PROBABILITY & STATISTICS

Homework assignment 4 - due with the seminar assignment

Instructions

Please turn in the homework with this cover page. You do not need to edit the solutions. Just make sure the handwriting is legible. You may discuss the problems with your peers but the final solutions should be your work.

Statement: With my signature I confirm that the solutions are the product of my own work. Name: ______________________ Signature: ____________________.

1. Do the following problems from Rice's book:

   Chapter 9: 13, 36, 61.

   Chapter 14: 7, 36.

   DO TWO TESTING AND TWO REGRESSION PROBLEMS BELOW.

2. Suppose $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are i.i.d. observations from a multivariate normal distribution $N(\mu, \mathbf{\Sigma})$ where $\mathbf{\Sigma}$ is known. Further assume that $\mathbf{R}$ is a given matrix and $\mathbf{r}$ a given vector. Use the likelihood ratio procedure to produce a test statistic for
$$H_0\colon \mathbf{R}\mu = \mathbf{r} \qquad \text{vs.} \qquad H_1\colon \mathbf{R}\mu \neq \mathbf{r}\,.$$
   Give explicit formulae for the test statistic and the critical values.

3. Assume that the data $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are an i.i.d. sample from the multivariate normal distribution of the form
$$\mathbf{X}_1 \sim \mathrm{N}\left(\begin{pmatrix}\boldsymbol{\mu}^{(1)}\\ \boldsymbol{\mu}^{(2)}\end{pmatrix}, \begin{pmatrix}\mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12}\\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22}\end{pmatrix}\right)\,.$$
   Assume that the parameters $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ are unknown. Assume the following theorem:

   If $\mathbf{A}(p\times p)$ is a given symmetric positive definite matrix then the positive definite matrix $\mathbf{\Sigma}$ that maximizes the expression
$$\frac{1}{\det(\mathbf{\Sigma})^{n/2}} \cdot \exp\left(-\frac{1}{2}\mathrm{Tr}\left(\mathbf{\Sigma}^{-1}\mathbf{A}\right)\right)$$
   is the matrix
$$\mathbf{\Sigma} = \frac{1}{n}\mathbf{A}\,.$$
   The testing problem is
$$H_0\colon \mathbf{\Sigma}_{12} = 0 \qquad \text{versus} \qquad H_1\colon \mathbf{\Sigma}_{12} \neq 0\,.$$

   a. Find the maximum likelihood estimates of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ in the unconstrained case.

   b. Find the maximum likelihood estimates of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ in the constrained case.

   c. Write the likelihood ratio statistic for the testing problem as explicitly as possible.

   d. What can you say about the distribution of the likelihood ratio statistic?

4. Assume the data pairs $(x_1, y_1), \ldots, (x_n, y_n)$ are an i.i.d. sample from the bivariate normal distribution with parameters

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} .$$

Assume the matrix $\boldsymbol{\Sigma}$ is invertible. We would like to test the hypothesis

$$\mathrm{H}_0 \colon \boldsymbol{\Sigma} \text{ has eigenvalues } \lambda \text{ and } 2\lambda \text{ for some } \lambda > 0$$

versus

$$\mathrm{H}_1 \colon \text{for the eigenvalues } \lambda \text{ and } \mu \text{ of } \boldsymbol{\Sigma} \text{ we have } \lambda/\mu \notin \{2, 1/2\} .$$

a. Find the maximum likelihood estimators of the parameters in the unrestricted case.

b. Show that every symmetric 2×2 matrix with eigenvalues λ and 2λ for $\lambda > 0$ is of the form

$$\begin{pmatrix} \lambda(1+a^2) & -\lambda ab \\ -\lambda ab & \lambda(1+b^2) \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 2\lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for a, b such that $a^2 + b^2 = 1$.

c. Find explicitly the likelihood ratio test for the above testing problem.

Hint: under the assumption $\alpha, \gamma > 0$ and $\alpha\gamma - \beta^2 > 0$ the minimum of the function $f(x, y) = \alpha x^2 + 2\beta xy + \gamma y^2$ subject to the side condition $x^2 + y^2 = 1$ is

$$\frac{\alpha + \gamma}{2} - \frac{\sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2} .$$

d. What can you say about the approximate distribution of the test statistic?

5. Let $\mathbf{Y} = \mathbf{X}\beta + \epsilon$ be a linear model where we assume $E(\epsilon) = \mathbf{0}$ in $\mathrm{var}(\epsilon) = \sigma^2 \boldsymbol{\Sigma}$ for a known invertible matrix $\boldsymbol{\Sigma}$.

a. Show that the BLUE for β is given by

$$\hat{\beta} = (\mathbf{X}^T \boldsymbol{\Sigma}^{-1} \mathbf{X})^{-1} \mathbf{X}^T \boldsymbol{\Sigma}^{-1} \mathbf{Y} .$$

Assume that $\mathbf{X}^T \boldsymbol{\Sigma}^{-1} \mathbf{X}$ is invertible and use the Gauss-Markov theorem.

b. Assume that the linear model is of the form

$$Y_{kl} = \alpha + \beta x_{kl} + u_k + \epsilon_{kl},$$

$k = 1, 2, \ldots, K$ in $l = 1, 2, \ldots, L_k$ where ϵ_{kl} are $N(0, \sigma^2)$ and u_k are $N(0, \tau^2)$ and all random quantities are independent. Assume that the ratio τ^2/σ^2 is known. Show that the BLUE is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \begin{pmatrix} \sum w_k & \sum w_k \bar{x}_k \\ \sum w_k \bar{x}_k & S_{xx} + \sum w_k \bar{x}_k^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum w_k \bar{y}_k \\ S_{xy} + \sum w_k \bar{x}_k \bar{y}_k \end{pmatrix},$$

where

$$\begin{aligned} w_k &= L_k \sigma^2 / (\sigma^2 + L_k \tau^2) \\ S_{xx} &= \sum_k \sum_l (x_{kl} - \bar{x}_k)^2 \\ S_{xy} &= \sum_k \sum_l (x_{kl} - \bar{x}_k)(y_{kl} - \bar{y}_k) \,. \end{aligned}$$

Hint: For $c \neq -1/n$ *one has* $(\mathbf{I} + c\mathbf{1}\mathbf{1}^T)^{-1} = \mathbf{I} - c(1 + nc)^{-1}\mathbf{1}\mathbf{1}^T$ *where* $\mathbf{1}^T = (1, 1, \ldots, 1)$.

c. What would you do if the ratio τ^2/σ^2 were unknown?

d. How would you test the hypothesis $H_0\colon \beta = 0$ versus $H_1\colon \beta \neq 0$? What is the distribution of the test statistic under the null-hypothesis?

6. Assume the data $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are a i.i.d. sample from the multivariate normal distribution with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. The problem is to test the hypothesis

$$H_0\colon \boldsymbol{\mu} = k\boldsymbol{\mu}_0 \qquad \text{versus} \qquad H_i\colon \boldsymbol{\mu} \text{ and } \boldsymbol{\mu_0} \text{ are not colinear.}$$

a. Assume that $\boldsymbol{\Sigma}$ is known and invertible. Find the likelihood ratio test statistic for the above testing problem. What is the approximate distribution of the test statistic if H_0 holds.

b. Find the exact distribution of the test statistic from a. if H_0 is true.

c. Assume $\boldsymbol{\Sigma}$ is unknown but assumed to be invertible. Find the likelihood ratio statistic in this case. You may assume the following: if $\mathbf{A}(p \times p)$ is a given symmetric positive definite matrix then the positive definite matrix $\boldsymbol{\Sigma}$ that maximizes the expression

$$\frac{1}{\det(\boldsymbol{\Sigma})^{n/2}} \cdot \exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}\right)\right)$$

is the matrix

$$\boldsymbol{\Sigma} = \frac{1}{n}\mathbf{A} \,.$$

d. What is the approximate distribution of the test statistic in this case?

7. Assume the usual linear regression model with

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \epsilon$$

where $\mathbf{X}$ is fixed and known and $E(\boldsymbol{\epsilon}) = 0$ and $\text{var}(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}$.

For a given n-dimensional vector $\mathbf{a} \neq 0$ we would like to find the best linear unbiased estimator $\hat{v} = \mathbf{LY}$ with the property $E(\mathbf{LY}) = 0$ of the quantity $v = \mathbf{a}^T\boldsymbol{\epsilon}$ in the sense that the mean squared error

$$E\left[(\hat{v} - v)^2\right]$$

is as small as possible.

a. Show that $\mathbf{LY} = \mathbf{a}^T\left(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\right)$ is the best estimator with the required properties. Here $\hat{\boldsymbol{\beta}}$ is the least squares estimator of the parameter $\boldsymbol{\beta}$.

b. Let $\hat{\boldsymbol{\beta}}$ be the least squares estimator of $\boldsymbol{\beta}$. Compute

$$E\left[\left(\mathbf{a}^T\hat{\boldsymbol{\epsilon}} - \mathbf{a}^T\boldsymbol{\epsilon}\right)^2\right],$$

where $\hat{\boldsymbol{\epsilon}} = (\mathbf{I} - \mathbf{H})\,\mathbf{Y}$ and $\mathbf{H} = \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T$.

c. Let $\mathbf{b}$ be an m-dimensional fixed vector and define

$$v = \mathbf{a}^T\boldsymbol{\epsilon} + \mathbf{b}^T\boldsymbol{\beta}\,.$$

We would like to find the linear estimator $\hat{v} = \mathbf{LY}$ of v such that $E(\hat{v}) = \mathbf{b}^T\boldsymbol{\beta}$ and the mean squared error

$$E\left[(\hat{v} - v)^2\right]$$

is as small as possible. Show first that for linear estimators with the given properties

$$E\left[(\mathbf{LY}) - v)^2\right] = E\left[(\mathbf{LY} - E(v|\mathbf{Y}))^2\right] + E\left[(E(v|\mathbf{Y}) - v)^2\right].$$

d. Assume $\boldsymbol{\epsilon}$ is multivariate normal which means that conditional expectations are linear. Use c. to argue that $\hat{v} = \mathbf{a}^T\hat{\boldsymbol{\epsilon}} + \mathbf{b}^T\hat{\boldsymbol{\beta}}$ is the best linear estimator of v in the above sense. Is the assumption about normality essential?

8. Assume the correct regression model is

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

for $E(\boldsymbol{\epsilon}) = 0$ and $\mathrm{var}(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}$. Assume the matrix $\mathbf{X}$ of dimensions $n \times m$ with $m < n$ has full rank. Denote by $\hat{\boldsymbol{\beta}}$ the ordinary least squares estimator of $\boldsymbol{\beta}$. Assume as known that the upper left corner of the inverse of

$$\begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix},$$

is

$$\boldsymbol{\Sigma}^{11} = \boldsymbol{\Sigma}_{11}^{-1} + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\left(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right)^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1},$$

and the lower right corner is

$$\boldsymbol{\Sigma}^{22} = (\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12})^{-1}.$$

a. Assume that we forget some independent variables and fit the regression model

$$\mathbf{Y} = \mathbf{X}_1\boldsymbol{\beta}_1^* + \boldsymbol{\epsilon}^*,$$

where $\mathbf{X} = [\mathbf{X}_1; \mathbf{X}_2]$ and $E(\boldsymbol{\epsilon}^*) = 0$ and $\mathrm{var}(\boldsymbol{\epsilon}^*) = \sigma^2\mathbf{I}$. Write $\boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}$. Assuming the "wrong" model we estimate $\boldsymbol{\beta}_1$ by $\hat{\boldsymbol{\beta}}_1^* = \left(\mathbf{X}_1^T\mathbf{X}_1\right)^{-1}\mathbf{X}_1^T\mathbf{Y}$. Let $\hat{\boldsymbol{\beta}}_1$ be the best unbiased linear estimator of $\boldsymbol{\beta}_1$ in the correct model. Show that

$$\mathrm{var}(\hat{\boldsymbol{\beta}}_1) - \mathrm{var}(\hat{\boldsymbol{\beta}}_1^*) = \sigma^2\mathbf{A}\mathbf{B}^{-1}\mathbf{A}^T,$$

where

$$\mathbf{A} = \left(\mathbf{X}_1^T\mathbf{X}_1\right)^{-1}\mathbf{X}_1^T\mathbf{X}_2 \quad \text{and} \quad \mathbf{B} = \mathbf{X}_2^T\mathbf{X}_2 - \mathbf{X}_2^T\mathbf{X}_1\mathbf{A}.$$

b. Show that the matrix $\mathbf{A}\mathbf{B}^{-1}\mathbf{A}^T$ is positive semi-definite. This means that $\hat{\boldsymbol{\beta}}_1^*$ has smaller variance than $\hat{\boldsymbol{\beta}}_1$. Why is this not in contradiction with the Gauss-Markov theorem? Explain your answer.

Hint: all the minors of a positive semi-definite matrix are positive semi-definite.

9. Assume the regression model

$$\begin{aligned} Y_{i1} &= \alpha + \beta x_{i1} + \epsilon_i \\ Y_{i2} &= \alpha + \beta x_{i2} + \eta_i \end{aligned}$$

for $i = 1, 2, \ldots, n$. In other words the observation come in pairs. Assume that $E(\epsilon_i) = E(\eta_i) = 0$, $\mathrm{var}(\epsilon_i) = \mathrm{var}(\eta_i) = \sigma^2$ and $\mathrm{corr}(\epsilon_i, \eta_i) = \rho \in (-1, 1)$. Assume that the pairs $(\epsilon_1, \eta_1), \ldots, (\epsilon_n, \eta_n)$ are uncorrelated. Furthermore assume that

$$\sum_{i=1}^{n} x_{i1} x_{i2} = 0\,.$$

a. Assume that ρ is known. Find the best linear unbiased estimate of the regression parameters α and β. Find an unbiased estimator of σ^2.

b. Assume that ρ is unknown and let $\hat{\alpha}$ and $\hat{\beta}$ be the ordinary least squares estimators of the regression parameters. Compute the standard errors of the two estimators.

c. Let $\hat{\epsilon}_i$ and $\hat{\eta}_i$ be the residuals from ordinary least squares. Express

$$E\left[\sum_{i=1}^{n} \left(\hat{\epsilon}_i^2 + \hat{\eta}_i^2\right)\right]$$

and

$$E\left[\sum_{i=1}^{n} \hat{\epsilon}_i \hat{\eta}_i\right]$$

with the elements of the hat matrix $\mathbf{H}$.

d. Give an estimate of $\mathrm{var}(\hat{\alpha})$ and $\mathrm{var}(\hat{\beta})$. Are the estimators unbiased?