University of Ljubljana, Faculty of Economics

Quantitative finance and actuarial science

Probability and statistics

Written examination

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Name and surname: ______________________ ID: | | | | | | | | |

Instructions

Read the problems carefull before staring your work. There are four problems. You can have a sheet with formulae and a mathematical handbook. Write your solutions on the paper provided. You have two hours.

Problem	a.	b.	c.	d.	
1.			•	•	
2.			•		
3.			•	•	
4.				•	
Total					

1. (25) Products are delivered in batches of size M. For quality control, n batches are selected by simple random sampling out of N batches delivered. In each selected batch a simple random sample of size m is selected. The percentage of defective items is to be estimated. The sampling procedures in selected batches are independent and independent of the selection procedures of batches.

a. (10) Is the sample percentage of defective items an unbiased estimator of the population percentage of defective items. Explain.

Solution: define

$$I_k = \begin{cases} 1 & \text{if batch } k \text{ is selected} \\ 0 & \text{else} \end{cases}$$

for $k = 1, 2, \ldots, N$. *Let* $\bar{Y}_k$ *be the sample proportion estimator based on a simple random sample of size* m *for* $k = 1, 2, \ldots, N$. *Let*

$$\bar{Y} = c \sum_{k=1}^{N} \bar{Y}_k I_k \,.$$

Computing expectations we get

$$E(\bar{Y}) = c \sum_{k=1}^{N} E(Y_k) E(I_k) = c \sum_{k=1}^{N} p_k \cdot \frac{n}{N}$$

where p_k *is the proportion of defective items in batch* k. *On the other hand we have*

$$p = \frac{1}{N} \sum_{k=1}^{N} p_k \,.$$

Letting $c = 1/n$ *makes* $\bar{Y}$ *an unbiased estimate of the overall proportion* p.

b. (15) Denote the proportion of defective items in the k-th batch by p_k for $k = 1, 2, \ldots, N$. Express the standard error of the sample percentage with these quantities.

Solution: compute

$$\begin{aligned} \operatorname{var}(\bar{Y}) &= \operatorname{var}\left(\frac{1}{n} \sum_{k=1}^{N} \bar{Y}_k I_k \right) \\ &= \frac{1}{n^2} \left(\sum_{k=1}^{N} \operatorname{var}\left(\bar{Y}_k I_k\right) + 2 \sum_{k<l} \operatorname{cov}\left(\bar{Y}_k I_k, \bar{Y}_l I_l\right) \right) . \end{aligned}$$

From the text it follows that all the $\bar{Y}_k$ *are independent of* $I_1, \ldots, I_N$. *We have*

$$\operatorname{var}(\bar{Y}_k I_k) = E\left(\bar{Y}_k^2 I_k^2\right) - E(\bar{Y}_k I_k)^2 \,.$$

By independence it follows

$$E\left(\bar{Y}_k^2 I_k^2\right) = E(\bar{Y}_k^2)E(I_k)\,.$$

From the known formula

$$\mathrm{var}(\bar{Y}_k) = \frac{p_k(1-p_k)}{n}\cdot\frac{M-m}{M-1}$$

we have

$$E(\bar{Y}_k^2) = \frac{p_k(1-p_k)}{n}\cdot\frac{M-m}{M-1} + p_k^2\,.$$

Note that $E(I_k) = E(I_k^2) = n/N$. *Furthermore, we have for* $k < l$

$$\mathrm{cov}\left(\bar{Y}_k I_k, \bar{Y}_l I_l\right) = E(\bar{Y}_k\bar{Y}_l I_k I_l) - E(\bar{Y}_k I_k)E(Y_l I_l)\,,$$

and by independence

$$\mathrm{cov}\left(\bar{Y}_k I_k, \bar{Y}_l I_l\right) = p_k p_l E(I_k I_l) - p_k p_l\cdot\frac{n^2}{N^2}\,.$$

From simple random sampling we know

$$E(I_k I_l) = -\frac{n(N-n)}{N^2(N-1)}\,.$$

The formula for covariance simplifies to

$$\mathrm{cov}\left(\bar{Y}_k I_k, \bar{Y}_l I_l\right) = -p_k p_l\cdot\frac{n(N-n)}{N^2(N-1)}\,.$$

Assembling all the quantities gives the variance.

2. (20) The observed values are pairs (x_i, y_i) for $i = 1, 2, \ldots, n$. Assume that the pairs are an i.i.d. sample from the distribution given by the density

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{\frac{(1+\beta^2)}{\alpha}x^2 - 2\beta xy + \alpha y^2}{2}} .$$

for $\alpha, \beta > 0$.

a. (10) Find the maximum likelihood estimates for the parameters α and β.

Solution: the log-likelihood function is

$$\ell\left(\alpha, \beta | \mathbf{x}, \mathbf{y}\right) = n \log(2\pi) - \frac{1}{2}\left(\frac{(1+\beta^2)}{\alpha}\sum_{i=1}^n x_i^2 - 2\beta \sum_{i=1}^n x_i y_i + \alpha \sum_{i=1}^n y_i^2\right) .$$

Denote

$$m_{xx} = \frac{1}{n}\sum_{k=1}^n x_k^2 \,, \qquad m_{xy} = \frac{1}{n}\sum_{k=1}^n x_k y_k \,, \qquad m_{yy} = \frac{1}{n}\sum_{k=1}^n y_k^2 \,,$$

and rewrite

$$\ell(\alpha, \beta | \mathbf{x}, \mathbf{y}) = n\left(-\log(2\pi) - \frac{(1+\beta^2)\, m_{xx}}{2\alpha} + \beta\, m_{xy} - \frac{\alpha\, m_{yy}}{2}\right) .$$

Taking derivatives we get

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{2}\left(\frac{(1+\beta^2)\, m_{xx}}{\alpha^2} - m_{yy}\right) , \qquad \frac{\partial \ell}{\partial \beta} = n\left(-\frac{\beta\, m_{xx}}{\alpha} + m_{xy}\right) .$$

Equating the derivatives to zero, we get

$$\hat{\alpha} = \frac{m_{xx}}{\sqrt{m_{xx}m_{yy} - m_{xy}^2}} \,, \qquad \hat{\beta} = \frac{m_{xy}}{\sqrt{m_{xx}m_{yy} - m_{xy}^2}} .$$

b. (5) Find the density of X.

Hint: note that you are integrating one of the normal densities.

Solution: we need to compute the marginal denisty. Integrating we get

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\frac{(1+\beta^2)}{\alpha}x^2 - 2\beta xy + \alpha y^2}{2}} dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\alpha\left(y - \frac{\beta}{\alpha}x\right)^2 + \frac{x^2}{\alpha}}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\alpha}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}\left(y - \frac{\beta}{\alpha}x\right)^2} dy \\
&= \frac{1}{\sqrt{2\pi}\sqrt{\alpha}} e^{-\frac{x^2}{2\alpha}} .
\end{aligned}$$

It follows that $X \sim \mathrm{N}(0, \alpha)$ and so $E(X^2) = \alpha$.

c. (10) Find the approximate standard errors for the maximum likelihood estimators of α and β.

Solution: let $n = 1$. In this case, the second derivatives are

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{(1+\beta^2)x^2}{\alpha^3}\,, \qquad \frac{\partial^2 \ell}{\partial \alpha\,\partial \beta} = \frac{\beta x^2}{\alpha^2}\,, \qquad \frac{\partial^2 \ell}{\partial \beta^2} = -\frac{x^2}{\alpha}\,.$$

Replace x by X and take expectations. Here we need $E(X^2) = \alpha$. The Fisher information matrix is

$$I(\alpha,\beta) = \begin{pmatrix} \frac{1+\beta^2}{\alpha^2} & -\frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} & 1 \end{pmatrix}\,.$$

Inverting we get

$$I^{-1}(\alpha,\beta) = \begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & 1+\beta^2 \end{pmatrix}\,.$$

The approximate standard errors are

$$\mathrm{se}(\hat{\alpha}) = \frac{\alpha}{\sqrt{n}} \quad \textit{and} \quad \mathrm{se}(\hat{\beta}) = \frac{\sqrt{1+\beta^2}}{\sqrt{n}}\,.$$

3. (25) Assume that your observations are pairs $(x_1, y_1), \ldots, (x_n, y_n)$. Assume the pairs are an i.i.d. sample from the density

$$f_{X,Y}(x, y) = e^{-x} \cdot \frac{1}{\sqrt{2\pi x}\sigma} e^{-\frac{(y-\theta x)^2}{2\sigma^2 x}}$$

for $\sigma > 0$, $x > 0$, $-\infty < y < \infty$. We would like to test the hypothesis

$$H_0\colon \theta = 0 \quad \text{versus} \quad H_1\colon \theta \neq 0\,.$$

a. (10) Find the maximum likelihood estimates for θ and σ.

Solution: the log-likelihood function is

$$\ell\left(\theta, \sigma | \mathbf{x}, \mathbf{y}\right) = \sum_{k=1}^{n} \left(-\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{1}{2}\sum_{k=1}^{n}\log x_k - \frac{(y_k - \theta x_k)^2}{2\sigma^2 x_k} \right)\,.$$

Take partial derivatives to get

$$\begin{aligned} \frac{\partial \ell}{\partial \theta} &= \sum_{k=1}^{n} \frac{(y_k - \theta x_k)}{\sigma^2} \\ \frac{\partial \ell}{\partial \sigma} &= -\frac{n}{\sigma} + \sum_{k=1}^{n} \frac{(y_k - \theta x_k)^2}{\sigma^3 x_k} \end{aligned}$$

Set the partial derivatives to 0. *From the first equation we have*

$$\hat{\theta} = \frac{\sum_{k=1}^{n} y_k}{\sum_{k=1}^{n} x_k}$$

and from the second

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{n} \frac{(y_k - \hat{\theta} x_k)^2}{x_k}\,.$$

b. (15) Find the likelihood ratio statistic for testing the above hypothesis. What is the approximate distribution of the test statistic under H_0?

Solution: if $\theta = 0$ *the log-likelihood functions attains its maximum for*

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{n} \frac{y_k^2}{x_k}\,.$$

It follows that

$$\lambda = -n\log\left(1 - \frac{\left(\sum_{k=1}^{n} y_k\right)^2}{\sum_{k=1}^{n} x_k \sum_{k=1}^{n} \frac{y_k^2}{x_k}}\right)\,.$$

The approximate distribution od λ *is* $\chi^2(1)$.

4. (25) Assume the regression model

$$Y_k = \beta x_k + \epsilon_k$$

for $k = 1, 2, \ldots, n$ where $\epsilon_1, \ldots, \epsilon_n$ are uncorrelated, $E(\epsilon_k) = 0$ and $\mathrm{var}(\epsilon_k) = \sigma^2$ for $k = 1, 2, \ldots, n$. Assume that $x_k > 0$ for all $k = 1, 2, \ldots, n$. Consider the following linear estimators of β:

$$\begin{aligned} \hat{\beta}_1 &= \frac{\sum_{k=1}^n x_k Y_k}{\sum_{k=1}^n x_k^2} \\ \hat{\beta}_2 &= \frac{1}{n}\sum_{k=1}^n \frac{Y_k}{x_k} \\ \hat{\beta}_3 &= \frac{\sum_{k=1}^n Y_k}{\sum_{k=1}^n x_k} \end{aligned}$$

a. (5) Are all estimators unbiased?

Solution: since $E(Y_k) = \beta x_k$ for all $k = 1, 2, \ldots, n$ all the estimators are unbiased.

b. (10) Which of the estimators has the smallest standard error? Justify your answer.

Solution: all the estimators are unbiased. Gauss-Markov tells us that the best estimator is the one given by least squares and that is $\hat{\beta}_1$.

c. (10) Write down the standard errors for all three estimators.

Solution: we first compute the theoretical variances. Since $Y_1, \ldots, Y_n$ are uncorrelated we have

$$\begin{aligned} \mathrm{var}(\hat{\beta}_1) &= \frac{\sigma^2}{\sum_{k=1}^n x_k^2} \\ \mathrm{var}(\hat{\beta}_2) &= \frac{\sigma^2 \sum_{k=1}^n x_k^{-2}}{n^2} \\ \mathrm{var}(\hat{\beta}_3) &= \frac{n\sigma^2}{\left(\sum_{k=1}^n x_k\right)^2}. \end{aligned}$$

We need an unbiassed eswtimate of σ^2. Theoretically, we have that

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{k=1}^n (Y_k - \hat{\beta} x_k)^2$$

is an unbiased estimator σ^2. This gives us an unbiased estimator of σ^2.