Kalman filter

Let $(\theta_i)_{i\geq 1}$ in $(\xi_i)_{i\geq 1}$ be two sequences of random vectors of dimensions k and l satisfying the equations

$$\begin{aligned}\theta_{n+1} &= \mathbf{a}_{1,n} + \mathbf{A}_{1,n}\theta_n + \mathbf{B}_{1,n}\eta_{1,n+1} + \mathbf{C}_{1,n}\eta_{2,n+1}\\ \xi_{n+1} &= \mathbf{a}_{2,n} + \mathbf{A}_{2,n}\theta_n + \mathbf{B}_{2,n}\eta_{1,n+1} + \mathbf{C}_{2,n}\eta_{2,n+1}\,, 1\end{aligned}$$

where $\mathbf{a}_{1,n}$, $\mathbf{a}_{2,n}$ are fixed known vectors of dimensions k and l, $\mathbf{A}_{1,n}$, $\mathbf{A}_{2,n}$ are matrices of dimensions $k\times k$ and $l\times k$, $\mathbf{B}_{1,n}$, $\mathbf{B}_{2,n}$ are matrices of dimensions $k\times k$ and $l\times k$, $\mathbf{C}_{1,n}$ and $\mathbf{C}_{2,n}$ matrices of dimensions $k\times l$ and $l\times l$ and $\eta_{1,n}$ and $\eta_{2,n}$ i.i.d. sequences of random vectors $N(\mathbf{0},\mathbf{I}_k)$ in $N(\mathbf{0},\mathbf{I}_l)$ independent of each other.

Assume that (θ_0,ξ_0) is a multivariate normal vector independent of $(\eta_{i,j})_{j\geq 1}, i = 1,2,\ldots$ such that

$$\begin{pmatrix}\theta_0\\ \xi_0\end{pmatrix} \sim N\left(\begin{pmatrix}\mu_0\\ \nu_0\end{pmatrix}, \begin{pmatrix}\mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12}\\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22}\end{pmatrix}\right).$$

a. Argue that the conditional distribution of θ_n, $(\theta_n,\xi_{n+1})^T$ and $(\theta_{n+1},\xi_{n+1})^T$ given $(\xi_1,\xi_2,\ldots,\xi_n)$ is multivariate normal. You do not need to give the distribution explicitely.

b. Denote

$$\mu_n = E(\theta_n|\xi_1,\xi_2,\ldots,\xi_n) \qquad \text{in} \qquad \gamma_n = \mathrm{var}\{\theta_n|\xi_1,\xi_2,\ldots,\xi_n\}\,.$$

We know that μ_n is the best forecast of θ_n based on $\xi_1,\xi_2,\ldots,\xi_n$. Show that the following recursive formulae are valid. Always assume that all the matrices one needs to invert are invertible.

$$\begin{aligned}\mu_{n+1} &= [\mathbf{a}_{1,n} + \mathbf{A}_{1,n}\mu_n] + [\mathbf{B}_{1,n}\mathbf{B}_{2,n}^T + \mathbf{C}_{1,n}\mathbf{C}_{2,n}^T + \mathbf{A}_{1,n}\gamma_n\mathbf{A}_{2,n}^T]\\ &\times [\mathbf{B}_{2,n}\mathbf{B}_{2,n}^T + \mathbf{C}_{2,n}\mathbf{C}_{2,n}^T + \mathbf{A}_{2,n}\gamma_n\mathbf{A}_{2,n}^T]^{-1}[\xi_{n+1} - \mathbf{a}_{2,n} - \mathbf{A}_{2,n}\mu_n]\end{aligned}$$

$$\begin{aligned}&\gamma_{n+1} =\\ &= [\mathbf{A}_{1,n}\gamma_n\mathbf{A}_{1,n}^T + \mathbf{B}_{1,n}\mathbf{B}_{1,n}^T + \mathbf{C}_{1,n}\mathbf{C}_{1,n}^T] -\\ &- [\mathbf{B}_{1,n}\mathbf{B}_{2,n}^T + \mathbf{C}_{1,n}\mathbf{C}_{2,n}^T + \mathbf{A}_{1,n}\gamma_n\mathbf{A}_{2,n}^T][\mathbf{B}_{2,n}\mathbf{B}_{2,n}^T + \mathbf{C}_{2,n}\mathbf{C}_{2,n}^T + \mathbf{A}_{2,n}\gamma_n\mathbf{A}_{2,n}^T]^{-1}\\ &\times [\mathbf{B}_{1,n}\mathbf{B}_{2,n}^T + \mathbf{C}_{1,n}\mathbf{C}_{2,n}^T + \mathbf{A}_{1,n}\gamma_n\mathbf{A}_{2,n}^T]^T\,.\end{aligned}$$

Remark: These recursion formulae are known as the Kalman-Bucy filter.

Hints:

(i) *If* $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})^T$ *is a random vector, such that the conditional distribution of* $\mathbf{Z}$ *given* $\mathbf{W}$ *is multivariate normal with parameters*

$$\begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}$$

then the conditional distribution of $\mathbf{Y}$ *given* $(\mathbf{X}, \mathbf{W})$ *is multivariate normal with parameters*

$$\begin{aligned} E(\mathbf{Y}|\mathbf{X}, \mathbf{W}) &= E(\mathbf{Y}|\mathbf{W}) + \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}(\mathbf{X} - E(\mathbf{X}|\mathbf{W})) \\ \mathrm{var}\{\mathbf{Y}|\mathbf{X}, \mathbf{W}\} &= \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\,. \end{aligned}$$

(ii) *Let*

$$E(\theta_{n+1}|\xi_1, \xi_2, \ldots, \xi_{n+1}) = E(\theta_{n+1}|(\xi_1, \xi_2, \ldots, \xi_n), \xi_{n+1})$$

Use (i).

c. Why is the word filter used?