Regression with dependent errors

Assume the regression model

$$\begin{aligned} Y_{i1} &= \alpha + \beta x_{i1} + \epsilon_i \\ Y_{i2} &= \alpha + \beta x_{i2} + \eta_i \end{aligned}$$

for $i = 1, 2, \ldots, n$. In other words the observation come in pairs. Assume that $E(\epsilon_i) = E(\eta_i) = 0$, $\mathrm{var}(\epsilon_i) = \mathrm{var}(\eta_i) = \sigma^2$ and $\mathrm{corr}(\epsilon_i, \eta_i) = \rho \in (-1, 1)$. Assume that the pairs $(\epsilon_1, \eta_1), \ldots, (\epsilon_n, \eta_n)$ are uncorrelated. Furthermore assume that

$$\sum_{i=1}^{n} x_{i1} x_{i2} = 0\,.$$

a. Assume that ρ is known. Find the best linear unbiased estimate of the regression parameters α and β. Find an unbiased estimator of σ^2.

b. Assume that ρ is unknown and let $\hat{\alpha}$ and $\hat{\beta}$ be the ordinary least squares estimators of the regression parameters. Compute the standard errors of the two estimators.

c. Let $\hat{\epsilon}_i$ and $\hat{\eta}_i$ be the residuals from ordinary least squares. Express

$$E\left[\sum_{i=1}^{n} \left(\hat{\epsilon}_i^2 + \hat{\eta}_i^2\right)\right]$$

and

$$E\left[\sum_{i=1}^{n} \hat{\epsilon}_i \hat{\eta}_i\right]$$

with the elements of the hat matrix $\mathbf{H}$.

d. Give an estimate of $\mathrm{var}(\hat{\alpha})$ and $\mathrm{var}(\hat{\beta})$. Are the estimators unbiased?