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ON G^2 CONTINUOUS SPLINE INTERPOLATION OF CURVES IN $\mathbb{R}^d$

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Abstract.

In this paper the problem of G^2 continuous interpolation of curves in $\mathbb{R}^d$ by polynomial splines of degree n is studied. The interpolation of the data points, and two tangent directions at the boundary is considered. The case $n = r + 2 = d$, where r is the number of interior points interpolated by each segment of the spline curve, is studied in detail. It is shown that the problem is uniquely solvable asymptotically, i.e., when the data points are sampled regularly and sufficiently dense, and lie on a regular, convex parametric curve in $\mathbb{R}^d$. In this case the optimal approximation order is also determined.

AMS subject classification: 65D05, 65D07.

Key words: Spline curve, G^2 continuity, interpolation, approximation order.

1 Introduction.

The problem of parametric polynomial interpolation of data points in $\mathbb{R}^d$ has already been studied in [1]–[5], [7], [9] and [10]. Perhaps [1] gave the first impetus to this subject. In [9], a nice general approach of increasing the approximation order by parametric polynomial interpolation methods has been developed. Here, we apply this general approach to the G^2 continuous spline case.
The geometric continuity is usually chosen in geometric design, since it refers to a particular parametrisation, and assures that the geometric invariants of the curve, i.e., the tangent direction, the curvature, etc., are continuous, but removes the influence of the parametrisation on the shape of the curve. Of course, it is usually sufficient to require G^2 continuity, since it is almost impossible to recognize the discontinuities of the higher order derivatives by the human eye.
The problem in its general form has already been considered in [5] and [7], and can be stated as follows. Let

$$\boldsymbol{B} := \boldsymbol{B}_n : [\zeta_0, \zeta_m] \to \mathbb{R}^d \qquad (1.1)$$

be the polynomial spline curve of degree n composed by m segments with breakpoint sequence

$$\zeta_0 < \zeta_1 < \ldots < \zeta_m.$$

Suppose points

$$\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_N \in \mathbb{R}^d, \quad \boldsymbol{T}_j \neq \boldsymbol{T}_{j+1}, \quad j = 0, 1, \ldots, N-1, \tag{1.2}$$

and tangent directions

$$\boldsymbol{d}_0, \ \boldsymbol{d}_N, \tag{1.3}$$

at the boundary points $\boldsymbol{T}_0$ and $\boldsymbol{T}_N$ are given. Find a polynomial spline $\boldsymbol{B}$ defined by (1.1) which is G^2 continuous and interpolates given points and tangent directions.

Locally, on the ℓ-th segment, $\boldsymbol{B}$ can be given as

$$\boldsymbol{B}(\zeta) =: \boldsymbol{B}^\ell\left(\frac{\zeta - \zeta_{\ell-1}}{\Delta\zeta_{\ell-1}}\right), \quad \zeta \in [\zeta_{\ell-1}, \zeta_\ell], \quad \ell = 1, \ldots, m, \tag{1.4}$$

where $\Delta\zeta_{\ell-1} := \zeta_\ell - \zeta_{\ell-1}$. Since $\boldsymbol{B}$ has to interpolate the data (1.2) and (1.3), its ℓ-th polynomial piece $\boldsymbol{B}^\ell$ should interpolate r interior and two boundary points. The interpolating conditions on the ℓ-th segment now read

$$\boldsymbol{B}^\ell(t_{\ell,j}) = \boldsymbol{T}_{(r+1)(\ell-1)+j} =: \boldsymbol{T}_{\ell,j}, \quad j = 0, 1, \ldots, r+1, \tag{1.5}$$

where

$$t_{\ell,0} := 0 < t_{\ell,1} < \cdots < t_{\ell,r} < t_{\ell,r+1} := 1,$$

and $(t_{\ell,j})_{j=1}^r$ are unknown parameter values which have to be determined. It remains to fulfil the G continuity conditions. The G^1 continuity can be written in the local parametrisation, i.e., t_ℓ, $0 \le t_\ell \le 1$, as

$$\begin{aligned} \frac{d}{dt_1}\boldsymbol{B}^1(0) &= \alpha_0\, \boldsymbol{d}_0, \\ \frac{d}{dt_{\ell+1}}\boldsymbol{B}^{\ell+1}(0) &= \alpha_\ell\, \frac{d}{dt_\ell}\boldsymbol{B}^\ell(1), \quad \ell = 1, 2, \ldots, m-1, \\ \frac{d}{dt_m}\boldsymbol{B}^m(1) &= \alpha_m\, \boldsymbol{d}_N, \end{aligned} \tag{1.6}$$

and the G^2 continuity relations read

$$\frac{d^2}{dt_{\ell+1}^2}\boldsymbol{B}^{\ell+1}(0) = \alpha_\ell^2\, \frac{d^2}{dt_\ell^2}\boldsymbol{B}^\ell(1) + \beta_\ell\, \frac{d}{dt_\ell}\boldsymbol{B}^\ell(1), \quad \ell = 1, \ldots, m-1, \tag{1.7}$$

where α_ℓ and β_ℓ are unknowns and $\alpha_\ell > 0$. Fig. 1.1 shows a particular polynomial piece $\boldsymbol{B}^\ell$ that joins its neighbors.

As already observed in [5], the assumption that the number of independent equations should be equal to the number of unknowns implies the following relation

$$d\,n - (d-1)\,r = 3\,d - 2. \tag{1.8}$$

This leads to two practically important cases, $n = r + 2 = d$ and $n = r + 1 = 2\,d - 1$, i.e., the interpolation by polynomial splines of low degree.

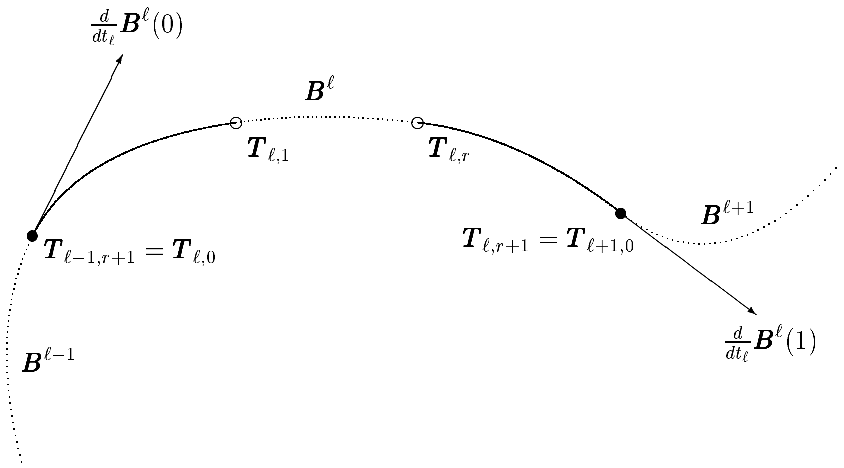

Figure 1.1: A particular segment of the spline curve and some of the important quantities.

The particular case $d = 3$ was asymptotically (i.e. for the data sampled on a regular curve dense enough) solved in [5]. Its extension to general d, but for segments only, can be found in [7]. Here we tackle the composite case. For the second case, i.e., $n = r + 1 = 2\,d - 1$, the approach explained here fails and will require some additional research.

The main result of the paper can be stated as follows.

THEOREM 1.1. *If the data points are sampled regularly and sufficiently dense, then there exists a G^2 polynomial spline curve $\boldsymbol{B}$, which interpolates given data points and tangent directions at the boundary. The approximation order is optimal, i.e., $r + 4 = n + 2$.*

The regularity of data point mentioned in the above theorem should be explained. It is clear that the solution depends on the structure of the data and we can not expect that the problem will be solvable for the arbitrary set of points. Since the asymptotic analysis will be done, the data points will be considered as points on a smooth curve. They will be sampled regularly in the sense that the distribution of points according to the arc length remains the same for all segments. And sufficiently dense means that the neighboring points are close enough to each other.

It is also clear that the approximation order is optimal since we have $r + 4$ interpolation conditions on each segment ($r + 2$ interpolated points and two conditions for two joining segments concerning G^1 and G^2 continuity, i.e., one condition for each segment at the breakpoint).

2 The system of nonlinear equations for $n = r + 2 = d$.

In this section, the system of nonlinear equations (1.5)–(1.7) will be transformed further for this particular case. Since $n = r + 2$, the polynomial curve

$$\boldsymbol{B}^{\ell}(t) := \boldsymbol{b}_{\ell}\, \omega_{\ell}(t) + \sum_{j=0}^{r+1} \mathcal{L}_{\ell,j}(t)\boldsymbol{T}_{\ell,j} \tag{2.1}$$

will satisfy the interpolation conditions (1.5) on $[\zeta_{\ell-1}, \zeta_{\ell}]$. Here

$$\omega_{\ell}(t) := \prod_{j=0}^{r+1}(t - t_{\ell,j}),\ \mathcal{L}_{\ell,j}(t) = \frac{\omega_{\ell}(t)}{(t - t_{\ell,j})\dot{\omega}_{\ell}(t_{\ell,j})},\ \dot{\omega}_{\ell} := \frac{d}{dt_{\ell}}\omega_{\ell}.$$

Provided $t_{\ell,j}$ are known, the unknown leading coefficient vector $\boldsymbol{b}_{\ell}$ can be expressed in two different ways

$$\begin{aligned} \boldsymbol{b}_{\ell} &= [t_{\ell,0}, t_{\ell,0}, t_{\ell,1}, \ldots, t_{\ell,r}, t_{\ell,r+1}]\boldsymbol{B}^{\ell} \\ &= [t_{\ell,0}, t_{\ell,1}, \ldots, t_{\ell,r}, t_{\ell,r+1}, t_{\ell,r+1}]\boldsymbol{B}^{\ell}. \end{aligned} \tag{2.2}$$

Since $t_{\ell,0} = 0$, $t_{\ell,r+1} = 1$, and for a smooth f

$$[x_0, x_0, x_1, \ldots, x_i]f = \sum_{j=0}^{i} \frac{1}{\dot{\omega}(x_j)}[x_0, x_j]f, \tag{2.3}$$

the equation (2.2) can be rewritten as

$$\begin{aligned} \boldsymbol{b}_{\ell} &= \frac{1}{\dot{\omega}_{\ell}(0)}\dot{\boldsymbol{B}}^{\ell}(0) + \sum_{j=1}^{r+1} \frac{1}{\dot{\omega}_{\ell}(t_{\ell,j})t_{\ell,j}}(\boldsymbol{T}_{\ell,j} - \boldsymbol{T}_{\ell,0}) \\ &= \frac{1}{\dot{\omega}_{\ell}(1)}\dot{\boldsymbol{B}}^{\ell}(1) + \sum_{j=0}^{r} \frac{1}{\dot{\omega}_{\ell}(t_{\ell,j})(t_{\ell,j} - 1)}(\boldsymbol{T}_{\ell,j} - \boldsymbol{T}_{\ell,r+1}). \end{aligned} \tag{2.4}$$

Inserting (2.1) into (1.7) and using (2.4), (1.6) in order to replace $\boldsymbol{b}_{\ell}$, $\boldsymbol{b}_{\ell+1}$ by $\dot{\boldsymbol{B}}^{\ell}(1)$, one concludes that

$$\begin{aligned} \dot{\boldsymbol{B}}^{\ell}(1) &= \gamma_{\ell} \left(\alpha_{\ell}^2 \left(\sum_{j=0}^{r} \frac{\ddot{\omega}_{\ell}(1)}{\dot{\omega}_{\ell}(t_{\ell,j})(t_{\ell,j} - 1)}(\boldsymbol{T}_{\ell,j} - \boldsymbol{T}_{\ell,r+1}) + \sum_{j=0}^{r+1} \ddot{\mathcal{L}}_{\ell,j}(1)\boldsymbol{T}_{\ell,j} \right) \right. \\ &\quad \left. - \left(\sum_{j=1}^{r+1} \frac{\ddot{\omega}_{\ell+1}(0)}{\dot{\omega}_{\ell+1}(t_{\ell+1,j})\, t_{\ell+1,j}}(\boldsymbol{T}_{\ell+1,j} - \boldsymbol{T}_{\ell+1,0}) + \sum_{j=0}^{r+1} \ddot{\mathcal{L}}_{\ell+1,j}(0)\boldsymbol{T}_{\ell+1,j} \right) \right). \end{aligned} \tag{2.5}$$

Here

$$\gamma_{\ell} = \left(\frac{\ddot{\omega}_{\ell+1}(0)}{\dot{\omega}_{\ell+1}(0)}\, \alpha_{\ell} - \frac{\ddot{\omega}_{\ell}(1)}{\dot{\omega}_{\ell}(1)}\, \alpha_{\ell}^2 - \beta_{\ell} \right)^{-1}.$$

If we write

$$\begin{aligned}
\boldsymbol{G}_0^\ell &= \sum_{j=1}^{r+1} \frac{\ddot{\omega}_{\ell+1}(0)}{\dot{\omega}_{\ell+1}(t_{\ell+1,j})\, t_{\ell+1,j}} (\boldsymbol{T}_{\ell+1,j} - \boldsymbol{T}_{\ell+1,0}) + \sum_{j=0}^{r+1} \ddot{\mathcal{L}}_{\ell+1,j}(0) \boldsymbol{T}_{\ell+1,j},\\
\boldsymbol{G}_1^\ell &= \sum_{j=0}^{r} \frac{\ddot{\omega}_{\ell}(1)}{\dot{\omega}_{\ell}(t_{\ell,j})(t_{\ell,j}-1)} (\boldsymbol{T}_{\ell,j} - \boldsymbol{T}_{\ell,r+1}) + \sum_{j=0}^{r+1} \ddot{\mathcal{L}}_{\ell,j}(1) \boldsymbol{T}_{\ell,j},
\end{aligned}$$

and

$$\boldsymbol{H}_\ell = \sum_{j=0}^{r} \frac{1}{\dot{\omega}_\ell(t_{\ell,j})\,(t_{\ell,j}-1)} (\boldsymbol{T}_{\ell,j} - \boldsymbol{T}_{\ell,r+1}) - \sum_{j=1}^{r+1} \frac{1}{\dot{\omega}_\ell(t_{\ell,j})\, t_{\ell,j}} (\boldsymbol{T}_{\ell,j} - \boldsymbol{T}_{\ell,0}),$$

then relation (1.6) leads by of (2.4), (2.5) to the following system of nonlinear equations

$$\begin{aligned}
\boldsymbol{F}_1 &:= \frac{\gamma_1}{\dot{\omega}_1(1)}(\alpha_1^2 \boldsymbol{G}_1^1 - \boldsymbol{G}_0^1) + \boldsymbol{H}_1 - \frac{\alpha_0}{\dot{\omega}_1(0)} \boldsymbol{d}_0 = \boldsymbol{0},\\
\boldsymbol{F}_\ell &:= \frac{\gamma_\ell}{\dot{\omega}_\ell(1)}(\alpha_\ell^2 \boldsymbol{G}_1^\ell - \boldsymbol{G}_0^\ell) + \boldsymbol{H}_\ell\\
&\quad - \frac{\alpha_{\ell-1}\gamma_{\ell-1}}{\dot{\omega}_\ell(0)}(\alpha_{\ell-1}^2 \boldsymbol{G}_1^{\ell-1} - \boldsymbol{G}_0^{\ell-1}), \quad \ell = 2, 3, \ldots, m-1,\\
\boldsymbol{F}_m &:= \frac{\alpha_m}{\dot{\omega}_m(1)} \boldsymbol{d}_N + \boldsymbol{H}_m - \frac{\alpha_{m-1}\gamma_{m-1}}{\dot{\omega}_m(0)}(\alpha_{m-1}^2 \boldsymbol{G}_1^{m-1} - \boldsymbol{G}_0^{m-1}).
\end{aligned} \tag{2.6}$$

Note that the equations for the first and for the last segment are different since $\boldsymbol{d}_0$ and $\boldsymbol{d}_N$ are given. Thus one has $m\,d$ nonlinear equations for $m\,d$ scalar unknowns

$$(t_{\ell,j})_{\ell=1,\,j=1}^{m,\,r}, \quad (\alpha_\ell)_{\ell=0}^{m}, \quad (\gamma_\ell)_{\ell=1}^{m-1}.$$

The best way to tackle such systems of nonlinear equations turned out to be the homotopy continuation methods, as was reported in [3]–[5].

3 Asymptotic analysis.

Since the system of nonlinear equations obtained in the previous section is difficult to analyze in full generality, the asymptotic analysis will be applied here. Under certain conditions the existence of the unique asymptotic solution of the system (2.6) will be proved. This will be done in the following way. First, the solution at the limit point will be found. Then the regularity of the Jacobian matrix of the analyzed system at the limit point will be proven. Since it is very difficult to do it in the general case, the regular distribution of the data points (i.e. the distribution of the parameter values at which the underlying curve match the data in the arc length parametrisation is equal for all the segments) will be required, which assures the Jacobian being the Toeplitz-like matrix. It makes it easier to analyze, since one can apply the general theory of systems of difference equations. After that the implicit function theorem will be used to establish the existence of the solution in the neighborhood of the limit solution.

Suppose that the data (1.2) and (1.3) are based upon a smooth regular parametric curve $\boldsymbol{f} : [0, L] \to \mathbb{R}^d$, parametrized by the arc length parameter. Obviously, (2.6) involves data and unknowns from three consecutive segments only. Let us recall the fact, observed in [3], that each block of equations can be simplified by some translation and rotation. So $\boldsymbol{f}$ can be, without loosing generality, locally parametrized by the arc length parameter s as

$$\boldsymbol{f}^{\ell}(s), \quad s \in [-h_{\ell-1}, h_\ell + h_{\ell+1}], \tag{3.1}$$

with $h_\ell := \delta_\ell\, h$ and bounded global mesh ratio $0 < \delta_0 \le \delta_\ell \le 1$, where

$$\boldsymbol{f}^{\ell}(0) = \boldsymbol{T}_{\ell,0} := \mathbf{0}, \quad \frac{d}{ds}\boldsymbol{f}^{\ell}(0) = \boldsymbol{e}_1.$$

Here $\boldsymbol{e}_i$ is the unit vector, i.e.,

$$\boldsymbol{e}_i(0) := [\underbrace{0, 0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{d-i}]^T.$$

The domain of the definition of the arc length parameter s in (3.1) was chosen to assure that the origin of the local coordinate system is in $\boldsymbol{T}_{\ell,0}$ at $s = 0$, and s runs over the three neighboring segments of the underlying curve $\boldsymbol{f}$ which are involved in the particular set of equations which has to be analyzed.

If the κ_i^ℓ, $i = 1, 2, \ldots, d-1$, are the first $d-1$ principal curvatures of $\boldsymbol{f}^\ell$, expanded locally as

$$\kappa_i^\ell(s) = \kappa_{i,0}^\ell + \frac{1}{1!}\kappa_{i,1}^\ell\, s + \frac{1}{2!}\kappa_{i,2}^\ell\, s^2 + \ldots,$$

and $\boldsymbol{f}^\ell$ is a regular curve in the sense that

$$\frac{d}{ds}\boldsymbol{f}^\ell, \ldots, \frac{d^{d-1}}{ds^{d-1}}\boldsymbol{f}^\ell$$

are linearly independent vectors in $\mathbb{R}^d$, then

$$\kappa_{i,0}^\ell \ge \text{const} > 0, \quad i = 1, 2, \ldots, d-2.$$

Additionally, we shall assume that $\kappa_{d-1,0}^\ell \ge \text{const} > 0$. With the aid of the Frenet-Serret formulae and the expansion of the principal curvatures, one obtains the local expansion

$$\begin{aligned}
\boldsymbol{f}^\ell(s) &= \boldsymbol{f}^\ell(0) + \frac{d}{ds}\boldsymbol{f}^\ell(0)\, s + \frac{1}{2!}\frac{d^2}{ds^2}\boldsymbol{f}^\ell(0)\, s^2 + \cdots \\
&= \boldsymbol{f}^\ell(0) + (s - \frac{1}{6}(\kappa_{1,0}^\ell)^2\, s^3 + \cdots)\, \boldsymbol{e}_1 \\
&+ (\frac{1}{2}\kappa_{1,0}^\ell\, s^2 + \frac{1}{6}\kappa_{1,1}^\ell\, s^3 + \cdots)\, \boldsymbol{e}_2 + (\frac{1}{6}\kappa_{1,0}^\ell\, \kappa_{2,0}^\ell\, s^3 + \cdots)\, \boldsymbol{e}_3 + \cdots
\end{aligned} \tag{3.2}$$

Using (3.1), the data points can be written as

$$\begin{aligned} \boldsymbol{T}_{\ell-1,j} &= \boldsymbol{f}^{\ell}(h_{\ell-1}\,(\eta_{\ell-1,j}-1)),\\ \boldsymbol{T}_{\ell,j} &= \boldsymbol{f}^{\ell}(h_{\ell}\,\eta_{\ell,j}),\\ \boldsymbol{T}_{\ell+1,j} &= \boldsymbol{f}^{\ell}(h_{\ell}+h_{\ell+1}\,\eta_{\ell+1,j}), \end{aligned} \tag{3.3}$$

where $\eta_{\ell,0} := 0 < \eta_{\ell,1} < \cdots < \eta_{\ell,r} < \eta_{\ell,r+1} := 1$, $\ell = 1,2,\ldots,m$, are given parameter values. The expansion (3.2) and the equations (3.3) finally give

$$\boldsymbol{T}_{\ell,j} = \left[h^i\,\delta_\ell^i\,\eta_{\ell,j}^i\,\frac{1}{i!}\prod_{q=1}^{i-1}\kappa_{q,0}^{\ell} + \mathcal{O}(h^{i+1})\right]_{i=1}^{d}. \tag{3.4}$$

Applying these expansions to the equations (2.6) and multiplying them by D_ℓ^{-1}, where

$$D_\ell = \operatorname{diag}\left(h, \frac{1}{2!}\,h^2\prod_{q=1}^{1}\kappa_{q,0}^{\ell},\ldots,\frac{1}{d!}\,h^d\prod_{q=1}^{d-1}\kappa_{q,0}^{\ell}\right),$$

the normalized system of nonlinear equations reads

$$\widetilde{\boldsymbol{F}}_\ell := D_\ell^{-1}\,\boldsymbol{F}_\ell = \boldsymbol{0}, \quad \ell = 1,2,\ldots,m. \tag{3.5}$$

Let

$$\begin{aligned} \widetilde{\boldsymbol{T}}_{\ell-1,j} &:= D_\ell^{-1}\boldsymbol{T}_{\ell-1,j} = [\,\delta_{\ell-1}^i\,(\eta_{\ell-1,j}-1)^i\,]_{i=1}^d + \mathcal{O}(h),\\ \widetilde{\boldsymbol{T}}_{\ell,j} &:= D_\ell^{-1}\boldsymbol{T}_{\ell,j} = [\,\delta_\ell^i\,\eta_{\ell,j}^i\,]_{i=1}^d + \mathcal{O}(h),\\ \widetilde{\boldsymbol{T}}_{\ell+1,j} &:= D_\ell^{-1}\boldsymbol{T}_{\ell+1,j} = [\,(\delta_\ell+\delta_{\ell+1}\,\eta_{\ell+1,j})^i\,]_{i=1}^d + \mathcal{O}(h), \end{aligned} \tag{3.6}$$

and

$$\widetilde{\alpha}_0 := \frac{\alpha_0}{h}, \quad \widetilde{\alpha}_m := \frac{\alpha_m}{h}.$$

The limit solution of the normalized system is given by the following lemma.

LEMMA 3.1. *As $h \to 0$, the solution of the system (3.5) is*

$$\begin{aligned} \widetilde{\alpha}_0^* &= \delta_1\\ \alpha_\ell^* &= \frac{\delta_{\ell+1}}{\delta_\ell}, \quad \ell = 1,2,\ldots,m-1\\ \widetilde{\alpha}_m^* &= \delta_m\\ t_{\ell,j}^* &= \eta_{\ell,j}, \quad \ell = 1,2,\ldots,m, \quad j = 1,2,\ldots,r.\\ \gamma_\ell^* &= \left(\frac{\delta_{\ell+1}}{\delta_\ell}\frac{\ddot{\omega}_{\ell+1}^*(0)}{\dot{\omega}_{\ell+1}^*(0)} - \frac{\delta_{\ell+1}^2}{\delta_\ell^2}\frac{\ddot{\omega}_\ell^*(1)}{\dot{\omega}_\ell^*(1)}\right)^{-1}, \quad \ell = 1,2,\ldots,m-1, \end{aligned} \tag{3.7}$$

where

$$\omega_\ell^*(t) = \prod_{j=0}^{r+1}(t - t_{\ell,j}^*).$$

PROOF. The proof is rather technical and some details will be omitted. The limit behavior of the relation (3.5) as $h \to 0$ has to be shown. Since all the calculations are similar, we will, e.g., show how

$$\lim_{h\to 0} D_\ell^{-1} \boldsymbol{G}_0^{\ell-1} = \sum_{j=1}^{r+1} \frac{\ddot{\omega}_\ell^*(0)}{\dot{\omega}_\ell^*(t_{\ell,j}^*)\, t_{\ell,j}^*} \lim_{h\to 0} (\widetilde{\boldsymbol{T}}_{\ell,j} - \widetilde{\boldsymbol{T}}_{\ell,0}) + \sum_{j=0}^{r+1} \ddot{\mathcal{L}}_{\ell,j}^*(0) \lim_{h\to 0} \widetilde{\boldsymbol{T}}_{\ell,j}, \tag{3.8}$$

can be simplified to

$$2\,\delta_\ell^2\, \boldsymbol{e}_2 - \frac{\dddot{\omega}_\ell^*(0)}{\dot{\omega}_\ell^*(0)} \delta_\ell \boldsymbol{e}_1.$$

Here

$$\mathcal{L}_{\ell,j}^*(t) = \frac{\omega_\ell^*(t)}{(t - t_{\ell,j}^*)\dot{\omega}_\ell^*(t_{\ell,j}^*)}.$$

Recall that (3.8) is a part of the nonlinear equation $\widetilde{\boldsymbol{F}}_\ell = \boldsymbol{0}$ considered at the limit point.
For the first part of expression (3.8), the relation (3.6) and well-known properties of the divided differences are used, which gives

$$\begin{aligned}
&\sum_{j=1}^{r+1} \frac{\ddot{\omega}_\ell^*(0)}{\dot{\omega}_\ell^*(t_{\ell,j}^*)\, t_{\ell,j}^*} \lim_{h\to 0} (\widetilde{\boldsymbol{T}}_{\ell,j} - \widetilde{\boldsymbol{T}}_{\ell,0}) = \ddot{\omega}_\ell^*(0) \sum_{j=1}^{r+1} \frac{1}{\dot{\omega}_\ell^*(t_{\ell,j}^*)} [\, \delta_\ell^i\, {t_{\ell,j}^*}^{i-1}\,]_{i=1}^d \\
= \;& \ddot{\omega}_\ell^*(0) [t_{\ell,0}^*, t_{\ell,1}^*, \ldots, t_{\ell,r+1}^*] [\, \delta_\ell^i \,\cdot^{i-1}\,]_{i=1}^d - \frac{\ddot{\omega}_\ell^*(0)}{\dot{\omega}_\ell^*(0)} [\delta_\ell^i\, {t_{\ell,0}^*}^{i-1}]_{i=1}^d \\
= \;& \ddot{\omega}_\ell^*(0)\, \delta_\ell^d\, \boldsymbol{e}_d - \frac{\ddot{\omega}_\ell^*(0)}{\dot{\omega}_\ell^*(0)}\, \delta_\ell\, \boldsymbol{e}_1.
\end{aligned}$$

For the second part recall also the identities

$$t^q = \sum_{j=0}^{r+1} {t_{\ell,j}^*}^q \mathcal{L}_{\ell,j}^*(t), \quad q = 0, 1, \ldots, r+1,$$

and

$$t^{r+2} = \omega_\ell^*(t) + \sum_{j=0}^{r+1} {t_{\ell,j}^*}^{r+2} \mathcal{L}_{\ell,j}^*(t),$$

which are applied to obtain

$$\sum_{j=0}^{r+1} \ddot{\widetilde{\mathcal{L}}}_{\ell,j}^*(0) \lim_{h\to 0} \widetilde{\boldsymbol{T}}_{\ell,j} = 2\,\delta_\ell^2\, \boldsymbol{e}_2 - \delta_\ell^d\, \ddot{\omega}_\ell^*(0)\, \boldsymbol{e}_d.$$

Similarly the other terms of (3.5) are handled and it is easy to show that they sum to zero which completes the proof of the lemma. □

It remains to prove the regularity of the Jacobian of the nonlinear system (3.5) at the limit solution. To simplify the calculation of the partial derivatives, it is more convenient to reverse the role of the unknowns and the parameters, as in [9]. The implicit function theorem can then be applied at the end to complete the proof. So let us assume for a while that $\boldsymbol{t}_\ell = (t_{\ell,j})_{\ell=1,j=1}^{m,r}$ are given parameters, and $\boldsymbol{\eta}_\ell = (\eta_{\ell,j})_{\ell=1,j=1}^{m,r}$ are the unknowns. Explicit computation of the Jacobian requires a particular ordering of the unknowns. The following one will be considered:

$$\widetilde{\alpha}_0, \boldsymbol{\eta}_1, \gamma_1, \alpha_1, \boldsymbol{\eta}_2, \gamma_2 \ldots, \gamma_{m-1}, \alpha_{m-1}, \boldsymbol{\eta}_m, \widetilde{\alpha}_m.$$

This ordering and the fact that there are only three neighboring segments involved in the particular set of equations of the nonlinear system (3.5), imply that the Jacobian is a block tridiagonal matrix. But the study of its regularity is still very difficult, and some further assumptions are needed. Suppose that the data points are regularly sampled. It means that the lengths of the segments of the curve $\boldsymbol{f}$ are all equal, i.e, $\delta_\ell = 1$ for all ℓ, and the components of the vector $\boldsymbol{t}_\ell$ (which are parameters now) are equally distributed on each segment, i.e.,

$$t_j := t_{\ell,j}, \quad \ell = 1, 2, \ldots, m, \quad j = 0, 1, \ldots, r+1.$$

The limit solution from lemma 3.1 then becomes

$$(3.9) \qquad \begin{aligned} \widetilde{\alpha}_0^* &= \alpha_\ell^* = \widetilde{\alpha}_m^* = 1, \quad \ell = 1, 2, \ldots, m-1, \\ \eta_{\ell,j}^* &= t_j, \quad \ell = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, r, \\ \gamma_\ell^* &= \gamma^* = \left(\frac{\ddot{\omega}(0)}{\dot{\omega}(0)} - \frac{\ddot{\omega}(1)}{\dot{\omega}(1)} \right)^{-1}, \ \omega(t) = \prod_{j=0}^{r+1} (t - t_j), \end{aligned}$$

and the Jacobian, say J_m, is a block tridiagonal Toeplitz-like matrix. This property will be used to prove the regularity of J_m at least for m large enough which will imply the theorem 1.1 stated in the introduction.

What follows now is the technical part of the proof of the theorem.

First the Jacobian J_m of the system at the limit point will be derived. If the notation $a := \dot{\omega}(1)/\dot{\omega}(0)$, $b := \ddot{\omega}(1)/\dot{\omega}(1)$, $c := 2\,\gamma^*$, and $u_j = t_j/(t_j - 1)$ is used, the columns of J_m arising from the ℓ-th segment $(1 < \ell < m)$ of the normalized system (3.5) and computed at the limit (3.9), are

$$\begin{aligned} \frac{\partial}{\partial \eta_{\ell-1,j}} \widetilde{\boldsymbol{F}}_\ell &= \frac{a\,c}{(t_j - 1)^2\, \dot{\omega}(t_j)} [\, i\, (t_j - 1)^{i-1}\,]_{i=1}^d, \\ \frac{\partial}{\partial \gamma_{\ell-1}} \widetilde{\boldsymbol{F}}_\ell &= -\frac{2}{\dot{\omega}(0)\, c} [1, 0, \ldots, 0]^T, \\ \frac{\partial}{\partial \alpha_{\ell-1}} \widetilde{\boldsymbol{F}}_\ell &= -\frac{1}{\dot{\omega}(0)} [1 - b\,c, 2\,c, 0, \ldots, 0]^T, \\ \frac{\partial}{\partial \eta_{\ell,j}} \widetilde{\boldsymbol{F}}_\ell &= \frac{1 - c\,(u_j + 1/u_j)}{t_j\,(t_j - 1)\, \dot{\omega}(t_j)} [\, i\, t_j^{i-1}\,]_{i=1}^d, \\ \frac{\partial}{\partial \gamma_\ell} \widetilde{\boldsymbol{F}}_\ell &= \frac{2}{c\, \dot{\omega}(1)} [\, i\,]_{i=1}^d, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial \alpha_\ell}\widetilde{\boldsymbol{F}}_\ell &= \frac{c}{\dot\omega(1)}[\, i\,(i-1-b)\,]_{i=1}^d,\\
\frac{\partial}{\partial \eta_{\ell+1,j}}\widetilde{\boldsymbol{F}}_\ell &= \frac{c}{a\,t_j^2\,\dot\omega(t_j)}[\, i\,(t_j+1)^{i-1}\,]_{i=1}^d.
\end{aligned}
$$

Other unknowns are not involved in the equations for the ℓ-th segment and the corresponding partial derivatives are zero. Similarly the columns of the first and the last diagonal blocks are derived. Multiplication of the obtained matrix by $L = \operatorname{diag}(L_1, L_2, \ldots, L_m)$, where $L_i = \operatorname{diag}(1, 1/2, \ldots, 1/d)$, $i = 1, 2, \ldots, m$, from the left, and by $R = \operatorname{diag}(R_1, R_2, \ldots, R_m)$,

$$
\begin{aligned}
R_1 &= \operatorname{diag}(-\dot\omega(0), \boldsymbol{v}^T, \dot\omega(1)\,c/2),\\
R_j &= \operatorname{diag}(-\dot\omega(0)/c, \boldsymbol{v}^T, \dot\omega(1)\,c/2), \quad j = 2, 3, \ldots, m-1,\\
R_m &= \operatorname{diag}(-\dot\omega(0)/c, \boldsymbol{v}^T, \dot\omega(1)),
\end{aligned}
$$

where $\boldsymbol{v} = [\, t_j\,(t_j-1)\,\dot\omega(t_j)/c\,]_{j=1}^r$, from the right, produces the matrices A_1, A, B, C and A_2,

$$
A_1 = \begin{bmatrix}
1 & (1/c-u_1)\,t_1^0 & \cdots & (1/c-u_r)\,t_r^0 & 1\\
0 & (1/c-u_1)\,t_1^1 & \cdots & (1/c-u_r)\,t_r^1 & 1\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & (1/c-u_1)\,t_1^{d-1} & \cdots & (1/c-u_r)\,t_r^{d-1} & 1
\end{bmatrix},
$$

$$
A = \begin{bmatrix}
1/c-b & a_{11} & a_{12} & \cdots & a_{1r} & 1\\
1 & a_{21} & a_{22} & \cdots & a_{2r} & 1\\
0 & a_{31} & a_{32} & \cdots & a_{3r} & 1\\
\vdots & \vdots & \vdots & & & \\
0 & a_{d1} & a_{d2} & \cdots & a_{dr} & 1
\end{bmatrix},
$$

where $a_{kj} := (1/c - u_j - 1/u_j)\,t_j^{k-1}$,

$$
B = \begin{bmatrix}
b/a & (t_1+1)^0/(a\,u_1) & \cdots & (t_r+1)^0/(a\,u_r) & 0\\
-(1-b)/a & (t_1+1)^1/(a\,u_1) & \cdots & (t_r+1)^1/(a\,u_r) & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
-(d-1-b)/a & (t_1+1)^{d-1}/(a\,u_1) & \cdots & (t_r+1)^{d-1}/(a\,u_r) & 0
\end{bmatrix},
$$

$$
C = \begin{bmatrix}
0 & a\,u_1\,(t_1-1)^0 & \cdots & a\,u_r\,(t_r-1)^0 & -a\\
0 & a\,u_1\,(t_1-1)^1 & \cdots & a\,u_r\,(t_r-1)^1 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & a\,u_1\,(t_1-1)^{d-1} & \cdots & a\,u_r\,(t_r-1)^{d-1} & 0
\end{bmatrix},
$$

and

$$
A_2 = \begin{bmatrix}
(1/c-b) & (1/c-1/u_1)\,t_1^0 & \cdots & (1/c-1/u_r)\,t_r^0 & 1\\
1 & (1/c-1/u_1)\,t_1^1 & \cdots & (1/c-1/u_r)\,t_r^1 & 1\\
0 & (1/c-1/u_1)\,t_1^2 & \cdots & (1/c-1/u_r)\,t_r^2 & 1\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & (1/c-1/u_1)\,t_1^{d-1} & \cdots & (1/c-1/u_r)\,t_r^{d-1} & 1
\end{bmatrix}.
$$

Note that all these matrices are quadratic since $d = r + 2$.

The transformed Jacobian, say $\widetilde{J}_m$, now becomes

$$\widetilde{J}_m := L\, J_m\, R = \begin{bmatrix} A_1 & B & 0 & 0 & \cdots & \cdots & 0 \\ C & A & B & 0 & \cdots & \cdots & 0 \\ 0 & C & A & \ddots & \ddots & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & A & B & 0 \\ \vdots & \vdots & \vdots & \ddots & C & A & B \\ 0 & 0 & 0 & \cdots & 0 & C & A_2 \end{bmatrix}. \tag{3.10}$$

Observe that the first and the last diagonal blocks are different from the others, since the tangent directions at the boundary are given. This is why the obtained matrix is block Toeplitz-like and not exactly block Toeplitz.

Matrices L and R are obviously invertible. Consequently, J_m is invertible if $\widetilde{J}_m = L\, J_m\, R$ is. It will be shown that $\widetilde{J}_m$ is invertible for m large enough. The fact that the matrix is block Toeplitz-like will be used. In this case the problem of the non-singularity of the matrix is closely connected with a particular system of difference equations. The solutions of this system depend mainly on the structure of the polynomial

$$\pi(\lambda) = \det(C + \lambda\, A + \lambda^2\, B), \quad \lambda \in \mathbb{C},$$

and the following lemma will be proved first.

LEMMA 3.2. *The determinant π is explicitly*

$$\pi(\lambda) = \det V(0, t_1, \ldots, t_r, 1)\, \lambda\, \pi_2(\lambda)^{r+1},$$

where

$$\pi_2(\lambda) = \frac{1}{a}\,\lambda^2 + \left(\frac{1}{c} - 2\right)\,\lambda + a,$$

and $V(0, t_1, \ldots, t_r, 1)$ is the Vandermonde matrix.

PROOF. Let

$$\begin{aligned} P &:= (C + \lambda\, A + \lambda^2\, B)\, \mathrm{diag}(1, t_1\,(t_1 - 1), \ldots, t_r\,(t_r - 1), 1) \\ &= \begin{bmatrix} \lambda\,(1/c - b) + \lambda^2\, b/a & p_{11} & \cdots & p_{1r} & \lambda - a \\ \lambda - \lambda^2\,(1 - b)/a & p_{21} & \cdots & p_{2r} & \lambda \\ -\lambda^2\,(2 - b)/a & p_{31} & \cdots & p_{3r} & \lambda \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\lambda^2\,(d - 1 - b)/a & p_{d1} & \cdots & p_{dr} & \lambda \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} p_{kj} &:= a\, t_j^2\,(t_j - 1)^{k-1} + \lambda\,(t_j\,(t_j - 1)/c - t_j^2 - (t_j - 1)^2)\, t_j^{k-1} + \\ &+ \lambda^2 (t_j - 1)^2\,(t_j + 1)^{k-1}/a. \end{aligned}$$

The approach based upon [8] will be used now. By the definition of the determinant, $\det P$ is a polynomial in variables $t_1, t_2, \ldots, t_r$. A brief look at the entries of the matrix P reveals that the total degree of its determinant is at most

$$\sum_{k=3}^{d}(k+1) = \frac{r^2+7r}{2}. \tag{3.11}$$

So, if all the zeros are guessed, one has to find the leading coefficient only. Note that

$$\begin{aligned} (p_{kj})_{k=1}^{d}\big|_{t_j\,=\,0} &= \frac{\lambda}{a}(\lambda-a,\lambda,\ldots,\lambda)^T, \\ (p_{kj})_{k=1}^{d}\big|_{t_j\,=\,1} &= -(\lambda-a,\lambda,\ldots,\lambda)^T \end{aligned}$$

are both proportional to the last column of P. Also, only one column of P depends on fixed t_j, and

$$\begin{aligned} \frac{\partial}{\partial t_j}(p_{kj})_{k=1}^{d}\big|_{t_j\,=\,0} &= -\frac{\lambda^2}{a}\left[2\,\frac{a}{\lambda}-\frac{a}{\lambda\,c}-2,-\frac{a}{\lambda}-1,0,1,2,\ldots,d-3\right]^T, \\ \frac{\partial}{\partial t_j}(p_{kj})_{k=1}^{d}\big|_{t_j\,=\,1} &= [2\,a,a-\lambda,-2\,\lambda,-3\,\lambda,\ldots,(-d+1)\,\lambda]^T \\ &+ \lambda\left(\frac{1}{c}-2\right)[1,1,\ldots,1]^T, \end{aligned}$$

and again the first, the $(j+1)$-th , and the last column are linearly dependent. Thus $\det P$ vanishes twofold at $t_j = 0, 1$. Since it vanishes also for $t_j = t_{j'}$, $j \neq j'$,

$$\det P = g\prod_{j=1}^{r} t_j^2\,(t_j-1)^2 \prod_{1\le j<k\le r}(t_k-t_j), \tag{3.12}$$

with g possibly depending only on other parameters, but not on $t_1, t_2 \ldots, t_r$, since the rest of the product is already of total degree

$$4r + \binom{r}{2} = \frac{r^2+7r}{2},$$

which equals (3.11). Even more, g must equal (for example) the coefficient of the term $t_1^4\,t_2^5\cdots t_r^{d+1}$ in $\det P$. Since this term is involved in the considered determinant only twice, namely in $p_{10}\,p_{2,r+1}\prod_{j=3}^{r+2}p_{j,j-2}$ and $p_{20}\,p_{1,r+1}\prod_{j=3}^{r+2}p_{j,j-2}$, an easy computation gives the desired coefficient

$$\begin{aligned} g &= \{(-1)^r\,(\lambda\,(1/c-b)+\lambda^2\,b/a)\,\lambda + (-1)^{r+1}\,(\lambda-\lambda^2\,(1-b)/a)\,(\lambda-a)\} \\ &\times \prod_{j=3}^{r+2}(\lambda^2/a+\lambda\,(1/c-2)+a) = (-1)^r\,\lambda\,(\lambda^2/a+\lambda\,(1/c-2)+a)^{r+1}. \end{aligned}$$

The result of the lemma follows after (3.12) is divided by $\prod_{j=1}^{r} t_j\,(t_j-1)$. □

Since the roots of π will be of particular interest, the following observation will be useful.

LEMMA 3.3. *There exist only three distinct real roots of* π, *namely* $\lambda_0 = 0$, λ_1 *and* λ_2. *Moreover,* $\lambda_1, \lambda_2 \neq a$ *and one of the roots, say* λ_2, *is dominant, i.e.,* $|\lambda_1| < |\lambda_2|$.

PROOF. By lemma 3.2, the roots of π consists of $\lambda_0 = 0$ and the roots of π_2. Since π_2 is quadratic polynomial and $(1/c-2)^2-4>0$ (because $c<0$), λ_1 and λ_2 are real and distinct. The relation $\lambda_1\,\lambda_2 = a^2 > 0$ then implies that λ_1 and λ_2 have the same sign and none of them equals a. Consequently $|\lambda_1| \neq |\lambda_2|$. □

These facts will be used to prove the following theorem.

THEOREM 3.4. *If* m *is large enough, then the matrix* J_m *is nonsingular.*

PROOF. It is by (3.10) enough to show that $\widetilde{J}_m$ is nonsingular if m is sufficiently large. Suppose that there is $\boldsymbol{x} \in \mathbb{R}^d$ and $\widetilde{J}_m\,\boldsymbol{x} = \boldsymbol{0}$. It will be shown that $\boldsymbol{x} = \boldsymbol{0}$ if m is large enough. Let us rewrite the equation $\widetilde{J}_m\,\boldsymbol{x} = \boldsymbol{0}$ in the block form

$$\widetilde{J}_m \begin{bmatrix} \boldsymbol{x}_{-m_1} \\ \boldsymbol{x}_{-m_1+1} \\ \vdots \\ \boldsymbol{x}_{m_2} \\ \boldsymbol{x}_{m_2+1} \end{bmatrix} = \boldsymbol{0}, \quad \boldsymbol{x}_\ell \in \mathbb{R}^d, \quad \ell = -m_1, -m_1+1, \ldots, m_2+1, \tag{3.13}$$

where $m_1 + m_2 = m-2$ and $m>2$. Since $\widetilde{J}_m$ is given by (3.10), equation (3.13) is equivalent to the following system of difference equations

$$\begin{aligned} A_1\boldsymbol{x}_{-m_1} + B\boldsymbol{x}_{-m_1+1} &= \boldsymbol{0} \\ C\boldsymbol{x}_{\ell-1} + A\boldsymbol{x}_\ell + B\boldsymbol{x}_{\ell+1} &= \boldsymbol{0}, \quad \ell = -m_1+1, \ldots, m_2 \\ C\boldsymbol{x}_{m_2} + A_2\boldsymbol{x}_{m_2+1} &= \boldsymbol{0}. \end{aligned} \tag{3.14}$$

In order to apply the general theory of difference equations ([6], p. 181–227), the system (3.14) will be first transformed to the system of difference equations of the first order. Let

$$\boldsymbol{y}_\ell = \begin{bmatrix} \boldsymbol{x}_\ell \\ \boldsymbol{x}_{\ell+1} \end{bmatrix}, \quad \ell = -m_1, -m_1+1, \ldots, m_2.$$

Now (3.14) is equivalent to

$$M\,\boldsymbol{y}_{\ell+1} = N\,\boldsymbol{y}_\ell, \quad \ell = -m_1, -m_1+1, \ldots, m_2-1, \tag{3.15}$$

where

$$M = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & I \\ -C & -A \end{bmatrix},$$

with I the identity in $\mathbb{R}^{d\times d}$, and the boundary conditions

$$[A_1\ B]\,\boldsymbol{y}_{-m_1} = \boldsymbol{0}, \tag{3.16}$$

$$[C\ A_2]\,\boldsymbol{y}_{m_2} = \boldsymbol{0}. \tag{3.17}$$

LEMMA 3.5. *Let $\pi(\lambda) := \det(N - \lambda M)$. Then π has three real roots, $\lambda_0 = 0$, λ_1 and λ_2 of multiplicity 1, $r+1$ and $r+1$ respectively. The general solution of (3.15) is*

$$\begin{aligned} \boldsymbol{y}_{-m_1} &= P J^{-m_1} \boldsymbol{c} + c_0 \boldsymbol{e}_1, \\ \boldsymbol{y}_\ell &= P J^\ell \boldsymbol{c}, \quad \ell = -m_1+1, -m_1+2, \ldots, m_2-1, \\ \boldsymbol{y}_{m_2} &= P J^{m_2} \boldsymbol{c} + c_\infty \boldsymbol{e}_{2d}, \end{aligned}$$

where $P = [P_1\, P_2]$ is the matrix of generalized eigenvectors and principal vectors of the matrix pencil $N - \lambda M$ corresponding to λ_1 and λ_2, $J = \operatorname{diag}(J_1, J_2)$ is the corresponding block Jordan matrix and $[c_0, \boldsymbol{c}^T, c_\infty]^T \in \mathbb{R}^{2d}$ is a vector of arbitrary constants.

PROOF. It is easy to verify that

$$\det(N - \lambda M) = \det(C + \lambda A + \lambda^2 B) = \pi(\lambda).$$

By lemma 3.2, π has three roots, $\lambda_0 = 0$, λ_1 and λ_2 of multiplicity 1, $r+1$ and $r+1$ respectively. Since the degree of π is $2r+3 = 2d-1 < 2d$, it has exactly one root at infinity, say λ_∞.
To find a general solution of (3.15), one has to construct the canonical forms of $N - \lambda_i M$, $i = 0, 1, 2$ and $\lambda_0 N - M$. It is then well known ([6], p. 225–227) that the general solution of (3.15) is

$$\begin{aligned} \boldsymbol{y}_{-m_1} &= P J^{-m_1} \boldsymbol{c} + c_0 \boldsymbol{z}, \\ \boldsymbol{y}_\ell &= P J^\ell \boldsymbol{c}, \quad \ell = -m_1+1, -m_1+2, \ldots, m_2-1, \\ \boldsymbol{y}_{m_2} &= P J^{m_2} \boldsymbol{c} + c_\infty \boldsymbol{w}, \end{aligned}$$

where $P = [P_1\, P_2]$ is the matrix of generalized eigenvectors and principal vectors corresponding to λ_1 and λ_2, $J = \operatorname{diag}(J_1, J_2)$ is the corresponding block Jordan matrix, $[c_0, \boldsymbol{c}^T, c_\infty]^T \in \mathbb{R}^{2d}$ is a vector of arbitrary constants, $\boldsymbol{z}$ is a generalized eigenvector corresponding to λ_0, and $\boldsymbol{w}$ is a generalized eigenvector corresponding to λ_∞. Since $N\boldsymbol{z} = M\boldsymbol{w} = \boldsymbol{0}$ and the first column of C and the last column of B are zero, we have $\boldsymbol{z} \in Lin\{\boldsymbol{e}_1\}$, $\boldsymbol{w} \in Lin\{\boldsymbol{e}_{2d}\}$. Consequently, we my assume that $\boldsymbol{z} = \boldsymbol{e}_1$ and $\boldsymbol{w} = \boldsymbol{e}_{2d}$.
Since it is also known that the matrix $\widetilde{P} = [\boldsymbol{z}\, P\, \boldsymbol{w}]$ is nonsingular, the solution space of (3.15) has dimension $2\,d$ and there are no other solutions. □

The result of the theorem 3.4 will follow if one can prove that the boundary conditions (3.16) and (3.17) imply $c_0 = c_\infty = 0$ and $\boldsymbol{c} = \boldsymbol{0}$ for m_1 and m_2 sufficiently large. Namely, if this is true, then $\boldsymbol{y}_\ell = \boldsymbol{0}$, $\ell = -m_1, -m_1+1, \ldots, m_2$, which implies $\boldsymbol{x}_\ell = \boldsymbol{0}$, $\ell = -m_1, -m_1+1, \ldots, m_2+1$ and $\ker \widetilde{J}_m = \{\boldsymbol{0}\}$.

It is in fact enough to show that $\boldsymbol{c} = \boldsymbol{0}$. Namely, if $\boldsymbol{c} = \boldsymbol{0}$, then the first boundary condition (3.16) becomes

$$[A_1\, B]\, \boldsymbol{y}_{-m_1} = c_0\, [A_1\, B]\, \boldsymbol{z} = c_0\, [A_1\, B]\, \boldsymbol{e}_1 = \boldsymbol{0}.$$

Since $[A_1\, B]\, \boldsymbol{e}_1$ is the first column of A_1, which is nonzero, c_0 must be zero. Similarly, the second boundary condition reads as

$$[C\, A_2]\, \boldsymbol{y}_{m_2} = c_\infty\, [C\, A_2]\, \boldsymbol{w} = c_\infty\, [C\, A_2]\, \boldsymbol{e}_{2d} = \boldsymbol{0}.$$

The last column of A_2 is clearly nonzero, which implies $c_\infty = 0$.
It remains to prove that $\boldsymbol{c}$ is zero. Suppose $\boldsymbol{c} \neq \boldsymbol{0}$. There exists an index i, $1 \leq i \leq 2d-2$, for which $c_i \neq 0$. Two cases have to be considered.

a) Let $1 \leq i \leq d-1$. Since by lemma 3.3 $|\lambda_1| < |\lambda_2|$, the dominant eigenvalue of J^{-1} is $1/\lambda_1$ and the power method asserts that if m_1 is large enough,

$$\frac{1}{||PJ^{-m_1}\,\boldsymbol{c}||_\infty}\,PJ^{-m_1}\,\boldsymbol{c} = \widetilde{\boldsymbol{u}} + \mathcal{O}(1/m_1),$$

where $\widetilde{\boldsymbol{u}}$ is a generalized eigenvector corresponding to λ_1. Suppose first that $|\lambda_1| \neq 1$. The normalized boundary condition (3.16) implies

$$[A_1\,B]\,\widetilde{\boldsymbol{u}} = \boldsymbol{0}. \tag{3.18}$$

We shall show that $\widetilde{\boldsymbol{u}}$ must be zero which is an obvious contradiction. Since $\widetilde{\boldsymbol{u}}$ is a generalized eigenvector corresponding to λ_1, the relation $(N - \lambda_1\,M)\widetilde{\boldsymbol{u}}$ holds, and

$$(C + \lambda_1\,A + \lambda_1^2\,B)\boldsymbol{u}_1 = \boldsymbol{0},\ \boldsymbol{u}_2 = \lambda_1\,\boldsymbol{u}_1, \quad \widetilde{\boldsymbol{u}} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}. \tag{3.19}$$

Together with (3.18) one now concludes

$$(C + \lambda_1(A - A_1))\boldsymbol{u}_1 = \boldsymbol{0}. \tag{3.20}$$

Let us define the matrix $S(\lambda) := C + \lambda(A - A_1)$.

LEMMA 3.6. *The determinant of the matrix $S(\lambda)$ is explicitly*

$$\det S(\lambda) = (-1)^r\,a\,\det V(0, t_1, t_2, \ldots, t_r, 1)\,\lambda\,(\lambda - a)^r.$$

PROOF. Determinant $\det S(\lambda)$ can be written as

$$\det \begin{bmatrix} \lambda\,(1/c - b - 1) & s_{11} & s_{12} & \cdots & s_{1r} & -a \\ \lambda & s_{21} & s_{22} & \cdots & s_{2r} & 0 \\ 0 & s_{31} & s_{32} & \cdots & s_{3r} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & s_{r+2,1} & s_{r+2,2} & \cdots & s_{r+2,r} & 0 \end{bmatrix},$$

where

$$s_{kj} = a\,u_j\,(t_j - 1)^{k-1} - \lambda\,\frac{1}{u_j}\,t_j^{k-1}.$$

Using some basic properties of determinants, it simplifies to

$$\det S(\lambda) = (-1)^r\,a\,\lambda\,\det[\,s_{kj}\,]_{k=3,\,j=1}^{r+2,\,r}.$$

The remaining determinant can be easily computed by following the ideas of the proof of lemma 3.2. ◻

By lemma 3.3, $\lambda_1 \neq 0, a$ and the determinant of $S(\lambda_1)$ must be nonzero. But the relations (3.20) and (3.19) then imply $\widetilde{\boldsymbol{u}} = \boldsymbol{0}$ which is a contradiction.

The same proof works in the case when $|\lambda_1| = 1$ and $\boldsymbol{c}_1 := (\boldsymbol{c})_{i=1}^{d-1}$ is not an eigenvector of J_1^{-1}. Thus it remains to consider the case when $|\lambda_1| = 1$ and $J_1^{-1}\,\boldsymbol{c}_1 = (1/\lambda_1)\,\boldsymbol{c}_1$. Since then $0 < \mathrm{const}_1 \leq ||P\,J^{-k}\,\boldsymbol{c}||_\infty \leq \mathrm{const}_2$, for all k, the normalized boundary condition (3.16) becomes

$$[A_1\;B]\,\widetilde{\boldsymbol{u}} + \widetilde{c}_0\,[A_1\;B]\,\boldsymbol{z} = \boldsymbol{0},$$

where $\widetilde{\boldsymbol{u}}$ is a generalized eigenvector corresponding to λ_1 and

$$\widetilde{c}_0 = c_0/||P\,J^{-m_1}\,\boldsymbol{c}||_\infty.$$

This relation, (3.19), and the fact that $[A_1\;B]\,\boldsymbol{z} = \boldsymbol{e}_1$ imply

$$(C + \lambda_1\,(A - A_1))\,\boldsymbol{u}_1 = \lambda_1\,\widetilde{c}_0\,\boldsymbol{e}_1. \tag{3.21}$$

Thus $\boldsymbol{u}_1 = -(\lambda_1/a)\,\widetilde{c}_0\,\boldsymbol{e}_d$ which leads either to $\widetilde{\boldsymbol{u}} = \boldsymbol{0}$ or contradicts (3.20) and the proof of the first case is complete.

b) Suppose $d \leq i \leq 2d-2$. The power method now assures that, as m_2 tends to infinity,

$$\frac{1}{||PJ^{m_2}\,\boldsymbol{c}||_\infty}\,PJ^{m_2}\boldsymbol{c}$$

converges to a generalized eigenvector corresponding to λ_2. The normalized boundary condition (3.17) becomes

$$\frac{1}{||PJ^{m_2}\,\boldsymbol{c}||_\infty}[C\;A_2]\,\boldsymbol{y}_{m_2} = [C\;A_2]\widetilde{\boldsymbol{v}} + \widetilde{c}_\infty\,\boldsymbol{w} + \mathcal{O}(1/m_2) = \boldsymbol{0}.$$

We use very similar arguments as in the first case. If $|\lambda_2| \neq 1$, the relation

$$[C\;A_2]\,\widetilde{\boldsymbol{v}} = \boldsymbol{0} \tag{3.22}$$

must hold. Since now $(N - \lambda_2\,M)\widetilde{\boldsymbol{v}} = \boldsymbol{0}$, we have

$$(C + \lambda_2\,A + \lambda_2^2\,B)\boldsymbol{v}_1 = \boldsymbol{0},\;\; \boldsymbol{v}_2 = \lambda_2\boldsymbol{v}_1, \quad \boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}, \tag{3.23}$$

and the relation (3.22) imply

$$(A - A_2 + \lambda_2\,B)\boldsymbol{v}_1 = \boldsymbol{0}. \tag{3.24}$$

Let $T(\lambda) := A - A_2 + \lambda\,B$ and $T_{d-1}(\lambda)$ denote the first $d-1$ columns of $T(\lambda)$. Suppose $\boldsymbol{e}$ is a vector of ones.

Lemma 3.7. *If $\widetilde{T}(\lambda) = [T_{d-1}(\lambda)\,\boldsymbol{e}]$, then*

$$\det \widetilde{T}(\lambda) = \frac{\det V(0, t_1, t_2, \ldots, t_r, 1)}{a^{r+1}}\,\lambda\,(\lambda - a)^r.$$

PROOF. The proof of this lemma is almost the same as the proof of lemma 3.2 and will be omitted. □

Observe that the last column of $T(\lambda)$ is zero for all λ, and recall that $\lambda_2 \neq 0, a$. The conclusion that $T(\lambda_2)$ has rank $d-1$ now follows directly from the previous lemma. Thus $T(\lambda_2)$ has one dimensional kernel spanned by $\boldsymbol{e}_d$. The only solutions of (3.24) are in $Lin\{\boldsymbol{e}_d\}$, which leads to $\boldsymbol{v}_1 = \boldsymbol{0}$ or $\boldsymbol{v}_1 = \text{const}\, \boldsymbol{e}_d$. The first solution produces an obvious contradiction $\widetilde{\boldsymbol{v}} = \boldsymbol{0}$ and the second one fails to satisfy (3.23).
If $|\lambda_2| = 1$, and $\boldsymbol{c}_2 := (\boldsymbol{c})_{i=d}^{2d-2}$ is not an eigenvector of J_2, the conclusion (3.24) is still true. Suppose that $|\lambda_2| = 1$ and $J_2\, \boldsymbol{c}_2 = \lambda_2\, \boldsymbol{c}_2$. If (3.17) is normalized by $||P\, J^{m_2}\, \boldsymbol{c}||_\infty$, which is bounded now, and the relations (3.23) and $[C\, A_2]\, \boldsymbol{w} = \boldsymbol{e}$ are used, the relation

$$(A - A_2 + \lambda_2\, B)\, \boldsymbol{v}_1 = \frac{\widetilde{c}_\infty}{\lambda_2}\, \boldsymbol{e}, \tag{3.25}$$

where $\widetilde{c}_\infty = c_\infty / ||P\, J^{m_2}\, \boldsymbol{c}||_\infty$, is obtained. Since by lemma 3.7 vector $\boldsymbol{e}$ is not in the image of $T(\lambda_2)$, (3.25) has a solution only if $\widetilde{c}_\infty = 0$, i.e., (3.24) must hold and a contradiction follows by previous conclusions.

In all cases $\boldsymbol{c} = \boldsymbol{0}$, thus c_0 and c_∞ must be zero too. This implies regularity of $\widetilde{J}_m$ for m_1 and m_2 large enough (i.e. m large enough) and the proof of the theorem 3.4 is complete. □

For the approximation order it is enough to look the error on the ℓ-th segment independently, i.e., $\text{dist}(\boldsymbol{B}^\ell, \boldsymbol{f})$. By (1.6) and (2.5) the tangent directions of $\boldsymbol{B}^\ell$ at the boundary points are $\mathcal{O}(h)$ approximations of the tangent directions of $\boldsymbol{f}$, which is easily obtained by straightforward computations. Consequently, there exists a smooth curve $\widetilde{\boldsymbol{f}}$ with positive first $d-1$ principal curvatures, which interpolates the given data points on the ℓ-th segment and the tangent directions of $\boldsymbol{B}^\ell$ at the boundary points. Additionally, it can be chosen in such a way that it interpolates $\boldsymbol{f}$ at additional two points on the ℓ-th segment too. Then

$$\text{dist}(\boldsymbol{B}^\ell, \boldsymbol{f}) \leq \text{dist}(\boldsymbol{B}^\ell, \widetilde{\boldsymbol{f}}) + \text{dist}(\widetilde{\boldsymbol{f}}, \boldsymbol{f}) = \mathcal{O}(h^{r+4}) + \mathcal{O}(h^{r+4}),$$

where the first part follows from the single segment case analysis in [7], and the second by the construction of $\widetilde{\boldsymbol{f}}$.
By theorem 3.4 the Jacobian of the nonlinear system (3.5) is nonsingular at the limit solution from lemma 3.1. The implicit function theorem now asserts, that the system (3.5) has a solution in the neighborhood of that limit solution, i.e., for h small enough, or equivalently, m large enough. This finally proves the results stated in the theorem 1.1.

4 Numerical example.

A numerical example will be given here to confirm the results obtained in the previous sections. Since the results for the curves in $\mathbb{R}^3$ have already been presented in [5] we shall consider the curve in $\mathbb{R}^4$.

Let the curve be given by

$$\boldsymbol{f} : [0,10] \to \mathbb{R}^4 : \tau \mapsto \boldsymbol{f}(\tau) := \begin{pmatrix} \cos\tau \\ \sin\tau \\ \ln(2+\tau) \\ \ln(1+\tau) \end{pmatrix}.$$

It can be verified that the first three curvatures of $\boldsymbol{f}$ are positive on $[0,10]$ and the curve satisfies our requirements. The interpolation points have been chosen on the curve at the equidistant values of the parameter τ. It is not a regular sampling in the sense of the theorem 1.1 but we still got a solution of the problem. This indicates that the regularity of data points could be omitted in the theorem. This fact has been confirmed by other examples too. The detailed numerical algorithm for solving the obtained nonlinear system together with the numerical results will appear elsewhere.

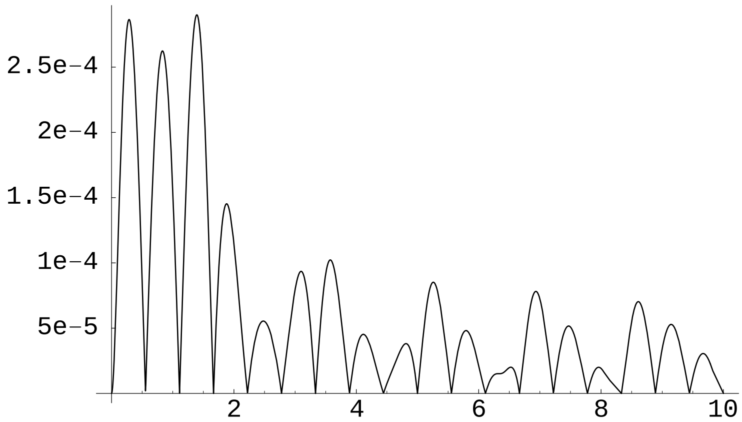

Figure 4.1: The estimated parametric error for the curve $\boldsymbol{f}$ and $m = 6$.

Table 4.1: The results for the curve $\boldsymbol{f}$.

m	Error	Rate	m	Error	Rate
6	$2.90 * 10^{-4}$	–	16	$1.38 * 10^{-6}$	5.67
8	$6.32 * 10^{-5}$	5.30	18	$7.07 * 10^{-7}$	5.70
10	$1.89 * 10^{-5}$	5.41	20	$3.87 * 10^{-7}$	5.71
12	$6.91 * 10^{-6}$	5.51	22	$2.24 * 10^{-7}$	5.73
14	$2.92 * 10^{-6}$	5.58	24	$1.36 * 10^{-7}$	5.75

The parametric distance (i.e. the error) between $\boldsymbol{f}$ and $\boldsymbol{B}$ was obtained by the method described in [2] and already used in [3]. The results of the interpolation are shown in Tab. 4.1. The first column is the number of segments of the spline curve, the second one the estimated error, and the third one the rate of convergence obtained from two consecutive m. Since in this case $r = d - 2 = 2$, the approximation order should be $r+4 = 6$, which is obviously confirmed by the results in the table. Fig. 4.1 shows the estimated parametric error for $m = 6$. It can be clearly seen that four points are interpolated on each segment implying the error being zero there.

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