FACULTY OF MATHEMATICS AND PHYSICS

Department of Mathematics

Financial mathematics 2

Written examination

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Instructions

Read carefully the problems before starting to solve them. There are 4 problems. You have two hours.

Problem	a.	b.	c.	d.	Total
1.				●	
2.			●	●	
3.			●	●	
4.			●	●	
Total					

SOLUTIONS

1. (20) Let X be the unique strong solution of the stochastic diferential equation

$$dX_t = 2\sqrt{X_t}dB_t + 3dt$$

with the initial condition $X_0 = x_0 > 0$. Assume as known that $X_t \in (0,\infty)$ for all $t \geq 0$. Let $0 < a < x_0 < b$ and define for $y > 0$

$$T_y = \inf\{t \geq 0 \colon X_t = y\}\,.$$

Let $T_{a,b} = T_a \wedge T_b$, and assume as known that $P(T_{a,b} < \infty) = 1$.

a. (5) Show that $Y_t = X_t^{-1/2}$ is a local martingale.

Solution: the function $f(x) = 1/\sqrt{x}$ is twice continuously differentiable on $(0,\infty)$, so we can apply Itô's formula to get

$$dY_t = -\frac{1}{2X_t^{3/2}}dX_t + \frac{3}{8X^{5/2}}d\langle X\rangle_t\,.$$

Rewrite to get

$$dY_t = -\frac{1}{X_t}dB_t + \left(-\frac{3}{2X_t^{3/2}} + \frac{4\cdot 3}{8X_t^{5/2}}X_t\right)dt$$

which simplifies to

$$dY_t = -\frac{1}{X_t}dB_t\,.$$

The process Y is a local martingale.

b. (10) Compute the probability $P(T_a < T_b)$.

Solution: the stopped local martingale $Y^{T_{a,b}}$ is bounded, and hence a martingale. Define for $y > 0$

$$\tilde{T}_y = \inf\{t \geq 0 \colon Y_t = y\}\,.$$

We have

$$P(T_a < T_b) = P\left(\tilde{T}_{1/\sqrt{a}} < \tilde{T}_{1/\sqrt{b}}\right)$$

Since we can use $Y^{T_{a,b}}$ instead of Y, we know that for any martingale we have

$$P\left(\tilde{T}_{1/\sqrt{a}} < \tilde{T}_{1/\sqrt{b}}\right) = \frac{1/\sqrt{x_0} - 1/\sqrt{b}}{1/\sqrt{a} - 1/\sqrt{b}} = \frac{\sqrt{ab/x_0} - \sqrt{a}}{\sqrt{b} - \sqrt{a}}\,.$$

c. (10) Compute $P(T_a = \infty)$.

Solution: since we know that $P(T_{a,b} < \infty)$, on the set $\{T_a = \infty\}$ we have $T_b < \infty$ for all b, and hence

$$\{T_a = \infty\} = \cap_{n>x}\{T_a > T_n\}\,.$$

The events in the intersection are decreasing, so

$$P(T_a = \infty) = \lim_{b\to\infty}\frac{\sqrt{b} - \sqrt{ab/x_0}}{\sqrt{b} - \sqrt{a}} = 1 - \sqrt{\frac{a}{x_0}}\,.$$

2. (25) Let B be standard Brownian motion and $f\colon [0,T] \to \mathbb{R}$ a continuous function. Let the process X satisfy the stochastic differential equation

$$dX_t = f(t)X_t dt + \sigma X_t dB_t$$

with $X_0 = x_0$ with σ a given constant.

a. (15) Let

$$Z_t = \exp\left(-\sigma B_t + \frac{\sigma^2}{2}t\right).$$

Compute $d\left(Z_t X_t\right)$.

Solution: Itô's formula gives

$$dZ_t = Z_t\left(-\sigma dB_t + \frac{\sigma^2}{2}dt\right) + \frac{\sigma^2}{2}Z_t dt\,.$$

We infer that $d\langle X, Z\rangle_t = -\sigma^2 X_t Z_t dt$. *The stochastic product rule gives*

$$\begin{aligned} d\left(Z_t X_t\right) &= Z_t dX_t + X_t dZ_t + d\langle X, Z\rangle_t \\ &= Z_t\left(f(t)X_t dt + \sigma X_t dB_t\right) + X_t Z_t\left(-\sigma dB_t + \sigma^2 dt\right) - \sigma^2 X_t Z_t \\ &= f(t) Z_t X_t dt\,. \end{aligned}$$

b. (10) Find X.

Solution: the equation in the first part is a deterministic differential equation for $Y_t = X_t Z_t$ *of the form*

$$dY_t = f(t)Y_t$$

with the initial condition $Y_0 = x_0$. *The solution is*

$$Y_t = x_0 \cdot \exp\left(\int_0^t f(s)ds\right).$$

It follows that

$$X_t = Z_t^{-1} Y_t = x_0 \exp\left(\sigma B_t + \int_0^t f(s)ds - \frac{\sigma^2}{2}t\right).$$

3. (25) Let B be standard Brownian motion. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ be a partition of the interval $[0, T]$. Let

$$X = \prod_{k=1}^{n} \left(B_{t_k} - B_{t_{k-1}} \right) .$$

By independence of increments, we have $E(X) = 0$ and $E(X^2) = \prod_{k=1}^{n}(t_k - t_{k-1})$.

a. (10) Define

$$H_t^k = \begin{cases} 1 & \text{for } t_{k-1} \le t < t_k \\ 0 & \text{else.} \end{cases}$$

and $M_t^k = \int_0^t H_s^k dB_s$ for $t \in [0, T]$. Compute $\int_0^T H_t^k dB_s$ and $\langle M^k, M^l \rangle$ for $k \neq l$.

Solution: the integrand H_t^k is elementary and hence $\int_0^T H_t^k dB_s = B_{t_k} - B_{t_{k-1}}$. We have $\langle H \cdot B, K \cdot B \rangle = HK \cdot \langle B, B \rangle$. In our case $H^k H^l = 0$, and hence $\langle M^k, M^l \rangle = 0$.

b. (15) Find the integrand H such that

$$X = \int_0^T H_t dB_t .$$

Hint: $X = F(M_T^1, \ldots, M_T^n) = \prod_{k=1}^{n} M_T^k$.

Solution: apply Itô's formula for the function $F(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n$. All second partial derivatives with respect to the same x_k are 0, and in the mixed terms the cross-covariances are 0. Only the first derivatives remain in the Itô formula. We have

$$\begin{aligned} X &= \prod_{k=1}^{n} M_T^k \\ &= \sum_{k=1}^{n} \int_0^T \frac{\partial F}{\partial x_k}(M_t^1, \ldots, M_t^n) dM_s^k \\ &= \sum_{k=1}^{n} \int_0^T \frac{\partial F}{\partial x_k}(M_t^1, \ldots, M_t^n) H_s^k dB_s . \end{aligned}$$

Finally,

$$H_t = \sum_{k=1}^{n} \left(\prod_{\substack{l=1 \\ l \neq k}}^{n} M_t^l \right) H_t^k .$$

4. (25) Assume that the stock price follows the Black-Scholes model with volatility σ and constant interest rate r, or in other words

$$S_t = S_0 \exp\left(\mu t + \sigma B_t - \frac{\sigma^2}{2} t\right) .$$

Let T be the expiration time, and let the option V_T pay

$$V_T = \begin{cases} S_T - b, & \text{if } S_T > b \\ a - S_T, & \text{if } S_T < a \\ 0, & \text{else} \end{cases}$$

for $0 < a < b$.

a. (10) Compute the initial price V_0.

Solution: we write

$$V_T = (S_T - b)_+ + (a - S_T)_+ = (S_T - b)_+ + (S_T - a)_+ - (S_T - a) .$$

Denote by $V_t^{e,k}$ *the value of the European call with strike price* k *at time* t. *We have that*

$$V_0 = E_Q(\tilde{V}_T) = V_0^{e,b} + V_0^{e,a} - (S_0 - ae^{-rT}) .$$

b. (15) Find H_t.

Solution: denote $H_t^{e,k}$ *be the stock component of the European call with strike price* k. *at time* t. *We have that*

$$H_t = H_t^{e,a} + H_t^{e,b} - 1 .$$