Investment in the money market, risk-free, price $A(t)$, usually $A(0) = 1$, position y.
- Investor's wealth at time t:

$$V(t) = \sum_{j=1}^{m} x_j S_j(t) + yA(t)$$

Assumptions (1)

Assumption (Randomness)

The future stock prices $S_1(t), S_2(t), \ldots, S_m(t)$, for every $t = 1, 2, \ldots,$ are random variables. $S_1(0), S_2(0), \ldots, S_m(0)$, and $A(t)$ for $t = 0, 1, 2, \ldots$ are deterministic.

Assumption (Positivity of Prices)

All prices are positive: $S_j(t) > 0$, $A(t) > 0$ for $j = 1, \ldots, m$ and $t = 0, 1, \ldots$.

Assumptions (2)

Assumption (Divisibility)

The numbers x_j for $j = 1, \ldots, m$ and y of shares and bonds may be integer or fractional.

Assumption (Short Selling)

The numbers x_j for $j = 1, \ldots, m$ and y of shares and bonds may be negative.

Assumption (Liquidity)

The numbers x_j for $j = 1, \ldots, m$ and y of shares and bonds are not bounded.

Assumptions (3)

Assumption (Solvency)

The investor's wealth is always non-negative:

$$V(t) \geq 0$$

Assumption (Discrete Prices)

Every future price $S_j(t)$, for $j = 1, \ldots, m$ and $t = 0, 1, \ldots,$ can take only finitely many values.

Portfolio vs. Strategy

- The elements of the portfolio may be altered at any moment.
- Formally: portfolio kept between moments $t-1$ and t is a vector of the form

$$(x_1(t), x_2(t), \ldots, x_m(t), y(t))$$

- Strategy is the sequence of such vectors (portfolios).

Value of the Portfolio

The value at time $t \geq 1$:

$$V(t) = \sum_{j=1}^{m} x_j(t) S_j(t) + y(t) A(t)$$

The value at time $t = 0$:

$$V(0) = \sum_{j=1}^{m} x_j(1) S_j(0) + y(1) A(0)$$

Example (Capinski, Zastawniak (2003), p.75)

Assume that $m = 2$ and the prices are as follows:

Price	$t = 0$	$t = 1$	$t = 2$
$S_1(t)$	60	65	75
$S_2(t)$	20	15	25
$A(t)$	100	110	121

Assume the initial portfolio is $(x_1(1), x_2(1), y(1)) = (20, 65, 5)$.
Consider two strategies, corresponding to two new possible portfolios: $(x_1(2), x_2(2), y(2)) = (x_1(1), x_2(1), y(1))$ and $(x_1(2), x_2(2), y(2)) = (15, 94, 4)$.

Self-Financing and Predictable Strategies

A strategy is **self-financing** if the portfolio constructed at any moment is fully financed by the investor's wealth:

$$\sum_{j=1}^{m} x_j(t+1)S_j(t)+y(t+1)A(t) = V(t) = \sum_{j=1}^{m} x_j(t)S_j(t)+y(t)A(t)$$

A strategy is **predictable** if the portfolio constructed at any moment t depends only on the market scenarios up to the moment t.

Risk-Free Position is Determined

Proposition

Given initial wealth $V(0)$ and predictable sequence $(x_1(t), x_2(t), \ldots, x_m(t))$ for $t = 1, 2, \ldots,$ it is always possible to find a (unique) sequence $y(t)$ of risk-free positions such that the sequence $(x_1(t), x_2(t), \ldots, x_m(t), y(t))$ is self-financing and predictable strategy.

Proof (sketch). We use recursion to calculate $y(1), y(2), \ldots$ - inductive proof.

Example (Capinski, Zastawniak (2003), p.78)

Assume that $m = 2$ and the prices are as follows:

Price	$t = 0$	$t = 1$	$t = 2$
$S_1(t)$	60	65	75
$S_2(t)$	20	15	25
$A(t)$	100	110	121

Assume that $V(0) = 200$ and the risky positions are $(x_1(1), x_2(1)) = (35.24, 24.18)$ and $(x_1(2), x_2(2)) = (-40.50, 10.13)$. Find the risk-free positions $y(1)$ and $y(2)$ of a predictable, self-financing strategy and corresponding values of the portfolios.

Example (Capinski, Zastawniak (2003), p.78)

Assume that $m = 2$ and the prices are as follows:

Price	$t = 0$	$t = 1$	$t = 2$
$S_1(t)$	60	65	75
$S_2(t)$	20	15	25
$A(t)$	100	110	121

Assume that $V(0) = 100$ and the initial portfolio is $(x_1(1), x_2(1), y(1)) = (-12, 31, 2)$. What are the values $V(1)$ and $V(2)$?

Handling with Insolvency

The portfolio from the example could not be constructed in practice:

- If one wants to short sell a risky asset, some deposit must be paid.
- If the value of the portfolio plus deposit reaches 0, the position has to be closed.

Admissible Strategy

A strategy is admissible if it is self-financing, predictable and for each $t = 0, 1, \ldots$ we have

$$V(t) \geq 0 \text{ with probability } 1$$

Example (Capinski, Zastawniak (2003), p.79)

Consider a market with one risk-free and one risky asset, where $A(0) = 10$, $A(1) = 11$, $S(0) = 10$ and $S(1) \in \{9, 13\}$. On the plane draw all the portfolios such that one-step strategy is admissible.

Assumptions (4)

Assumption (No-Arbitrage Principle)

The arbitrage is impossible. In other words, there is no admissible strategy such that $V(0) = 0$*, for which* $V(t) > 0$ *with probability* $p > 0$ *for some* $t = 1, 2, \ldots$.

Example (Capinski, Zastawniak (2003), p.80)

Show that the No-Arbitrage Principle would be violated if there was a self-financing predictable strategy with initial value $V(0) = 0$ and final value $0 \neq V(2) \geq 0$, such that $V(1) < 0$ with positive probability.

Example (Capinski, Zastawniak (2003), p.80)

Consider a market with one risk-free asset and one risky asset that follows the binomial tree model. Suppose that whenever stock goes up, you can be sure it will go down at the next step. Find a self-financing (but not necessarily predictable) strategy with $V(0) = 0$, $V(1) \geq 0$ and $0 \neq V(2) \geq 0$.

Example (Capinski, Zastawniak (2003), p.80-81)

There are three possible scenarios, and the possible prices of an asset are:

Scenario	$S(0)$	$S(1)$	$S(2)$
ω_1	100	120	144
ω_2	100	120	96
ω_3	100	90	96

Assume $A(0) = 100$, $A(1) = 110$ and $A(2) = 121$. Is there an opportunity of arbitrage if short selling is allowed? What if it is not allowed? What if $S_{\omega_2}(1) \notin \{90, 120\}$? What if the prices are as before, short selling is allowed, but the positions must be integers? What if transaction costs of 5% are applied when the stock is traded?

No-Arbitrage Principle and the Binomial Tree Model

Proposition

The binomial tree model admits no arbitrage if and only if $d < r < u$.

Proof (sketch).

- Begin with one-step model. Consider cases $r \leq d$, $r \geq u$ and $d < r < u$.
- Several steps: each fragment of the tree is the one-step tree. First $V(t) > 0$ in the sequence $1, 2, \dots, t$ contradicts with one-step case, so there is no arbitrage if $d < r < u$. If there is no arbitrage, then also $V(1) = 0$ and $d < r < u$.

No-Arbitrage Principle and the Risk-Neutral Probability

Corollary

The binomial tree model admits no arbitrage if and only if there exists a risk-neutral probability $0 < p_\star < 1$.

Proof (sketch). From the definition:

$$r = p_\star u + (1 - p_\star)d$$

we have $0 < p_\star < 1$ if and only if $d < r < u$.

Fundamental Theorem of Asset Pricing

Theorem

The No-Arbitrage Principle is equivalent to the existence of probability $P_\star$ on the set of scenarios Ω such that $P_\star(\omega) > 0$ for each scenario $\omega \in \Omega$ and the discounted stock prices $\tilde{S}_j(t) = S_j(t)/A(t)$ satisfy

$$E_\star(\tilde{S}_j(t+1)|S(t)) = \tilde{S}_j(t)$$

for any $j = 1, \ldots, m$ and $t = 0, 1, \ldots,$ where $E_\star(\cdot|S(t))$ denotes the conditional expectation with respect to probability $P_\star$ computed once the stock price $S(t)$ becomes known at time t.

Proof - omitted.

Martingales

- Such a sequence is called a martingale with respect to $P_\star$.
- $P_\star$ is called risk-neutral or martingale probability.
- $E_\star$ is called risk neutral or martingale expectation.

Example (Capinski, Zastawniak (2003), p.84)

There are four possible scenarios, $A(0) = 100$, $A(1) = 110$, $A(2) = 121$ and the possible prices of an asset are:

Scenario	$S(0)$	$S(1)$	$S(2)$
ω_1	90	100	112
ω_2	90	100	106
ω_3	90	80	90
ω_4	90	80	80

- Present the tree of price movements.
- Assume the risk-neutral probabilities $p_\star$, $q_\star$ and $r_\star$, corresponding to the nodes of the tree. Write the equations following from the martingale property and compute $p_\star$, $q_\star$ and $r_\star$.
- Calculate the risk-neutral probability of every scenario.

Introduction

- Generalization of previously discussed models.
- m risky primary securities, prices $S_1(t), S_2(t), \ldots, S_m(t)$, positions $x_1, x_2, \ldots, x_m$.
- Investment in the money market, risk-free, price $A(t)$, usually $A(0) = 1$, position y.
- k risky derivative securities, prices $D_1(t), D_2(t), \ldots, D_k(t)$, positions $z_1, z_2, \ldots, z_k$.
- Investor's wealth at time t:

$$V(t) = \sum_{j=1}^{m} x_j S_j(t) + yA(t) + \sum_{j=1}^{k} z_j D_j(t)$$

Assumptions (1)

Assumption (Randomness)

The future asset prices $S_1(t), S_2(t), \ldots, S_m(t)$ *and* $D_1(t), D_2(t), \ldots, D_k(t)$, *for every* $t = 1, 2, \ldots,$ *are random variables.*

Assumption (Positivity of Prices)

All prices of primary securities are positive: $S_j(t) > 0$, $A(t) > 0$ *for* $j = 1, \ldots, m$ *and* $t = 0, 1, \ldots$.

Assumptions (2)

Assumption (Divisibility)

The numbers x_j for $j = 1, \dots, m$, y and z_j for $j = 1, \dots, k$ of assets may be integer or fractional.

Assumption (Short Selling)

The numbers x_j for $j = 1, \dots, m$, y and z_j for $j = 1, \dots, k$ of assets may be negative.

Assumption (Liquidity)

The numbers x_j for $j = 1, \dots, m$, y and z_j for $j = 1, \dots, k$ of assets are not bounded.

Assumptions (3)

Assumption (Solvency)

The investor's wealth is always non-negative:

$$V(t) \geq 0$$

Assumption (Discrete Prices)

Every future price $S_j(t)$*, for* $j = 1, \dots, m$ *and* $t = 0, 1, \dots$ *and every future price* $D_j(t)$*, for* $j = 1, \dots, k$ *and* $t = 0, 1, \dots$ *can take only finitely many values.*

Portfolio vs. Strategy

- Portfolio kept between moments $t-1$ and t is a vector of the form

$$(x_1(t), x_2(t), \ldots, x_m(t), y(t), z_1(t), z_2(t), \ldots, z_k(t))$$

- Strategy is the sequence of such vectors (portfolios).

Value of the Portfolio

The value at time $t \geq 1$:

$$V(t) = \sum_{j=1}^{m} x_j(t)S_j(t) + y(t)A(t) + \sum_{j=1}^{k} z_j(t)D_j(t)$$

The value at time $t = 0$:

$$V(0) = \sum_{j=1}^{m} x_j(1)S_j(0) + y(1)A(0) + \sum_{j=1}^{k} z_j(1)D_j(0)$$

Self-Financing and Predictable Strategies

A strategy is **self-financing** if the portfolio constructed at any moment is fully financed by the investor's wealth:

$$\sum_{j=1}^{m} x_j(t+1)S_j(t) + y(t+1)A(t) + \sum_{j=1}^{k} z_j(t+1)D_j(t) = V(t)$$
$$= \sum_{j=1}^{m} x_j(t)S_j(t) + y(t)A(t) + \sum_{j=1}^{k} z_j(t)D_j(t)$$

A strategy is **predictable** if the portfolio constructed at any moment t depends only on the market scenarios up to the moment t.

Admissible Strategy

A strategy is admissible if it is self-financing, predictable and for each $t = 0, 1, \ldots$ we have

$$V(t) \geq 0 \text{ with probability } 1$$

Assumptions (4)

Assumption (No-Arbitrage Principle)

The arbitrage is impossible. In other words, there is no admissible strategy such that $V(0) = 0$, for which $V(t) > 0$ with probability $p > 0$ for some $t = 1, 2, \ldots$.

Fundamental Theorem of Asset Pricing

Theorem

The No-Arbitrage Principle is equivalent to the existence of probability $P_\star$ on the set of scenarios Ω such that $P_\star(\omega) > 0$ for each scenario $\omega \in \Omega$ and the discounted prices of primary and derivative securities $\tilde{S}_j(t) = S_j(t)/A(t)$ and $\tilde{D}_j(t) = D_j(t)/A(t)$ satisfy

$$E_\star(\tilde{S}_j(t+1)|S(t)) = \tilde{S}_j(t)$$

for $j = 1, \dots, m$ and

$$E_\star(\tilde{D}_j(t+1)|S(t)) = \tilde{D}_j(t)$$

for $j = 1, \dots, k$, for every $t = 0, 1, \dots,$ where $E_\star(\cdot|S(t))$ denotes the conditional expectation with respect to probability $P_\star$ computed once the stock price $S(t)$ becomes known at time t.

Proof - omitted.

Example (Capinski, Zastawniak (2003), p.89)

There are four possible scenarios, $A(0) = 100$, $A(1) = 110$, $A(2) = 121$ and the possible prices of an asset are:

Scenario	$S(0)$	$S(1)$	$S(2)$
ω_1	90	100	112
ω_2	90	100	106
ω_3	90	80	90
ω_4	90	80	80

- Consider a call option with strike price $C = 85$ at time 2. Calculate the option prices at times 0 and 1.
- Consider a put option with strike price $P = 85$ at time 2. Calculate the option prices at times 0 and 1.

Measuring the risk

- One wants to compare the portfolios with same expected return but different spread of the possible values.
- The measure should cover two aspects: the distances to a reference value ($E(K)$) and the probabilities.
- Best measure: standard deviation σ_K (it changes linearly with the rates of return, while the variance changes with square of the rates).

Example

Compute the expected return, variance and standard deviation of the following three investment strategies, where the return depends on the market scenario:

Scenario	Probability	K_1	K_2	K_3
ω_1	0.3	-1%	-2%	-5%
ω_2	0.7	2%	4%	10%

Example (Capinski, Zastawniak (2003), p.93)

Compute the expected return and risk of the following three investment strategies, where the return depends on the market scenario:

Scenario	Probability	K_1	K_2	K_3
ω_1	0.25	12%	11%	2%
ω_2	0.75	12%	13%	22%

Example

In the case of one-period rates of return the ordinary rates of return are better than the logarithmic ones - compare the variances of both in the following situation:

Scenario	Probability	K_1	K_2
ω_1	0.5	12%	2%
ω_2	0.5	8%	-2%

Example (Capinski, Zastawniak (2003), p.94)

Assume we want to invest 1000€ either only in one of the shares, or equally in both. Calculate the risk in all the three situations.

Scenario	Probability	K_1	K_2
ω_1	0.5	10%	−5%
ω_2	0.5	−5%	10%

Weights

- Percentages of the values of securities:

$$w_j = \frac{x_j S_j(0)}{V(0)}$$

- They can change even if the numbers x_j do not.
- They always sum up to 1.
- If short selling is allowed, one of them can be negative (and thus the other one greater than 1). Otherwise, both are positive.

Example (Capinski, Zastawniak (2003), pp.94-95)

Assume that the prices are as follows:

Price	$t = 0$	$t = 1$
$S_1(t)$	30	35
$S_2(t)$	40	39
$A(t)$	10	11

- Calculate the initial and final values of the portfolio and the weights, if $x_1 = 20$ and $x_2 = 10$.
- Calculate the initial and final numbers of stocks, the final value of the portfolio and the final weights, if $V(0) = 1000$, $w_1 = 1.2$, $w_2 = -0.2$ and no security deposit is necessary.
- Repeat the calculations from the previous point, if the security deposit of 50% of the sum raised by shorting stock 2 is required (borrow the money).

Return on a Portfolio

Proposition

The return on a portfolio is described with the following formula:

$$K_V = w_1 K_1 + w_2 K_2$$

Proof (sketch).

- Initial value:

$$V(0) = x_1 S_1(0) + x_2 S_2(0)$$

- Final value:

$$V(1) = x_1 S_1(0)(1 + K_1) + x_2 S_2(0)(1 + K_2)$$

Logarithmic Return on a Portfolio

Proposition

The logarithmic return on a portfolio is described with the following formula:

$$k_V = \ln\left(w_1 e^{k_1} + w_2 e^{k_2}\right)$$

Proof (sketch). Use the formula for ordinary return.

Expected Return on a Portfolio

Proposition

The expected return on a portfolio is described with the following formula:

$$\mu_V = E(K_V) = w_1 E(K_1) + w_2 E(K_2)$$

Proof (sketch). Use the formula for ordinary return and the linearity of expectation.

Variance of a Portfolio

Proposition

The variance of a portfolio is described with the following formula:

$$\sigma_V^2 = Var(K_V) = w_1^2 Var(K_1) + w_2^2 Var(K_2) + 2w_1 w_2 Cov(K_1, K_2)$$

$$= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \sigma_1\sigma_2\rho_{12} = \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

Proof (sketch).

- Substitute $K_V = w_1 K_1 + w_2 K_2$ to $Var(K_V) = E(K_V^2) - E^2(K_V)$.
- Use the formula for the correlation coefficient:

$$\rho_{12} = \frac{Cov(K_1, K_2)}{\sigma_1\sigma_2}$$

Example (Capinski, Zastawniak (2003), pp.99-100)

There are three possible scenarios and the possible prices of assets are:

Scenario	Probability	K_1	K_2
ω_1	0.4	-10%	20%
ω_2	0.2	0%	20%
ω_3	0.4	20%	10%

Calculate the expected returns and risks of both assets and of the portfolio in which:

- $w_1 = 0.4$ and $w_2 = 0.6$,
- $w_1 = 0.8$ and $w_2 = 0.2$,
- $w_1 = -0.5$ and $w_2 = 1.5$.

Variance of a Portfolio

Proposition

If short sales are not allowed, then

$$\sigma_V^2 \leq \max\{\sigma_1^2, \sigma_2^2\}$$

Proof (sketch).

- Observe that

$$w_1\sigma_1 + w_2\sigma_2 \leq (w_1 + w_2)\max\{\sigma_1, \sigma_2\} = \max\{\sigma_1, \sigma_2\}$$

- We have $\rho_{12} \leq 1$
- Thus

$$\sigma_V^2 \leq (w_1\sigma_1 + w_2\sigma_2)^2 \leq (\max\{\sigma_1, \sigma_2\})^2$$

Portfolio with Variance 0

Proposition

- *If short sales are not allowed, then the portfolio with variance equal to 0 exists only if* $\rho = -1$.
- *If short sales are allowed, then the portfolio with variance equal to 0 exists also if* $\rho = 1$ *and* $\sigma_1 \neq \sigma_2$.

Proof (sketch).

- Part 1 and part 2 if $w_1, w_2 \geq 0$:

$$\sigma_V^2 > (w_1\sigma_1 - w_2\sigma_2)^2 \geq 0$$

- Part 2 if $w_1 < 0$ or $w_2 < 0$:

$$\sigma_V^2 > (w_1\sigma_1 + w_2\sigma_2)^2 \geq 0$$

- Calculate the values of w_1 and w_2.

Portfolio with Variance 0 - illustration

Assume that $w_2 = s$ and $w_1 = 1 - s$. If $\rho = -1$, then:

$$\sigma_V = |(1-s)\sigma_1 - s\sigma_2|$$
$$\mu_V = (1-s)\mu_1 + s\mu_2$$

If $\rho = 1$, then:

$$\sigma_V = |(1-s)\sigma_1 + s\sigma_2|$$
$$\mu_V = (1-s)\mu_1 + s\mu_2$$

(FIGURES)

Example (Capinski, Zastawniak (2003), p.102)

Assume there are only two scenarios ω_1 and ω_2 and two securities with returns K_1 and K_2. Show that:

- $K_1 = aK_2 + b$ for some constants a and b,
- $\rho_{12} \in \{-1, 1\}$.

Portfolio with Minimum Variance

Proposition

The weight of the security 2 minimizing the variance is equal to

$$w_2 = s_0 = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$$

If short sales are not allowed, then $s_{min} = 0$ *if* $s_0 < 0$ *and* $s_{min} = 1$ *if* $s_0 > 1$.

Proof (sketch).

- Compute the derivative of σ_V^2 with respect to s and equate it to 0. Verify the sign of second derivative.
- If $s_0 \notin \langle 0, 1 \rangle$, then σ_V^2 is monotone on $\langle 0, 1 \rangle$.

Portfolio with Minimum Variance

Corollary

Let

$$\rho^\star = \frac{\min\{\sigma_1, \sigma_2\}}{\max\{\sigma_1, \sigma_2\}}$$

If $-1 \leq \rho_{12} < \rho^\star$*, then there exists a portfolio without short selling such that* $\sigma_V < \min\{\sigma_1, \sigma_2\}$*. If* $\rho_{12} = \rho^\star$*, then* $\sigma_V \geq \min\{\sigma_1, \sigma_2\}$ *for every portfolio. If* $\rho^\star < \rho_{12} \leq 1$*, then* $\sigma_V \geq \min\{\sigma_1, \sigma_2\}$ *for every portfolio without short selling, but there is a portfolio with short selling such that* $\sigma_V < \min\{\sigma_1, \sigma_2\}$*.*

Proof (sketch). Observe that s_0 is non-increasing with ρ_{12} (derivative is non-positive). Check the values on the bounds of every interval, observe that it involves short selling or not, calculate the minimum variance. (FIGURES, SIMULATION)

Example (Capinski, Zastawniak (2003), pp.99-100)

There are three possible scenarios and the possible prices of assets are:

Scenario	Probability	K_1	K_2
ω_1	0.4	-10%	20%
ω_2	0.2	0%	20%
ω_3	0.4	20%	10%

Find the portfolio with the smallest variance, if

- short selling is allowed,
- short selling is not allowed.

Portfolio with Risk-Free Asset

Let $\mu_2 = r_F$ be the rate of return on a risk-free security, and so $\sigma_2 = 0$. Then we have:

$$\sigma_V^2 = w_1^2 \sigma_1^2$$

and thus

$$\begin{aligned} \sigma_V &= |w_1| \sigma_1 \\ \mu_V &= (1 - s)\mu_1 + s\mu_2 \end{aligned}$$

(FIGURE)

Notation

- Weights vector:

$$w = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix}$$

- Unity vector:

$$u = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$$

- Vector of expected returns:

$$m = \begin{bmatrix} \mu_1 & \dots & \mu_n \end{bmatrix}$$

- Covariance matrix:

$$C = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \dots & \dots & \dots \\ c_{n1} & \dots & c_{nn} \end{bmatrix}$$

Expected Return and Risk

Proposition

The expected return and variance are given with the formulae:

$$\mu_V = mw^T$$

$$\sigma_V^2 = wCw^T$$

Proof (sketch). Use the definition of product of matrices.

Example (Capinski, Zastawniak (2003), pp.99-100)

There are three possible scenarios and the possible prices of assets are:

Scenario	Probability	K_1	K_2	K_3
ω_1	0.4	-10%	20%	12%
ω_2	0.2	0%	20%	15%
ω_3	0.4	20%	10%	8%

Calculate the expected returns and risks of all the assets and of the portfolio in which:

- $w_1 = 0.4$ and $w_2 = 0.3$,
- $w_1 = 0.8$ and $w_2 = 0.3$,
- $w_1 = -0.5$ and $w_2 = 0.8$.

Minimum Variance Portfolio

Proposition

The weights minimizing the variance are equal to

$$w = \frac{uC^{-1}}{uC^{-1}u^T}$$

Proof (sketch).

- Equate to 0 the gradient of Lagrange function (constraint: $uw^T = 1$). Calculate w, substitute to the constraint, calculate multiplier.
- Check the first order necessary condition (C is positive definite, thus σ_V^2 is convex).

Minimum Variance Line

Proposition

Let

$$D_0 = \begin{vmatrix} uC^{-1}u^T & uC^{-1}m^T \\ mC^{-1}u^T & mC^{-1}m^T \end{vmatrix}, D_1 = \begin{vmatrix} 1 & uC^{-1}m^T \\ \mu & mC^{-1}m^T \end{vmatrix}, D_2 = \begin{vmatrix} uC^{-1}u^T & 1 \\ mC^{-1}u^T & \mu \end{vmatrix}$$

The weights minimizing the variance under the condition $mw^T = \mu$ *are equal to*

$$w = \frac{D_1 uC^{-1} + D_2 mC^{-1}}{D_0}$$

Proof (sketch).

- Equate to 0 the gradient of Lagrange function (constraints: $uw^T = 1$, $mw^T = \mu$). Calculate w, substitute to the constraints, calculate multipliers.
- Check the first order necessary condition (C is positive definite, thus σ_V^2 is convex).

Example (Capinski, Zastawniak (2003), p.111)

Assume that $\mu_1 = 0.2$, $\mu_2 = 0.13$, $\mu_3 = 0.17$, $\sigma_1 = 0.25$, $\sigma_2 = 0.28$, $\sigma_3 = 0.20$, $\rho_{12} = 0.3$, $\rho_{13} = 0.15$ and $\rho_{23} = 0$. Find the portfolio with the smallest variance, if

- there are no additional constraints,
- the expected return has to be equal to 20%.

Markowitz Bullet

By substituting w with

$$w = \frac{D_1 u C^{-1} + D_2 m C^{-1}}{D_0}$$

in the formulae for expected return and variance, we obtain the parametric equation of a parabola. It is so-called "Markowitz bullet".
(FIGURE)

Efficient Frontier

Definition

We say that portfolio with parameters (σ_1, μ_1) dominates portfolio with parameters (σ_2, μ_2) if

$$\mu_1 \geq \mu_2 \wedge \sigma_1 \leq \sigma_2 \wedge (\sigma_1, \mu_1) \neq (\sigma_2, \mu_2)$$

The set of all non-dominated portfolios is called the ***efficient frontier****.*

(FIGURE)

Minimum Variance Line

Proposition

Assume that two portfolios w^1 and w^2, $w^1 \neq w^2$, belong to the minimum variance line. Then the portfolio w^3 belongs to the minimum variance line if and only if for some $c \in R$

$$w^3 = cw^1 + (1-c)w^2$$

Proof (sketch). We already know that the optimal portfolio is of the form $w = a\mu_V + b$, for certain vectors a, b. Thus we have some $\mu_V^1 \neq \mu_V^2$. As $c\mu_V^1 + (1-c)\mu_V^2$ is a real line (all the possible returns are reached), the given equation covers whole minimum variance line.

Effective Frontier

Proposition

Every portfolio on the effective frontier, except the minimum variance portfolio, satisfies the condition

$$\gamma w C = m - \mu u$$

for some real numbers $\gamma > 0$ *and* μ.

Effective Frontier

Proof (sketch). Let the parameters of w be (σ_V, μ_V). Draw the tangent to the efficient frontier at point (σ_V, μ_V). It intersects vertical line at some μ and the slope is

$$g(w) = \frac{mw^T - \mu}{\sqrt{wCw^T}}$$

This slope is maximum over all lines with intersection μ, under the constraint $uw^T = 1$. Lagrange function is $g(w) + \lambda(uw^T - 1)$. Equating gradient to 0 gives the desired equation with $\gamma = g(w)/\sigma_V$ and $\mu = -\lambda\sigma_V$.

Example (Capinski, Zastawniak (2003), p.117)

Assume that $\mu_1 = 0.2$, $\mu_2 = 0.13$, $\mu_3 = 0.17$, $\sigma_1 = 0.25$, $\sigma_2 = 0.28$, $\sigma_3 = 0.20$, $\rho_{12} = 0.3$, $\rho_{13} = 0.15$ and $\rho_{23} = 0$. Consider the portfolio on the efficient frontier with $\mu_V = 21\%$. Compute the values of γ and μ such that $\gamma w C = m - \mu u$.

Introduction

- Classical (Markowitz) approach is numerically complex and unstable (e.g. C^{-1}).
- Every investor has the same data, so the same effective frontier (so-called capital market line).
- There is risk-free security with parameters $(0, r_F)$. Tangent to Markowitz bullet containing this point is an efficient frontier.

(FIGURE)

Market Portfolio

- The tangency point - the market portfolio (σ_M, μ_M). In practice approximated by a suitable stock exchange index.
- Every investor has the same capital market line, thus everyone has the same proportions of ridky securities.
- We have

$$\frac{\mu - r_F}{\sigma} = \frac{\mu_M - r_F}{\sigma_M}$$

  and so

$$\mu = r_F + \frac{\mu_M - r_F}{\sigma_M}\sigma$$

  (last term is called the **risk premium**).

Example (Capinski, Zastawniak (2003), p.119)

Assume that $\mu_1 = 0.1$, $\mu_2 = 0.15$, $\mu_3 = 0.20$, $\sigma_1 = 0.28$, $\sigma_2 = 0.24$, $\sigma_3 = 0.25$, $\rho_{12} = -0.1$, $\rho_{13} = 0.25$ and $\rho_{23} = 0.2$. Assume also that $r_F = 5\%$. Using the relation $\gamma w C = m - \mu u$, find the weights of the risky assets (start with finding γw).

Line of Best Fit

- For every portfolio with return K_V we can plot the points (K_M, K_V) for all the scenarios.
- We can find the line of best fit, $K_V = \alpha_V + \beta_V K_M$, minimizing the sum of squares of the residuals.
- Formulae:

$$\beta_V = \frac{Cov(K_V, K_M)}{\sigma_M^2}, \alpha_V = \mu_V - \beta_V \mu_M$$

Example (Capinski, Zastawniak (2003), p.121)

There are four possible scenarios and the possible rates of return are:

Scenario	Probability	K_V	K_M
ω_1	0.1	-5%	10%
ω_2	0.3	0%	14%
ω_3	0.4	2%	12%
ω_4	0.2	4%	16%

Find the line of best fit.

Beta Factor

- β_V is called the **beta factor**. It measures the reaction of K_V on the change of K_M.
- On the other hand we have

$$\sigma_V^2 = Var(\varepsilon_V) + \beta_V^2 \sigma_M^2$$

where

$$\varepsilon_V = K_V - (\alpha_V + \beta_V K_M)$$

- $Var(\varepsilon_V)$ is the **diversifiable risk**, $\beta_V^2 \sigma_M^2$ is the **systematic (undiversifiable) risk**. The market portfolio involves only the latter.

Example (Capinski, Zastawniak (2003), p.122)

Show that the β factor of the portfolio consisting of n securities equals to

$$\beta_V = w_1\beta_1 + \ldots + w_n\beta_n$$

First prove and then use the fact that $Cov(a_1X_1 + a_2X_2, Y) = a_1Cov(X_1, Y) + a_2Cov(X_2, Y)$.

Security Market Line

Theorem

The expected return on a portfolio is a linear function of the beta factor:

$$\mu_V = r_F + (\mu_M - r_F)\beta_V$$

The last term $(\mu_M - r_F)\beta_V$ is the **risk premium**. This time it works for all the portfolios, not only for those belonging to the capital market line.

Security Market Line

Proof (sketch). We have

$$\gamma w_M C = m - \mu u$$

Thus

$$\beta_V = \frac{Cov(K_V, K_M)}{\sigma_M^2} = \frac{w_M C w_V^T}{w_M C w_M^T} = \frac{\frac{1}{\gamma}(m - \mu u) w_V^T}{\frac{1}{\gamma}(m - \mu u) w_M^T} = \frac{\mu_V - \mu}{\mu_M - \mu}$$

Consider risk-free security. Substituting (β_V, μ_V) with $(\beta_F = 0, r_F)$, we obtain $\mu = r_F$ and finally

$$\beta_V = \frac{\mu_V - r_F}{\mu_M - r_F}$$

Example (Capinski, Zastawniak (2003), p.123)

We are going to show that the characteristic lines of all securities intersect at a common point. We have $\mu_V = r_F + (\mu_M - r_F)\beta_V$, so $\alpha_V = \mu_V - \beta_V \mu_M = r_F - r_F \beta_V$. Thus the line $K_V = \alpha_V + \beta_V K_M$ contains the point $(K_M, K_V) = (r_F, r_F)$.

Self-Adjustments

CAPM describes the equilibrium.

- If $\mu_V > r_F + (\mu_M - r_F)\beta_V$ (the security is underpriced), then the investors will want to increase the position, the demand and so the price will increase and the expected return will decrease.
- If $\mu_V < r_F + (\mu_M - r_F)\beta_V$ (the security is overpriced), then the investors will want to decrease the position, the demand and so the price will decrease and the expected return will increase.

Thank You

THANK YOU :-)

Financial Mathematics 2022

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