FACULTY OF MATHEMATICS AND PHYSICS

Department of Mathematics

Financial mathematics 2

Written examination

April 17th, 2026

Name and surname: ____________________ Student number: ☐☐☐☐☐☐☐☐

Instructions

Read carefully the problems before starting to solve them. There are 4 problems. You have two hours.

Problem	a.	b.	c.	d.	Total
1.				•	
2.			•	•	
3.			•	•	
4.				•	
Total					

1. (20) Let B be standard Brownian motion. For $\lambda \in \mathbb{R}$ define

$$M_t = e^{2\lambda^2 \int_0^t B_s^2 ds} \cos\left(\lambda\left(B_t^2 - t\right)\right) \quad \text{and} \quad N_t = e^{2\lambda^2 \int_0^t B_s^2 ds} \sin\left(\lambda\left(B_t^2 - t\right)\right) .$$

a. (10) Show that M and N are local martingales.

Solution: the process M is a product of two semimartingales of which one has finite total variation. The stochastic product rule and Itô's formula, and taking into account that

$$d\left(e^{2\lambda^2 \int_0^t B_s^2 ds}\right) = e^{2\lambda^2 \int_0^t B_s^2 ds}\left(2\lambda^2 B_t^2\right) dt$$

and

$$d\left(B_t^2 - t\right) = 2B_t dB_t$$

and as a consequence

$$d\langle B_t^2 - t\rangle_t = 4B_t^2 dt\,,$$

we get

$$\begin{aligned} dM_t &= \\ &= e^{2\lambda^2 \int_0^t B_s^2 ds}\left(2\lambda^2 B_t^2\right)\cos\left(\lambda\left(B_t^2 - t\right)\right) dt \\ &\qquad + e^{2\lambda^2 \int_0^t B_s^2 ds}\left[-\lambda \sin\left(\lambda\left(B_t^2 - t\right)\right) 2B_t dB_t - \frac{1}{2}\lambda^2 \cos\left(\lambda\left(B_t^2 - t\right)\right) 4B_t^2 dt\right] \\ &= -2\lambda e^{2\lambda^2 \int_0^t B_s^2 ds} \sin\left(\lambda\left(B_t^2 - t\right)\right) B_t dB_t\,. \end{aligned}$$

It follows that M is a local martingale. For N the derivation is similar.

b. (5) For $a > 0$ define

$$T_a = \inf\{t \geq 0 \colon 4\int_0^t B_s^2 ds > a\}\,.$$

Argue that T_a is a stopping time and that the processes

$$\tilde{M}_t = M_{t\wedge T_a} \quad \text{and} \quad \tilde{N}_t = N_{t\wedge T_a}$$

are martingales.

Solution: if we know B on the interval $[0,t]$, we know whether $4\int_0^t B_s^2 ds > a$ or not, hence T_a is a stopping time. Since

$$e^{2\lambda^2 \int_0^{t\wedge T_a} B_s^2 ds} \leq e^{\frac{1}{2}\lambda^2 a}\,,$$

$\tilde{M}$ and $\tilde{N}$ are bounded local martingales and hence martingales.

c. (10) Assume as known that $P(T_a < \infty) = 1$. Show that

$$E\left(M_{t\wedge T_a}\right) = 1 \quad \text{and} \quad E\left(N_{t\wedge T_a}\right) = 0\,.$$

Argue that

$$E\left(M_{T_a}\right) = 1 \quad \text{and} \quad E\left(N_{T_a}\right) = 0\,.$$

Solution: since $\tilde{M}$ and $\tilde{N}$ are martingales, for $t > 0$ we have

$$E\left(M_{t\wedge T_a}\right) = 1 \quad \textit{and} \quad E\left(N_{t\wedge T_a}\right) = 0\,.$$

As $t \to \infty$, by assumption of finiteness of T_a we have $M_{t\wedge T_a} \to M_{T_a}$. All random variables are bounded so by dominated convergence

$$E\left(M_{t\wedge T_a}\right) \to E\left(M_{T_a}\right)\,,$$

as $t \to \infty$. The argument for N is identical.

2. (25) Let B be standard Brownian motion and $f, g\colon [0,T] \to \mathbb{R}$ continuous functions. Let the process X satisfy the stochastic differential equation

$$dX_t = f(t)X_t dt + g(t)X_t dB_t$$

with $X_0 = x_0$.

a. (15) Let

$$Z_t = \exp\left(-\int_0^t g(u)dB_u + \frac{1}{2}\int_0^t g(u)^2 du\right) .$$

Compute $d\left(Z_t X_t\right)$.

Solution: Itô's formula gives

$$dZ_t = Z_t\left(-g(t)dB_t + \frac{1}{2}g(t)^2 dt\right) + \frac{1}{2}Z_t g(t)^2 dt$$

We infer that $d\langle X, Z\rangle_t = -g(t)^2 Z_t X_t$. The stochastic product rule gives

$$\begin{aligned} & d\left(Z_t X_t\right) \\ &= Z_t dX_t + X_t dZ_t + d\langle X, Z\rangle_t \\ &= Z_t\left(f(t)X_t dt + g(t)X_t dB_t\right) + X_t Z_t\left(-g(t)dB_t + g(t)^2 dt\right) - g(t)^2 X_t Z_t \\ &= f(t)Z_t X_t dt\,. \end{aligned}$$

b. (10) Find X.

Solution: the equation in the first part is a deterministic differential equation for $Y_t = X_t Z_t$ of the form

$$dY_t = f(t)Y_t$$

with the initial condition $Y_0 = x_0$. The solution is

$$Y_t = x_0 \cdot \exp\left(\int_0^t f(s)ds\right) .$$

It follows that

$$X_t = Z_t^{-1}Y_t = x_0 \exp\left(\int_0^t g(u)dB_u + \int_0^t f(s)ds - \frac{1}{2}\int_0^t g(u)^2 du\right) .$$

3. (25) Let B be standard Brownian motion and assume as known that for $\lambda \in \mathbb{R}$ we have

$$E\left(e^{\lambda B_t}\right) = e^{\frac{\lambda^2 t}{2}}\,.$$

Let $T > 0$ be fixed, and let $X = e^{\lambda B_T}$.

a. (10) Compute

$$E\left(X|\mathcal{F}_t\right)$$

for $0 \leq t \leq T$, where $\mathcal{F}_t$ is the natural filtration of B.

Solution: compute

$$\begin{aligned} E\left(X|\mathcal{F}_t\right) &= E\left(e^{\lambda B_T}|\mathcal{F}_t\right) \\ &= E\left(e^{\lambda(B_T - B_t + B_t)}|\mathcal{F}_t\right) \\ &= e^{\lambda B_t} E\left(e^{\lambda(B_T - B_t)}|\mathcal{F}_t\right) \\ &= e^{\lambda B_t} \cdot e^{\frac{\lambda^2 (T-t)}{2}}\,. \end{aligned}$$

b. (15) Find the adapted integrand H such that

$$X = E(X) + \int_0^t H_s dB_s\,.$$

Solution: note that

$$E\left(X|\mathcal{F}_t\right) = F(B_t, t)\,,$$

where F is twice continuously differentiable. We know that

$$H_t = \frac{\partial F}{\partial x}(B_t, t)\,.$$

We compute

$$\frac{\partial F}{\partial x}(x, t) = \lambda e^{\lambda x + \frac{\lambda^2 (T-t)}{2}}\,.$$

4. (25) Let S be the price of the stock in the Black-Scholes model with interest rate $r \in (0,\infty)$ and volatility $\sigma \in (0,\infty)$. Fix the exercise time $T \in (0,\infty)$ and denote with Q the martingale measure on the time interval $[0,T]$, such that on $[0,T]$ we have $dS_t = rS_t dt + \sigma S_t dW_t$ for Q-Brownian motion W.

Assume constants $0 < b \leq a < \infty$. A financial derivative has a payoff $V_T = f(S_T)$ at time T, where for $x \in (0,\infty)$ we have

$$f(x) = \begin{cases} x-a, & \text{if } x > a, \\ b-x, & \text{if } x < b, \\ 0, & \text{otherwise.} \end{cases}$$

a. (5) Sketch the function $f(x)$. Write $f(S_T)$ as a linear combination of the payoff of two European options and a forward contract (all of them at terminal time T and with appropriate strike prices).

   Explanation: Forward contract has a payoff $S_T - k$ at time T for an exercise price k.

   Solution: Sketch is elementary. We have $f(S_T) = (S_T - a)^+ + (b - S_T)^+ = (S_T - a)^+ + (S_T - b)^+ - (S_T - b)$.

b. (10) For $0 \leq t \leq T$ determine

$$V_t := E_Q\left[e^{-r(T-t)} f(S_T) | \mathcal{F}_t\right] .$$

   Solution: We know that for $0 \leq t < T$ and $k \in (0,\infty)$, we have a.s.

$$Q(t;k) := E_Q\left[e^{-r(T-t)}(S_T - k)^+ | \mathcal{F}_t\right] = S_t \Phi(d_1) - k e^{-r(T-t)} \Phi(d_2)\,,$$

   where

$$d_1 = \frac{\log\left(S_t/k\right) + r(T-t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

   and

$$d_2 = d_1 - \sigma\sqrt{T-t}\,.$$

   From the linearity of the conditional expectation, we have a.s.

$$V_t = Q(t;a) + Q(t;b) - \left(S_t - b e^{-r(T-t)}\right).$$

   Here, we took into account that the discounted price process is a Q-martingale.

c. (10) For $0 \leq t < T$ derive the H_t component of the hedging portfolio for the option that pays $f(S_T)$ at time T.

   Hint: figure out how to hedge a forward contract.

   Solution: From the linearity, it follows that the desired H is a sum of the Hs of European call options with strike prices a and b, reduced by the H of the forward *contract with strike price b. Since the hedging portfolio of the forward contract is simply a unit of a stock, we get*

$$
\begin{aligned}
H_t = \Phi&\left(\frac{\log\left(S_t/a\right) + r(T-t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) \\
&+ \ \Phi\left(\frac{\log\left(S_t/b\right) + r(T-t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) - 1\,.
\end{aligned}
$$

FAKULTETA ZA MATEMATIKO IN FIZIKO

Oddelek za matematiko

Finančna matematika 2

Pisni izpit

17. april 2026

Ime in priimek: ________________________ Vpisna št: | | | | | | | | |

Navodila

Pazljivo preberite besedilo naloge, preden se lotite reševanja. Naloge so 4. Na razpolago imate 2 uri.

Naloga	a.	b.	c.	d.
1.				•
2.			•	•
3.			•	•
4.			•	•
Skupaj				

1. (25) Naj bo B standardno Brownovo gibanje. Definirajte za $\lambda \in \mathbb{R}$

$$M_t = e^{2\lambda^2 \int_0^t B_s^2 ds} \cos\left(\lambda\left(B_t^2 - t\right)\right) \quad \text{in} \quad N_t = e^{2\lambda^2 \int_0^t B_s^2 ds} \sin\left(\lambda\left(B_t^2 - t\right)\right) .$$

a. (10) Pokažite, da sta procesa M in N lokalna martingala.

Rešitev: proces M je produkt dveh semimartingalov, od katerih ima prvi omejeno totalno variacijo. Računamo po pravilu za stohastično odvajanje produkta in po Itôvi formuli, pri čemer upoštevamo

$$d\left(e^{2\lambda^2 \int_0^t B_s^2 ds}\right) = e^{2\lambda^2 \int_0^t B_s^2 ds}\left(2\lambda^2 B_t^2\right) dt$$

in

$$d\left(B_t^2 - t\right) = 2B_t dB_t$$

ter posledično

$$d\langle B_t^2 - t\rangle_t = 4B_t^2 dt\,.$$

Sledi

$$\begin{aligned} dM_t &= \\ &= e^{2\lambda^2 \int_0^t B_s^2 ds}\left(2\lambda^2 B_t^2\right)\cos\left(\lambda\left(B_t^2 - t\right)\right) dt \\ &\quad + e^{2\lambda^2 \int_0^t B_s^2 ds}\left[-\lambda \sin\left(\lambda\left(B_t^2 - t\right)\right) 2B_t dB_t - \frac{1}{2}\lambda^2 \cos\left(\lambda\left(B_t^2 - t\right)\right) 4B_t^2 dt\right] \\ &= -2\lambda e^{2\lambda^2 \int_0^t B_s^2 ds} \sin\left(\lambda\left(B_t^2 - t\right)\right) B_t dB_t\,. \end{aligned}$$

Sledi, da je M lokalni martingal. Račun za N je skoraj enak.

b. (5) Definirajte za $a > 0$

$$T_a = \inf\{t \geq 0\colon 4\int_0^t B_s^2 ds > a\}\,.$$

Utemeljite, da je T_a čas ustavljanja in sta procesa

$$\tilde{M}_t = M_{t\wedge T_a} \quad \text{in} \quad \tilde{N}_t = N_{t\wedge T_a}$$

martingala.

Rešitev: če poznamo trajektorijo B na intervalu $[0,t]$, lahko tudi ugotovimo ali je $4\int_0^t B_s^2 ds > a$, torej je T_a čas ustavljanja. Vemo, da sta $\tilde{M}$ in $\tilde{N}$ lokalna martingala. Ker je

$$e^{2\lambda^2 \int_0^{t\wedge T_a} B_s^2 ds} \leq e^{\frac{1}{2}\lambda^2 a}\,,$$

sta tudi omejena, zato sta martingala.

c. (10) Privzemite kot znano, da je $P(T_a < \infty) = 1$. Pokažite, da je

$$E\left(M_{t\wedge T_a}\right) = 1 \quad \text{in} \quad E\left(N_{t\wedge T_a}\right) = 0\,.$$

Sklepajte, da je

$$E\left(M_{T_a}\right) = 1 \quad \text{in} \quad E\left(N_{T_a}\right) = 0\,.$$

Rešitev: ker sta $\tilde{M}$ in $\tilde{N}$ martingala, za vsak $t > 0$ velja

$$E\left(M_{t\wedge T_a}\right) = 1 \quad \textit{in} \quad E\left(N_{t\wedge T_a}\right) = 0\,.$$

Ko $t \to \infty$, zaradi privzetka o končnosti T_a velja $M_{t\wedge T_a} \to M_{T_a}$. Zaradi omejenosti lahko uporabimo izrek o dominirani konvergenci, tako da

$$E\left(M_{t\wedge T_a}\right) \to E\left(M_{T_a}\right)\,,$$

ko $t \to \infty$. Trditev za M sledi. Dokaz za N je enak.

2. (25) Naj bo B standardno Brownovo gibanje in naj bosta $f, g\colon [0,T] \to \mathbb{R}$ zvezni funkciji. Naj proces X zadošča stohastični diferencialni enačbi

$$dX_t = f(t)X_t dt + g(t)X_t dB_t,$$

z $X_0 = x_0$.

a. (15) Naj bo

$$Z_t = \exp\left(-\int_0^t g(u)dB_u + \frac{1}{2}\int_0^t g(u)^2 du\right) .$$

Izračunajte $d\left(Z_t X_t\right)$.

Rešitev: Itôva formula da

$$dZ_t = Z_t\left(-g(t)dB_t + \frac{1}{2}g(t)^2 dt\right) + \frac{1}{2}Z_t g(t)^2 dt$$

Opazimo $d\langle X, Z\rangle_t = -g(t)^2 Z_t X_t$. *Stohastični per-partes da*

$$\begin{aligned} & d\left(Z_t X_t\right) \\ &= Z_t dX_t + X_t dZ_t + d\langle X, Z\rangle_t \\ &= Z_t\left(f(t)X_t dt + g(t)X_t dB_t\right) + X_t Z_t\left(-g(t)dB_t + g(t)^2 dt\right) - g(t)^2 X_t Z_t \\ &= f(t) Z_t X_t dt\,. \end{aligned}$$

b. (10) Določite proces X.

Rešitev: enačba iz prvega dela naloge je deterministična navadna diferencialna enačba $Y_t = X_t Z_t$ *oblike*

$$dY_t = f(t)Y_t$$

z začetnim pogojem $Y_0 = x_0$. *Rešitev je*

$$Y_t = x_0 \cdot \exp\left(\int_0^t f(s)ds\right) .$$

Sledi

$$X_t = Z_t^{-1} Y_t = x_0 \exp\left(\int_0^t g(u)dB_u + \int_0^t f(s)ds - \frac{1}{2}\int_0^t g(u)^2 du\right) .$$

3. (25) Naj bo B standardno Brownovo gibanje in privzemite kot znano, da je za $\lambda \in \mathbb{R}$

$$E\left(e^{\lambda B_t}\right) = e^{\frac{\lambda^2 t}{2}}\,.$$

Naj bo $T > 0$ fiksen in naj bo $X = e^{\lambda B_T}$.

a. (10) Izračunajte

$$E\left(X|\mathcal{F}_t\right)$$

za $0 \leq t \leq T$, kjer je $\mathcal{F}_t$ naravna filtracija B.

Rešitev: računamo

$$\begin{aligned} E\left(X|\mathcal{F}_t\right) &= E\left(e^{\lambda B_T}|\mathcal{F}_t\right) \\ &= E\left(e^{\lambda(B_T - B_t + B_t)}|\mathcal{F}_t\right) \\ &= e^{\lambda B_t} E\left(e^{\lambda(B_T - B_t)}|\mathcal{F}_t\right) \\ &= e^{\lambda B_t} \cdot e^{\frac{\lambda^2 (T-t)}{2}}\,. \end{aligned}$$

b. (15) Poiščite prilagojen integrand H, da bo

$$X = E(X) + \int_0^t H_s dB_s\,.$$

Rešitev: opazimo, da je

$$E\left(X|\mathcal{F}_t\right) = F(B_t, t)\,,$$

kjer je F dvakrat zvezno parcialno odvedljiva. Sledi, da je

$$H_t = \frac{\partial F}{\partial x}(B_t, t)\,.$$

Z odvajanjem dobimo

$$\frac{\partial F}{\partial x}(x, t) = \lambda e^{\lambda x + \frac{\lambda^2 (T-t)}{2}}\,.$$

4. (25) Naj bo S cena delnice v Black-Scholesovem modelu z obrestno mero $r \in (0,\infty)$ in volatilnostjo $\sigma \in (0,\infty)$. Fiksirajmo zapadlost $T \in (0,\infty)$ in označimo s Q martingalsko mero za interval $[0,T]$, tako da je na $[0,T]$, $dS_t = rS_t dt + \sigma S_t dW_t$ za Q-Brownovo gibanje W.

Naj bodo dane konstante $0 < b \leq a < \infty$. Izveden finančni instrument ima izplačilo $V_T = f(S_T)$ ob času T, kjer je za $x \in (0,\infty)$,

$$f(x) = \begin{cases} x-a, & \text{če je } x > a \\ b-x, & \text{če je } x < b \\ 0, & \text{sicer} \end{cases}.$$

a. (5) Skicirajte funkcijo $f(x)$. Zapišite $f(S_T)$ kot ustrezno linearno kombinacijo izplačil dveh evropskih nakupnih opcij ter terminske pogodbe (vseh z zapadlostmi T in ustreznimi izvršilnimi cenami).

Pojasnilo: terminske pogodbe so matematično oblike $S_T - k$ za neko izvršilno ceno k.

Rešitev: Skica grafa je elementarna. Velja $f(S_T) = (S_T - a)^+ + (b - S_T)^+ = (S_T - a)^+ + (S_T - b)^+ - (S_T - b)$.

b. (10) Določite za $0 \leq t \leq T$

$$V_t := E_Q\left[e^{-r(T-t)} f(S_T) | \mathcal{F}_t\right].$$

Rešitev: Vemo, da je za $0 \leq t < T$ in $k \in (0,\infty)$, s.g.

$$Q(t;k) := E_Q\left[e^{-r(T-t)}(S_T - k)^+ | \mathcal{F}_t\right] = S_t \Phi(d_1) - k e^{-r(T-t)} \Phi(d_2),$$

kjer je

$$d_1 = \frac{\log(S_t/k) + r(T-t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

in

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Iz linearnosti pričakovane vrednosti sledi, da je s.g.

$$V_t = Q(t;a) + Q(t;b) - (S_t - b e^{-r(T-t)}).$$

Upoštevali smo, da je diskontirani proces cene delnice Q-martingal.

c. (10) Za $0 \leq t < T$ navedite komponento H_t varovalnega portfelja za opcijo, ki izplača $f(S_T)$ v času T.

Namig: pomislite, kako moramo varovati terminsko pogodbo.

Rešitev: Iz linearnosti sledi, da je iskani H vsota H-jev za evropski nakupni opciji z izvršnima cenama a in b zmanjšan za H terminske pogodbe z izvršno ceno b. Se pravi

$$\begin{aligned} H_t &= \Phi\left(\frac{\log\left(S_t/a\right) + r(T-t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\qquad + \Phi\left(\frac{\log\left(S_t/b\right) + r(T-t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) - 1\,. \end{aligned}$$

Za terminsko pogodbo je varovalni portfelj enota temelja-delnice.