FACULTY OF MATHEMATICS AND PHYSICS

Department of Mathematics

Financial mathematics 2

Written examination

April 15th, 2024

Name and surname: ____________________ Student number: | | | | | | | | |

Instructions

Read carefully the problems before starting to solve them. There are 4 problems. You have two hours.

Problem	a.	b.	c.	d.	Total
1.					
2.			•	•	
3.			•	•	
4.			•	•	
Total					

1. (25) Let B and D be independent standard Brownian motions. We define

$$T = \inf\{t \geq 0\colon B_t^2 + D_t^2 = 1\}\,.$$

Assume that $E(T^2) < \infty$.

a. (5) Show that $B_t^2 + D_t^2 - 2t$ is a martingale and use it to compute $E(T)$. Justify your steps.

Solution: $B_t^2 + D_t^2 - 2t$ *is a martingale, since it is the sum of two. By optional sampling we have*

$$E\left(B_{t\wedge T}^2 + D_{t\wedge T}^2 - 2(T\wedge t)\right) = 0\,.$$

By assumption

$$B_{t\wedge T}^2 + D_{t\wedge T}^2 \to 1$$

as $t \to \infty$ *and the sum of squares is bounded by* 1. *So the expectation converges to* 1. *On the other hand* $t\wedge T \uparrow T$, *so by monotone convergence* $E(t\wedge T) \to E(T)$. *It follows that* $E(T) = \frac{1}{2}$.

b. (10) Show that

$$R_t = \left[B_t^2 + D_t^2\right]^2 - 2B_t^2D_t^2 - 6t(B_t^2 + D_t^2) + 6t^2$$

is a local martingale.

Solution: by independence $\langle B, D\rangle = 0$. *Denote* $Z_t = B_t^2 + D_t^2$. *We use Itô's formula to compute*

$$\begin{aligned} dR_t &= 4Z_tB_tdB_t + 4Z_tD_tdD_t + 2(3B_t^2 + D_t^2)dt + 2(B_t^2 + 3D_t^2)dt \\ &\quad -4B_tD_t^2dB_t - 4B_t^2D_tdD_t - 2D_t^2dt - 2B_t^2dt \\ &\quad -6Z_tdt - 6t\left(2B_tdB_t + 2D_tdD_t + 2dt\right) + 12tdt \\ &= 4B_t^3dB_t + 4D_t^3dD_t - 6t\left(2B_tdB_t + 2D_tdD_t\right)\,. \end{aligned}$$

Integrals of continuous integrads against continuous local martingales are continuous local martingales.

c. (10) Assume as known that

$$E\left(B_T^2D_T^2\right) = \frac{1}{8}\,.$$

Use the fact that R is a local martingale to compute $E(T^2)$. Justify your steps.

Solution: note R^T *is a bounded continuous local martingale, hence a martingale. Martingales have constant expectation. Then*

$$E\left(R_{t\wedge T}\right) = E(R_0) = 0\,.$$

In $R_{t\wedge T}$*, all the terms are dominated by integrable random variables and* $R_{t\wedge T}$ *coverges to* R_T *as* $t\uparrow\infty$*. It follows that*

$$0 = 1 - 2E(B_T^2 D_T^2) - 6E(T) + 6E(T^2)\,.$$

Finally, we have $E(T^2) = \frac{3}{8}$*.*

2. (25) Let W be a standard Brownian motion and let the process X satisfy the stochastic differential equation

$$dX_t = \sqrt{1 + X_t^2} dW_t + \frac{1}{2} X_t dt$$

with initial condition $X_0 = x_0$. Assume as known that the stochastic differential equation has a unique strong solution for which

$$E\left(X_t^2\right) \leq K e^{Ct}$$

for some constants $C, K < \infty$.

a. (10) Show that

$$d\left(e^{-\frac{t}{2}} X_t\right) = e^{-\frac{t}{2}} \sqrt{1 + X_t^2} dW_t \,.$$

Solution: by the product rule we have

$$d\left(e^{-\frac{t}{2}} X_t\right) = -\frac{1}{2} e^{-\frac{t}{2}} X_t dt + e^{-\frac{t}{2}} dX_t \,.$$

Substituting dX_t from the stochastic differential equation we get

$$d\left(e^{-\frac{t}{2}} X_t\right) = e^{-\frac{t}{2}} \sqrt{1 + X_t^2} dW_t \,.$$

b. (15) Find $E(X_t^2)$. Justify your steps. Assume as known that the solution of the differential equation

$$g' = e^{-t} + g$$

is of the form

$$g(t) = -\frac{1}{2} e^{-t} + c e^t$$

for some constant c.

Solution: from the first part we have

$$e^{-\frac{t}{2}} X_t = x_0 + \int_0^t e^{-\frac{s}{2}} \sqrt{1 + X_s^2} dW_s \,.$$

By the above inequality

$$E\left(\int_0^t e^{-s}(1 + X_s^2) ds\right) < \infty$$

so $e^{-\frac{t}{2}}X_t$ *is a square integrable martingale. We have*

$$\begin{aligned}
&E\left(e^{-t}X_t^2\right) \\
&= x_0^2 + 2x_0 E\left[\int_0^t e^{-\frac{s}{2}}\sqrt{1+X_s^2}dW_s\right] \\
&\qquad + E\left[\left(\int_0^t e^{-\frac{s}{2}}\sqrt{1+X_s^2}dW_s\right)^2\right] .
\end{aligned}$$

Applying the fact that the expectation of the stochastic integral is 0 *and Itô's isometry we get*

$$e^{-t}E(X_t^2) = x_0^2 + \int_0^t e^{-s}(1 + E(X_s^2))ds\,.$$

Denote $g(t) = e^{-t}E(X_t^2)$*. Differentiating we get*

$$g' = e^{-t} + g\,.$$

The initial condition for this differential equation is $g(0) = x_0^2$*. The solution is*

$$g(t) = -\frac{1}{2}e^{-t} + \left(x_0^2 + \frac{1}{2}\right)e^t\,.$$

It follows that

$$E(X_t^2) = \left(x_0^2 + \frac{1}{2}\right)e^{2t} - \frac{1}{2}\,.$$

3. (25) Let B and D be Brownian motions with $\langle B, D\rangle = 0$. Let f, g be bounded continuous functions. By the martingale representation theorem, there are integrands H and K for $0 \le t \le T$, such that

$$f(B_T) = E\left[f(B_T)\right] + \int_0^T H_s dB_s \quad \text{and} \quad g(D_T) = E\left[g(D_T)\right] + \int_0^T K_s dD_s$$

with

$$E\left[\int_0^T H_s^2 dt\right] < \infty \quad \text{and} \quad E\left[\int_0^T K_s^2 dt\right] < \infty\,.$$

a. (10) Let

$$M_t = E\left[f(B_T)\right] + \int_0^t H_s dB_s \quad \text{and} \quad N_t = E\left[g(D_T)\right] + \int_0^t K_s dD_s\,.$$

Find an integral expression for $M_t N_t$ for $0 \le t \le T$.

Solution: the processes M_t and N_t are martingales. By the stochastic partial integration rule we have

$$M_t N_t = M_0 N_0 + \int_0^t M_s dN_s + \int_0^t N_s dM_s + \langle M, N\rangle_t\,.$$

The rules for stochastic integration give:

- $\langle M, N\rangle = HK \cdot \langle B, D\rangle = 0.$
- $M \cdot N = MH \cdot B.$
- $N \cdot M = NK \cdot D.$

We rewrite

$$M_t N_t = M_0 N_0 + \int_0^t M_s K_s dD_s + \int_0^t N_s H_s dB_s\,.$$

b. (15) Show that

$$E\left[f(B_T) g(D_T)\right] = E\left[f(B_T)\right] E\left[g(D_T)\right]\,.$$

Justify your steps.

Solution: since M_T is bounded by assumption, we have that

$$|M_t| = |E(M_T \mid \mathcal{F}_t)|$$

is bounded by a constant C, say, so

$$E\left[\int_0^T M_s^2 K_s^2 dt\right] \le C^2 E\left[\int_0^T K_s^2 dt\right] < \infty\,.$$

This means that $t \mapsto \int_0^t M_s K_s dD_s$ *is a square integrable martingale with zero expectation. The same argument applies to* $\int_0^t N_s H_s dB_s$. *Taking expectations on both sides for* $t = T$, *we get*

$$E\left[M_T N_T\right] = f(M_0) g(N_0)\,,$$

or in other words,

$$E\left[f(B_T) g(D_T)\right] = E\left[f(B_T)\right] E\left[g(D_T)\right]\,.$$

4. (25) Let $T > 0$ and assume the Black-Scholes model for S_t:

$$S_t = S_0 e^{\mu t + \sigma B_t - \frac{\sigma^2}{2} t}.$$

Assume the interest rate r and volatility σ are given and fixed. Let f be a given non-negative continuous function such that $\int_0^T f(s)ds = 1$. The weighted geometric Asian option is defined by

$$V_T = \left(\exp\left(\int_0^T f(s)\log S_s ds\right) - k\right)_+.$$

Use the known fact that for $Z \sim \mathrm{N}(a, b^2)$ and $c > 0$ we have

$$E\left[\left(e^Z - c\right)_+\right] = e^{a+\frac{b^2}{2}}\Phi\left(\frac{a + b^2 - \log c}{b}\right) - c\Phi\left(\frac{-\log c + a}{b}\right),$$

where $\Phi(z)$ is the distribution function of the standard normal variable.

a. (10) Let W be Brownian motion and $F(t) = \int_0^t f(s)ds$. Verify that

$$\int_0^T f(s)W_s ds = \int_0^T (F(T) - F(s))dW_s$$

and show that

$$\operatorname{var}\left(\int_0^T f(s)W_s ds\right) = \int_0^T (F(T) - F(s))^2 ds$$

Solution: denote $X = \int_0^T f(s)W_s ds$. *By the stochastic per partes integration formula we have*

$$[(F(T) - F(s))W_s]\,|_0^T = -\int_0^T f(s)W_s ds + \int_0^T (F(T) - F(s))dW_s.$$

The expression on the left-hand side equals 0. *Since* $E(X) = 0$, *Itô's isometry yields*

$$\operatorname{var}(X) = E(X^2) = \int_0^T (F(T) - F(s))^2 ds.$$

b. (15) Compute V_0.

Solution: we have

$$V_0 = E_Q\left(\tilde{V}_T\right),$$

where under Q

$$\log(S_t) = \log(S_0) + rt + \sigma W_t - \frac{\sigma^2}{2}t$$

and W is Brownian motion under Q. Then

$$\tilde{V}_T = \left(\exp\left(-rT + \int_0^T f(s)\log S_s ds\right) - e^{-rT}k\right)_+ .$$

The random variable

$$Z = -rT + \int_0^T f(s)\log S_s ds$$

is a normal variable under Q with expected value

$$a = -rT + \log(S_0)F(T) + \left(r - \frac{\sigma^2}{2}\right)\int_0^T sf(s)ds$$

and variance

$$b^2 = \sigma^2 \int_0^T (F(T) - F(s))^2 ds\,.$$

Take

$$c = e^{-rT}k$$

Then

$$V_0 = e^{-rT}\left[e^{a+\frac{b^2}{2}}\Phi\left(\frac{a + b^2 - \log c}{b}\right) - c\Phi\left(\frac{-\log c + a}{b}\right)\right] .$$

follows.