FACULTY OF MATHEMATICS AND PHYSICS

Department of Mathematics

Financial mathematics 2

Written examination

August 18^{th}, 2025

Name and surname: ________________ Student number:

Instructions

Read carefully the problems before starting to solve them. There are 4 problems. You have two hours.

Problem	a.	b.	c.	d.	Total
1.			•	•	
2.			•	•	
3.				•	
4.				•	
Total					

1. (25) Naj bo B standardno Brownovo gibanje glede na filtracijo $(\mathcal{F}_t)_{t\geq 0}$. Za $a > 0$ definirajte čas ustavljanja

$$T_a = \inf\{t \geq 0\colon |B_t| = a\}\,.$$

Privzemite, da je $P(T_a < \infty) = 1$. Definirajte

$$X = \int_0^{T_a} B_t^2 dt\,.$$

a. (12) Pokažite, da je

$$M_t = B_t^4 - 6\int_0^t B_s^2 ds$$

martingal.

Rešitev: po Itôvi formuli je za $f(x) = x^4$

$$B_t^4 = 4\int_0^t B_s^3 dB_s + 6\int_0^t B_s^2 ds\,.$$

Sledi, da je

$$B_t^4 - 6\int_0^t B_s^2 ds = 4\int_0^t B_s^3 dB_s\,.$$

Desna stran je lokalni martingal. Ker velja, da je $E(B_t^6) = t^3 E(B_1^6)$ in je zadnja pričakovana vrednost končna, velja

$$E\left(\int_0^t B_s^6 ds\right) = \frac{t^4}{4}E(B_1^6) < \infty\,.$$

Po konstrukciji Itôvega integrala je $\int_0^t B_s^3 dB_s$ martingal.

b. (13) Izračunajte $E(X)$. Utemeljite vse korake.

Rešitev: po izreku o opcijskem ustavljanju je za fiksen $t > 0$

$$E\left(M_{T_a\wedge t}\right) = E\left(M_0\right) = 0\,.$$

Sledi, da je

$$E\left(B_{T_a\wedge t}^4\right) = 6E\left(\int_0^{T_a\wedge t} B_s^2 ds\right)\,.$$

Ko $t \to \infty$, konvergira $B_{T_a\wedge t}^4$ proti $B_{T_a}^4$, integrandi pa so omejeni z a^4, zato po izreku o dominirani konvergenci velja

$$E\left(B_{T_a\wedge t}^4\right) \to a^4\,.$$

Ko $t \to \infty$, *konvergira* $\int_0^{T_a \wedge t} B_s^6 ds$ *proti* $\int_0^{T_a} B_s^6 ds$ *in sicer monotono. Ker so integrandi nenegativni, po izreku o monotoni konvergenci velja*

$$E\left(\int_0^{T_a \wedge t} B_s^2 ds\right) \to E\left(\int_0^{T_a} B_s^2 ds\right) .$$

Sledi

$$E\left(\int_0^{T_a} B_s^6 ds\right) = \frac{a^4}{6} .$$

2. (25) Naj bo B standardno Brownovo gibanje. Semimartingal Z naj ustreza enačbi

$$dZ_t = F(t)Z_t dt + G(t)dM_t\,,$$

kjer sta $F(t)$ in $G(t)$ zvezno odvedljivi funkciji in je

$$M_t = \exp\left(B_t - \frac{1}{2}t\right)\,.$$

Začetni pogoj naj bo $Z_0 = z_0$.

a. (10) Naj bo

$$Y_t = \exp\left(-\int_0^t F(u)du\right) Z_t\,.$$

Izračunajte dY_t in zapišite Z eksplicitno.

Rešitev: računamo po stohastični formuli za odvod produkta in opazimo, da je prvi člen na desni v definiciji procesa Y zvezno odvedljiv. Dobimo

$$\begin{aligned} dY_t &= -\exp\left(-\int_0^t F(u)du\right) F(t)Z_t dt + \exp\left(-\int_0^t F(u)du\right) dZ_t \\ &= -\exp\left(-\int_0^t F(u)du\right) F(t)Z_t dt \\ &\qquad + \exp\left(-\int_0^t F(u)du\right)\left(F(t)Z_t dt + G(t)dM_t\right) \\ &= \exp\left(-\int_0^t F(u)du\right) G(t)M_t dB_t\,. \end{aligned}$$

Sledi

$$Y_t = z_0 + \int_0^t \exp\left(-\int_0^u F(s)ds\right) G(u)M_u dB_u$$

in posledično

$$Z_t = \exp\left(\int_0^t F(u)du\right)\left(z_0 + \int_0^t \exp\left(-\int_0^u F(s)ds\right) G(u)M_u dB_u\right)\,.$$

b. (15) Izračunajte $\mathrm{var}(Z_t)$. Utemeljite vse korake.

Rešitev: označimo

$$H_u = \exp\left(-\int_0^u F(s)ds\right) G(u)M_u\,.$$

Vemo, da je

$$\begin{aligned} E(M_u^2) &= E\left[\exp(2B_t - t)\right] \\ &= E\left[\exp(2B_t - 2t + t)\right] \\ &= e^t\,. \end{aligned}$$

Za integrand H velja

$$\int_0^t E(H_s^2)ds = \int_0^t \exp\left(-2\int_0^u F(s)ds\right) G(u)^2 e^u du < \infty\,.$$

Integral zgornje meje je martingal, zato je pričakovana vrednost enaka 0 *in sledi*

$$Z_t - E(Z_t) = \exp\left(\int_0^t F(u)du\right)\int_0^t \exp\left(-\int_0^u F(s)ds\right) G(u) M_u dB_u\,.$$

Itôva izometrija nam da

$$E\left[(Z_t - E(Z_t))^2\right] = \exp\left(2\int_0^t F(u)du\right)\int_0^t \exp\left(-2\int_0^u F(s)ds\right) G(u)^2 e^u du\,.$$

3. (25) Naj bo B standardno Brownovo gibanje in naj bo $(\mathcal{F}_t)_{t\geq 0}$ filtracija, ki jo generira. Naj bo

$$A_t = \int_0^t B_s^2 ds\,.$$

a. (15) Za $0 \leq t < T$ izračunajte

$$E\left(A_T | \mathcal{F}_t\right)\ .$$

Rešitev: računamo

$$\begin{aligned} A_T &= A_t + \int_t^T B_s^2 ds \\ &= A_t + \int_t^T \left(B_t + (B_s - B_t)\right)^2 ds \\ &= A_t + B_t^2(T-t) + 2B_t \int_0^{T-t} (B_{t+u} - B_t) du + \int_0^{T-t} (B_{t+u} - B_t)^2 du\,. \end{aligned}$$

Po markovski lastnosti je $(B_{t+u} - B_t \colon u \geq 0)$ *Brownovo gibanje neodvisno of* $\mathcal{F}_t$. *Opazimo še, da je*

$$E\left[\int_0^{T-t} (B_{t+u} - B_t) du\right] = 0$$

in

$$E\left[\int_0^{T-t} (B_{t+u} - B_t)^2 du\right] = \int_0^{T-t} u du = \frac{(T-t)^2}{2}\,.$$

Po pravilih za pogojno pričakovano vrednost bo

$$E\left(A_T | \mathcal{F}_t\right) = A_t + B_t^2(T-t) + \frac{(T-t)^2}{2}\,.$$

b. (10) Poiščite prilagojen proces H, da bo veljalo

$$E\left[\int_0^T H_s^2 ds\right] < \infty$$

in

$$A_T = E\left(A_T\right) + \int_0^T H_s dB_s\,.$$

Namig: $M_t = E(A_T|\mathcal{F}_t) = F(B_t, A_t, t)$ *je martingal. Uporabite Itôvo formulo.*

Rešitev: iz prvega dela sledi, da je

$$E\left(A_T | \mathcal{F}_t\right) = F(B_t, A_t, t)\,.$$

Ker je

$$F(x,y,t) = y + x^2(T-t) + \frac{(T-t)^2}{2}\,,$$

je funkcija dovolj gladka, da uporabimo Itôvo formulo. Sledi

$$\begin{aligned} F(B_t, A_t, t) - F(0,0,0) = \\ = \quad & \int_0^t \frac{\partial F}{\partial x}(B_s, A_s, s)dB_s + \int_0^t \frac{\partial F}{\partial y}(B_s, A_s, s)dA_s \\ & \qquad + \int_0^t \frac{\partial F}{\partial t}(B_s, A_s, s)ds + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(B_s, A_s, s)ds\,. \end{aligned}$$

Na levi je martingal. To pomeni, da se morajo na desni členi, ki imajo končno totalno variacijo, sešteti v konstanto. Sledi, da je

$$F(B_t, A_t, t) = \frac{T^2}{2} + \int_0^t \frac{\partial F}{\partial x}(B_s, A_s, s)dB_s\,.$$

Sledi

$$H_t = 2B_t(T-t)\,.$$

Očitno je zadoščeno tudi pogoju za integrabilnost.

4. (25) Naj bo S cena delnice v Black-Scholesovem modelu z obrestno mero r in volatilnostjo $\sigma > 0$. Fiksirajmo zapadlost $T \in (0, \infty)$ in označimo s Q martingalsko mero za interval $[0, T]$, tako da je pod Q na $[0, T]$ diskontiran proces $\tilde{S}$ martingal, W pa Brownovo gibanje. Začetna cena delnice je $S_0 > 0$.

Geometrijska azijska opcija z izvršilno ceno $K \in (0, \infty)$ in zapadlostjo T izplača

$$Y = \left(S_0 \exp\left(\frac{1}{T} \int_0^T \log(e^{-rt} S_t / S_0) dt \right) - K \right)_+ .$$

ob času T. Kot znano privzemite, da je pod mero Q za $0 \leq t < T$ slučajna spremenljivka $\int_t^T (W_s - W_t) ds$ neodvisna od $\mathcal{F}_t$ in ima $\mathrm{N}\left(0, (T-t)^3/3\right)$ porazdelitev. Kot znano tudi privzemite, da je za $Z \sim \mathrm{N}(a, b^2)$ in $c > 0$

$$E\left[\left(e^Z - c \right)_+ \right] = e^{a + \frac{b^2}{2}} \Phi\left(\frac{a + b^2 - \log c}{b} \right) - c \Phi\left(\frac{-\log c + a}{b} \right) ,$$

kjer je $\Phi(z)$ porazdelitvena funkcija standardizirano normalne porazdelitve.

a. (10) Izračunajte začetno vrednost V_0 zgornje opcije.

Solution: since for $t \in [0, T]$, $S_t e^{-rt}/S_0 = \mathcal{E}(\sigma W)_t$, we have

$$V_0 = e^{-rT} Q(S_0 \exp(A_T) - K)^+,$$

where $A_T = \int_0^T (\sigma W_t - \sigma^2 t/2) dt/T \sim_Q N(-\sigma^2 T/4, \sigma^2 T/3) = N(-\sigma^2 T/12 - \sigma^2 T/6, \sigma^2 T/3)$. The analogous expression for the plain vanilla European option, strike K, expiry T, is

$$e^{-rT} Q(S_0 \exp(C_T) - K)^+,$$

with $C_T \sim_Q N(rT - \sigma^2 T/2, \sigma^2 T)$. It follows that in order to obtain V_0, we may simply effect the following changes in the formula for the initial value of the plain vanilla European option (in this order): multiply by e^{rT}, send $\sigma^2 \to \sigma^2/3$, send $r \to -\sigma^2/12$, finally multiply back by e^{-rT}. This yields:

$$V_0 = e^{-rT}(S_0 e^{-\sigma^2 T/12} \Phi(d_1) - K \Phi(d_2))$$

with

$$d_1 = \frac{\log(S_0/K) + \frac{\sigma^2 T}{12}}{\sigma \sqrt{T/3}}$$

and

$$d_2 = d_1 - \sigma \sqrt{T/3} .$$

b. (10) Izračunajte vrednostni proces $V = (V_t)_{t \in [0,T]}$ zgornje opcije.

Solution: $V_T = Y$. For $t < T$

$$V_t = e^{-r(T-t)} Q_{\mathcal{F}_t} \left[(S_0 \exp(A_T) - K)_+ \right]$$

and it follows from the lemma on conditioning, the simple Markov property of Brownian motion, part a., and simple rearrangements, that we need only change in the expression from part b. $S_0 \to S_0 e^{\frac{1}{T}\int_0^T W_{s\wedge t} - \frac{\sigma^2(s\wedge t)}{2}ds}$ *and* $T \to T - t$.

c. (5) Kako bi s samo-financirajočim trgovanjem v delnici in denarnem računu replicirali vrednostni proces V? Ni potrebno navesti eksplicitnih formul za poziciji v denarnem računu in delnici, opišite pa kako bi do njiju prišli.

Solution: one sees that $\int_0^T W_s^t ds = W_t(T-t)+\int_0^t W_s ds = W_t T - \int_0^t s dW_s$. W *is a Brownian motion under* Q, *and* $M_t = \int_0^t s dW_s$ *is a continuous square-integrable martingale under* Q. *Seeing* V_t *from the preceding part, once multiplied by* e^{-rt} *to obtain the discounted value process, as a suitable function* F *of* W_t, M_t *and time* t, *we apply to it Itô's formula. Then, at least on* $[0,T)$, *one would write* $\tilde{V}_t = D_t V_t$, *a martingale, as the sum of* $\tilde{V}_0 = D_0 V_0 = V_0$, *of continuous local martingales (more specifically of integrals against* W *and* M*) and of continuous adapted FV processes. By the usual argument it would follow that the latter vanish. The integration against* M *is, by associativity of stochastic integration, equivalent to integration against* W. *Finally, integration against the latter is (again by associativity of stochastic integration) equivalent to integration against* $\tilde{S} = DS$. *Then such a replication of* $\tilde{V} = DV$ *by* $\tilde{S} = DS$ *is in turn equivalent to a self-financing replication of* V *by trading in the stock and in the money market account. Doing this on* $[0,T)$, *by the a.s. convergence of* $F(\dots) \to Y$ *as* $t \uparrow T$, *is sufficient.*