Contingency tables

A 2×2 contingency table is of the form

n_{11}	n_{12}
n_{21}	n_{22}

We assume that n_{ij} is the count of units in a sample that posses property i and property j. Assume that the sample is simple random *with* replacement of size $n = n_{11} + n_{12} + n_{21} + n_{22}$. If N_{ij} is the random number of units with property i and property j in the sample then

$$\begin{aligned} & P(N_{11} = n_{11}, N_{12} = n_{12}, N_{21} = n_{21}, N_{22} = n_{22}) = \\ = \quad & \frac{n!}{n_{11}!n_{12}!n_{21}!n_{22}!} p_{11}^{n_{11}} p_{12}^{n_{12}} p_{21}^{n_{21}} p_{22}^{n_{22}} \end{aligned}$$

for parameters $p_{11} + p_{21} + p_{12} + p_{22} = 1$.

a. Find the maximum likelihood estimates for the parameters if the parameter space is

$$\Omega = \left\{(p_{11}, p_{12}, p_{21}, p_{22}) : p_{ij} \geq 0, \sum_{ij} p_{ij} = 1\right\}.$$

What is the dimension of this parameter sapce?

b. Find the maximum likelihood estimates for the parameters if the parameter space is restricted to

$$\Omega_0 = \left\{(p_{11}, p_{12}, p_{21}, p_{22}) : p_{ij} \geq 0, \sum_{ij} p_{ij} = 1, p_{11}p_{22} - p_{12}p_{21} = 0\right\}.$$

c. Find the likelihood ratio statistic for the testing problem $H_0 : \theta \in \Omega_0$ versus $H_1 : \theta \in \Omega \backslash \Omega_0$. When would you reject H_0 if the probability of Type I error is to be α?

d. The usual χ^2-test for contingency tables is given by

$$\chi^2 = \sum_{i,j} \frac{(n_{ij} - \hat{n}_{ij})^2}{\hat{n}_{ij}}$$

where $\hat{n}_{ij} = n\hat{p}_{ij}$ and $\hat{p}_{ij}$ is the estimate from b. Show that the usual χ^2-test and the Wilks's λ are approximations of each other.

Hint: Expand the logarithms into Taylor series up to the second degree terms.

e. Would a similar argument hold for contingency tables of higher dimensions?