FACULTY OF MATHEMATICS AND PHYSICS

Department of Mathematics

Financial mathematics 2

Written examination

August 30th, 2024

Name and surname: ______________________ Student number: | | | | | | | | |

Instructions

Read carefully the problems before starting to solve them. There are 4 problems. You have two hours.

Problem	a.	b.	c.	d.	Total
1.			•	•	
2.			•	•	
3.			•	•	
4.					
Total					

1. (25) A continuous adapted process X satisfies the stochastic differential equation

$$dX_t = dB_t + X_t dt\,.$$

Given the initial condition $X_0 = x_0$, the equation has a unique strong solution.

a. (10) Let $F : \mathbb{R} \to \mathbb{R}$ be a differentiable function for which

$$F'(x) = e^{-x^2}.$$

Show that $Y_t = F(X_t)$ is a martingale.

Solution: the function F is twice continuously differentiable with

$$F''(x) = -2xe^{-x^2}\,.$$

By Itô formula we have ($y_0 := F(x_0)$)

$$\begin{aligned} Y_t &= y_0 + \int_0^t F'(X_s)dX_s + \frac{1}{2}\int_0^t F''(X_s)d\langle X\rangle_s \\ &= y_0 + \int_0^t e^{-X_s^2}(dB_s + X_s ds) - \frac{1}{2}\int_0^t 2X_s e^{-X_s^2} ds \\ &= y_0 + \int_0^t e^{-X_s^2} dB_s\,. \end{aligned}$$

The process Y is the stochastic integral of a bounded integrand with respect to Brownian motion and therefore a martingale.

b. (15) Let $x_0 > 0$ and $T_0 = \inf\{t \geq 0\colon X_t = 0\}$. Express $P(T_0 = \infty)$ using the function $F(x) = \int_0^x e^{-s^2} ds$.

Hint: consider $T_{0,a} = \inf\{t \geq 0\colon X_t \in \{0, a\}\}$ first for $a > x_0$. Assume as known that $P(T_{0,a} < \infty) = 1$ for all $a > x_0$, and that $\int_0^\infty e^{-s^2} ds = \frac{1}{2}\sqrt{\pi}$.

Solution: the function $F(x)$ is strictly increasing. Since $F(X_t)$ is a martingale, we have that

$$P(T_a < T_0) = P\left(F(X_{T_{0,a}}) = F(a)\right) = \frac{F(x_0)}{F(a) - F(0)}\,.$$

By continuity we have

$$\{T_0 < \infty\} = \cup_{n > x_0}\{T_0 < T_n\}\,.$$

The events in the union are increasing, hence

$$P(T_0 < \infty) = \lim_{n\to\infty} P(T_0 < T_n)\,.$$

It follows that

$$P(T_0 = \infty) = \frac{2F(x_0)}{\sqrt{\pi}}\,.$$

2. (25) A continuous adapted process X satisfies the stochastic differential equation

$$dX_t = \left(\frac{2X_t}{1+t} - (1+t)^2\right) dt + (1+t)^2 dB_t\,,$$

where B is standard Brownian motion. The initial condition is $X_0 = x_0 \in \mathbb{R}$.

a. (15) Let $X_t = (1+t)^2 Y_t$. Find a stochastic differential equation for Y.

Solution: the rule for stochastic derivatives of products gives

$$dX_t = 2(1+t)Y_t dt + (1+t)^2 dY_t\,.$$

We rewrite the SDE for X as

$$dX_t = \left(2Y_t(1+t) - (1+t)^2\right) dt + (1+t)^2 dB_t\,.$$

Equating both expressions for dX_t we get, after some cancellations,

$$dY_t = -dt + dB_t\,.$$

b. (10) Find the solution X of the original stochastic differential equation. Compute $E(X_t)$ and $\mathrm{var}(X_t)$.

Solution: since $Y_0 = x_0$, we get

$$Y_t = x_0 - t + B_t\,,$$

and hence

$$X_t = (1+t)^2\left(x_0 - t + B_t\right)\,.$$

We have

$$E(X_t) = (1+t)^2(x_0 - t) \qquad \textit{and} \qquad \mathrm{var}(X_t) = t(1+t)^4\,.$$

3. (25) Let B be standard Brownian motion. Denote by $\chi_{(a,b]}$ the characteristic function of the interval $(a,b]$.

a. (10) For $0 < a < b \leq T$ define

$$M_t = 2\int_0^t \chi_{(0,a]}(s)B_s dB_s \qquad \text{and} \qquad N_t = 2\int_0^t \chi_{(a,b]}(s)B_s dB_s\,.$$

Compute $M_T N_T$ and $\langle M, N\rangle_t$.

Solution: we know that

$$B_t^2 = t + 2\int_0^t B_s dB_s\,.$$

From this representation we have that

$$M_T = B_a^2 - a \qquad \textit{and} \qquad N_T = B_b^2 - B_a^2 - (b-a)\,.$$

From the definition it follows that $\langle M, N\rangle_t = 0$.

b. (15) Let $0 < a < b \leq T$. Express the integrand H, such that

$$B_a^2 B_b^2 = E(B_a^2 B_b^2) + \int_0^T H_s dB_s\,,$$

with M, N, t, B and T.

Hint: you may use without proof that

$$E(B_T^4 \mid \mathcal{F}_t) = 3(T-t)^2 + 6(T-t)B_t^2 + B_t^4\,.$$

Solution: by the stochastic partial integration rule we have

$$M_T N_T = \int_0^T M_t dN_t + \int_0^T N_t dM_t\,,$$

or in other words

$$(B_a^2 - a)(B_b^2 - B_a^2 - (b-a)) = 2\int_0^T M_t \chi_{(a,b]}(t)B_t dB_t + 2\int_0^T N_t \chi_{(0,a]}(t)B_t dB_t\,.$$

Let

$$K_t = 2M_t\chi_{(a,b]}(t)B_t + 2N_t\chi_{(0,a]}(t)B_t\,.$$

We rewrite

$$a(b-a) - a(B_b^2 - B_a^2) - (b-a)B_a^2 + B_a^2B_b^2 - B_a^4 = \int_0^T K_t dB_t\,.$$

Furthermore, we have

$$a(B_b^2 - B_a^2) - a(b-a) = \int_0^T 2a\chi_{(a,b]}(t)B_t dB_t$$

and

$$(b-a)B_a^2 - (b-a)a = \int_0^T 2(b-a)\chi_{(0,a]}(t)B_t dB_t\,.$$

We need the representation of B_a^4. *Since*

$$E(B_T^4 \mid \mathcal{F}_t) = F(B_t, t)$$

with F *twice continuously differentiable, we have*

$$B_a^4 = E(B_a^4) + \int_0^T \chi_{(0,a]}(t)\left(12(T-t)B_t + 4B_t^3\right) dB_t\,.$$

The integrand follows by linearity of stochastic integrals.

4. (25) Assume the Black-Scholes model for the share price process. Assume a constant interest rate r, and denote the maturity by T.

a. (5) Let the option be given by $V_T = S_T$. What is V_t? What are the components (H_t^0, H_t) of the hedging portfolio?

Solution: under Q the share price and the option value are martingales. It follows that

$$\begin{aligned} \tilde{V}_t &= E_Q\left(\tilde{V}_T|\mathcal{F}_t\right) \\ &= E_Q\left(\tilde{S}_T|\mathcal{F}_t\right) \\ &= \tilde{S}_t\,. \end{aligned}$$

As a consequence $(H_t^0, H_t) = (0,1)$ for $0 \le t \le T$.

b. (5) For a European call with strike price k we have for $t < T$

$$V_t = S_t\Phi(d_1) - e^{-r(T-t)}k\Phi(d_2)$$

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$$d_1 = \frac{\log(S_t/k) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}\,.$$

Further, we have

$$H_t = \Phi(d_1)\,.$$

Use the formula

$$(k-x)_+ = (x-k)_+ - (x-k)$$

to derive the price and the components of the hedging portfolio for the European put option $V_T^p = (k - S_T)_+$.

Solution: we have

$$\begin{aligned} V_t &= e^{-r(T-t)}E_Q\left[V_T|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}E_Q\left[(k-S_T)_+|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}E_Q\left[(S_T-k)_+ - (S_T-k)|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}E_Q\left[(S_T-k)_+ - (S_T-k)|\mathcal{F}_t\right] \\ &= S_t\Phi(d_1) - e^{-r(T-t)}k\Phi(d_1) - \left(S_t - ke^{-r(T-t)}\right) \\ &= -S_t\Phi(-d_1) + ke^{-r(T-t)}\Phi(-d_2)\,. \end{aligned}$$

By linearity we have $H_t = \Phi(d_1) - 1 = -\Phi(-d_1)$.

c. (10) The *Butterfly option* is defined as

$$V_T = f(S_T)\,,$$

where for $0 < a < k < b$

$$f(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{k-a} & a \le x < k \\ \frac{b-x}{b-k} & k \le x < b \\ 0 & x > b \end{cases}$$

Compute V_t.

Hint: look at the functions

$$\lambda\left[(x-k)_+ - (x-b)_+\right] \quad \textit{and} \quad \mu\left[(k-x)_+ - (a-x)_+\right]$$

with suitable λ *and* μ*.*

Solution: let $V_t^{n,k}$ *be the price of European call with strike* k*, and let* $V_t^{p,k}$ *be the price of a European put with strike price* k*. We have*

$$f(x) = 1 - \frac{1}{b-k}\left[(x-k)_+ - (x-b)_+\right] - \frac{1}{k-a}\left[(k-x)_+ - (a-x)_+\right]\,.$$

The price of the Butterfly option *equals*

$$V_t = 1 - \frac{1}{b-k}\left[V_t^{n,k} - V_t^{n,b}\right] - \frac{1}{k-a}\left[V_t^{p,k} - V_t^{p,a}\right]\,.$$

d. (5) Find H_t for the *Butterfly option.*

Solution: for the option that always pays 1*, we have* $(H_t^0, H_t) = (e^{-r(T-t)}, 0)$*. The overall hedging portfolio for the* Butterfly option *is then the linear combination of all the hedging portfolios.*