FACULTY OF MATHEMATICS AND PHYSICS

Department of Mathematics

Financial mathematics 2

Written examination

February 7th, 2025

Name and surname: ____________________ Student number: | | | | | | | | |

Instructions

Read carefully the problems before starting to solve them. There are 4 problems. You have two hours.

Problem	a.	b.	c.	d.	Total
1.					
2.			•	•	
3.			•	•	
4.			•	•	
Total					

1. (25) Let B be Brownian motion. For $\lambda \in \mathbb{R}$ define

$$M_t = e^{2\lambda^2 \int_0^t B_s^2 ds} \cos\left(\lambda\left(B_t^2 - t\right)\right) \quad \text{and} \quad N_t = e^{2\lambda^2 \int_0^t B_s^2 ds} \sin\left(\lambda\left(B_t^2 - t\right)\right) .$$

a. (10) Show that M and N are local martingales.

Solution: the process M is the product of two semimartingales of which one has bounded variation. We have

$$d\left(e^{2\lambda^2 \int_0^t B_s^2 ds}\right) = e^{2\lambda^2 \int_0^t B_s^2 ds}\left(2\lambda^2 B_t^2\right) dt$$

and

$$d\left(B_t^2 - t\right) = 2B_t dB_t .$$

As a consequence we have

$$d\langle B_t^2 - t\rangle_t = 4B_t^2 dt .$$

We have

$$\begin{aligned} dM_t &= \\ &= e^{2\lambda^2 \int_0^t B_s^2 ds}\left(2\lambda^2 B_t^2\right)\cos\left(\lambda\left(B_t^2 - t\right)\right) dt \\ &\quad + e^{2\lambda^2 \int_0^t B_s^2 ds}\left[-\lambda \sin\left(\lambda\left(B_t^2 - t\right)\right) 2B_t dB_t - \frac{1}{2}\lambda^2 \cos\left(\lambda\left(B_t^2 - t\right)\right) 4B_t^2 dt\right] \\ &= -2\lambda e^{2\lambda^2 \int_0^t B_s^2 ds} \sin\left(\lambda\left(B_t^2 - t\right)\right) B_t dB_t . \end{aligned}$$

We have that M is a local martingale as an integral with respect to B.

b. (5) For $a > 0$ define

$$T_a = \inf\{t \geq 0 \colon 4\int_0^t B_s^2 ds > a\} .$$

Argue that T_a is a stopping time and the processes

$$\tilde{M}_t = M_{t\wedge T_a} \quad \text{in} \quad \tilde{N}_t = N_{t\wedge T_a}$$

are martingales.

Solution: from the trajectory of B on $[0,t]$, we can establish whether $4\int_0^t B_s^2 ds > a$ or not, hence $\{T_a \leq t\} \in \mathcal{F}_t$. We have

$$e^{2\lambda^2 \int_0^{t\wedge T_a} B_s^2 ds} \leq e^{\frac{1}{2}\lambda^2 a} ,$$

which means that $\tilde{M}$ and $\tilde{N}$ are bounded local martingales and hence martingales.

c. (5) Assume $P(T_a < \infty) = 1$. Show that

$$E\left(M_{t\wedge T_a}\right) = 1 \quad \text{and} \quad E\left(N_{t\wedge T_a}\right) = 0\,.$$

Argue that

$$E\left(M_{T_a}\right) = 1 \quad \text{and} \quad E\left(N_{T_a}\right) = 0\,.$$

Solution: $\tilde{M}$ and $\tilde{N}$ are martingales and hence

$$E\left(M_{t\wedge T_a}\right) = 1 \quad \textit{and} \quad E\left(N_{t\wedge T_a}\right) = 0\,.$$

As $t \to \infty$, by assumption we have $M_{t\wedge T_a} \to M_{T_a}$. Applying the dominated convergence theorem we have

$$E\left(M_{t\wedge T_a}\right) \to E\left(M_{T_a}\right)\,,$$

as $t \to \infty$. The argument for N is similar.

d. (5) Compute

$$E\left[\cos\left(\lambda\left(B_{T_a}^2 - T_a\right)\right)\right]\,.$$

and

$$E\left[\sin\left(\lambda\left(B_{T_a}^2 - T_a\right)\right)\right]\,.$$

Solution: we have

$$E\left[\cos\left(\lambda\left(B_{T_a}^2 - T_a\right)\right)\right] = e^{-\frac{\lambda^2}{2}a}$$

and

$$E\left[\sin\left(\lambda\left(B_{T_a}^2 - T_a\right)\right)\right] = 0\,.$$

This implies that $B_{T_a} - T_a \sim \mathrm{N}(0, a)$.

2. (25) The semimartingales X and Y satisfy the equations

$$\begin{aligned} dX_t &= X_t dB_t^{(1)} + X_t dB_t^{(2)} \\ dY_t &= -Y_t dt + Y_t dB_t^{(1)}, \end{aligned}$$

where $B^{(1)}$ and $B^{(2)}$ are independent Brownian motions. Assume that $X_0 = Y_0 = 1$.

a. (10) Compute $d\,(X_t Y_t)$.

Solution: we have

$$\begin{aligned} d\,(X_t Y_t) &= X_t dY_t + Y_t dX_t + d\langle X, Y\rangle_t \\ &= X_t \left(-Y_t dt + Y_t dB_t^{(1)}\right) + Y_t \left(X_t dB_t^{(1)} + X_t dB_t^{(2)}\right) + X_t Y_t dt \\ &= 2X_t Y_t dB_t^{(1)} + X_t Y_t dB_t^{(2)}. \end{aligned}$$

b. (15) Find $X_t Y_t$.

Hint: the process

$$W_t = \frac{2B_t^{(1)} + B_t^{(2)}}{\sqrt{5}}$$

is Brownian motion.

Solution: from the first part we get

$$X_t Y_t = 1 + 2\int_0^t X_s Y_s dB_s^{(1)} + \int_0^t X_s Y_s dB_s^{(2)}.$$

Rewrite to get

$$X_t Y_t = 1 + \sqrt{5}\int_0^t X_s Y_s dW_s.$$

The solution of this equation is the exponential martingale

$$X_t Y_t = \exp\left(\sqrt{5}W_t - \frac{5}{2}t\right)$$

or

$$X_t Y_t = \exp\left(2B_t^{(1)} + B_t^{(2)} - \frac{5}{2}t\right).$$

3. (25) Let B be Brownian motion and $T > 0$ fixed. Assume as known that

$$E\left(\cos(\lambda B_t)\right) = e^{-\frac{\lambda^2 t}{2}} \qquad \text{and} \qquad E\left(\sin(\lambda B_t)\right) = 0\,.$$

a. (10) Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by the Brownian motion. Compute

$$E\left(\sin\left(\lambda B_T\right) \mid \mathcal{F}_t\right)$$

for $0 \leq t \leq T$.

Solution: we compute

$$\begin{aligned}
& E\left(\sin\left(\lambda B_T\right) \mid \mathcal{F}_t\right) \\
&= E\left(\sin\left(\lambda(B_T - B_t + B_t)\right) \mid \mathcal{F}_t\right) \\
&= E\left(\sin(\lambda(B_T - B_t)\right)\cos(B_t) + E\left(\cos(\lambda(B_T - B_t))\right)\sin(\lambda B_t) \\
&= e^{-\frac{\lambda^2(T-t)}{2}}\sin(\lambda B_t)\,.
\end{aligned}$$

b. (15) Find the integrand H such that

$$\sin(\lambda B_T) = \int_0^T H_s dB_s\,.$$

Solution: we have

$$E\left(\sin\left(\lambda B_T\right) \mid \mathcal{F}_t\right) = f(B_t, t)\,.$$

It follows that

$$H_s = \frac{\partial f}{\partial x}(B_s, s) = \lambda e^{-\frac{\lambda^2(T-s)}{2}}\cos(\lambda B_s)\,.$$

4. (25) Assume the Black-Scholes model

$$dS_t = S_t(\mu dt + \sigma dB_t)\,.$$

for the stock price. Asusme that r and σ are given constants. Denote the exercise time by T.

a. (5) Let $V_T = S_T$. What is V_0? What are (H_t^0, H_t)?

Solution: since under Q the price process $\tilde{S}_t$ and $\tilde{V}_t$ are martingales, we have

$$\begin{aligned}\tilde{V}_t &= E_Q\left(\tilde{V}_T|\mathcal{F}_t\right)\\ &= E_Q\left(\tilde{S}_T|\mathcal{F}_t\right)\\ &= \tilde{S}_t\,.\end{aligned}$$

It follows that $(H_t^0, H_t) = (0,1)$ for $0 \leq t \leq T$.

b. (10) For a European call option with strike price k we have for $t < T$

$$V_t = S_t\Phi(d_1) - e^{-r(T-t)}k\Phi(d_2)$$

with

$$d_1 = \frac{\log(S_t/k) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

and

$$H_t = \Phi(d_1)\,.$$

Use

$$(k-x)_+ = (x-k)_+ - (x-k)$$

to derive V_t and H_t for the European put option $V_T = (k - S_T)_+$.

Solution: we have

$$\begin{aligned}V_t &= e^{-r(T-t)}E_Q\left[V_T|\mathcal{F}_t\right]\\ &= e^{-r(T-t)}E_Q\left[(k-S_T)_+|\mathcal{F}_t\right]\\ &= e^{-r(T-t)}E_Q\left[(S_T-k)_+ - (S_T-k)|\mathcal{F}_t\right]\\ &= e^{-r(T-t)}E_Q\left[(S_T-k)_+ - (S_T-k)|\mathcal{F}_t\right]\\ &= S_t\Phi(d_1) - e^{-r(T-t)}k\Phi(d_2) - \left(S_t - ke^{-r(T-t)}\right)\\ &= -S_t\Phi(-d_1) + ke^{-r(T-t)}\Phi(-d_2)\,.\end{aligned}$$

By linearity $H_t = \Phi(d_1) - 1 = -\Phi(-d_1)$.

c. (10) The *Butterfly* option is given by

$$V_T = f(S_T)\,,$$

where for $0 < a < k < b$

$$f(x) = \begin{cases} 0 & x < k-a \\ \frac{x-a}{k-a} & a \leq x < k \\ \frac{b-x}{b-k} & k < x < b \\ 0 & x > b \end{cases}$$

Compute V_t.

Hint: look at the functions

$$\lambda\left[(x-k)_+ - (x-b)_+\right] \quad \textit{and} \quad \mu\left[(k-x)_+ - (a-x)_+\right]$$

and choose λ *and* μ*.*

Solution: let $V_t^{c,k}$ *be the prices of the European call with strike price* k *and* $V_t^{p,k}$ *the price of the European put with strike price* k*. From the hint we have that the* Butterfly *is given by* $V_T = f(S_T)$ *where*

$$f(x) = 1 - \frac{1}{b-k}\left[(x-k)_+ - (x-b)_+\right] - \frac{1}{k-a}\left[(k-x)_+ - (a-x)_+\right]\,.$$

For the prices of the Butterfly it follows that

$$V_t = 1 - \frac{1}{b-k}\left[V_t^{c,k} - V_t^{c,b}\right] - \frac{1}{k-a}\left[V_t^{p,k} - V_t^{p,a}\right]\,.$$

d. (5) Find H_t for the Butterfly option.

Solution: use linearity and the fact that for the option $V_T = 1$*,the replicating portfolio equals* $(e^{-r(T-t)}, 0)$*.*