

# Lattices on simplicial partitions

Gašper Jaklič<sup>a,c,\*</sup>, Jernej Kozak<sup>a,b</sup>, Marjeta Krajnc<sup>b</sup>,  
Vito Vitrih<sup>c</sup>, Emil Žagar<sup>a,b</sup>

<sup>a</sup>*University of Ljubljana, FMF, Jadranska 19, Ljubljana, Slovenia*

<sup>b</sup>*IMFM, Jadranska 19, Ljubljana, Slovenia*

<sup>c</sup>*University of Primorska, PINT, Muzejski trg 2, Koper, Slovenia*

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## Abstract

In this paper,  $(d + 1)$ -pencil lattices on simplicial partitions in  $\mathbb{R}^d$  are studied. The barycentric approach naturally extends the lattice from a simplex to a simplicial partition, providing a continuous piecewise polynomial interpolant over the extended lattice. The number of degrees of freedom is equal to the number of vertices of the simplicial partition. The constructive proof of this fact leads to an efficient computer algorithm for the design of a lattice.

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## 1 Introduction

It is well-known that the multivariate Lagrange polynomial interpolation problem is much harder than the univariate one. While the existence and the uniqueness of the interpolant in the univariate case are guaranteed by the fact that the interpolation points are pairwise distinct, this is far away to be true in the multivariate case. Recall that the Lagrange interpolation problem at  $\binom{n+d}{d}$  interpolation points is correct in the space of polynomials in  $d$  variables of total degree  $\leq n$ ,  $\Pi_n^d$ , iff the points do not lie on an algebraic hypersurface of degree  $\leq n$ . In practice, this condition is hard to verify, thus alternatively, prescribed configurations of interpolation points, that guarantee correctness of the interpolation problem, are needed.

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\* Corresponding author.

*Email address:* `gasper.jaklic@fmf.uni-lj.si` (Gašper Jaklič).

The most often used such configurations are *lattices*, introduced in [2], where the interpolation points are generated as intersections of particular hyperplanes. *Principal lattices* (see [2,3], e.g.) are generated as intersections of  $d + 1$  pencils of parallel hyperplanes. In [8], these lattices have been generalized to the case of not necessarily parallel hyperplanes intersecting in so called *centers*. These lattices are known as  $(d + 1)$ -*pencil lattices of order  $n$* . Some further generalizations can be found also in [1]. It is well-known that lattices admit correct interpolation in  $\Pi_n^d$  since they satisfy the GC condition (cf. [2]).

In [5], the barycentric approach has been used for  $(d + 1)$ -pencil lattices in order to obtain the explicit positions of lattice points on a given simplex in  $\mathbb{R}^d$  and to construct the interpolant in the Lagrange form. This representation of  $(d + 1)$ -pencil lattices is useful in many practical applications, such as an explicit interpolation of multivariate functions, finite element methods in solving partial differential equations, numerical methods for multidimensional integrals ([6])... In this paper, the results of [5] are extended to  $(d + 1)$ -pencil lattices on simplicial partitions. It is shown, that it is possible to construct a  $(d + 1)$ -pencil lattice on a given simplicial partition with  $V$  vertices, such that the lattice points on common faces of the partition agree, and that there are  $V$  degrees of freedom, that can be used as shape parameters. This provides a continuous piecewise polynomial Lagrange interpolant over the given simplicial partition.

The paper is organized as follows. In Section 2 a definition of a  $(d + 1)$ -pencil lattice, based on control points, is recalled, and the notation is introduced. Section 3 provides the tools, necessary for extending the lattice from a simplex to a simplicial partition. In Section 4 the main result is presented. The paper is concluded by an example in Section 5.

## 2 Preliminaries

A definition of a lattice, based upon control points, introduced in [5], will be used. First, let us recall some basic facts about the lattices and introduce the notation.

A simplex in  $\mathbb{R}^d$  is a convex hull of  $d + 1$  vertices  $\mathbf{T}_i$ ,  $i = 0, 1, \dots, d$ . Since for our purpose the ordering of the vertices of the simplex will be important, the notation

$$\Delta := \langle \mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_d \rangle,$$

which defines a simplex with a prescribed order of the vertices  $\mathbf{T}_i$ , will be used.

A  $(d + 1)$ -pencil lattice of order  $n$  on  $\Delta$  is a set of  $\binom{n+d}{d}$  points, generated by

particular  $d + 1$  pencils of  $n + 1$  hyperplanes, such that each lattice point is an intersection of  $d + 1$  hyperplanes, one from each pencil. Furthermore, each pencil intersects at a center

$$\mathbf{C}_i \subset \mathbb{R}^d, \quad i = 0, 1, \dots, d,$$

a plane of codimension two. The lattice is based upon affinely independent

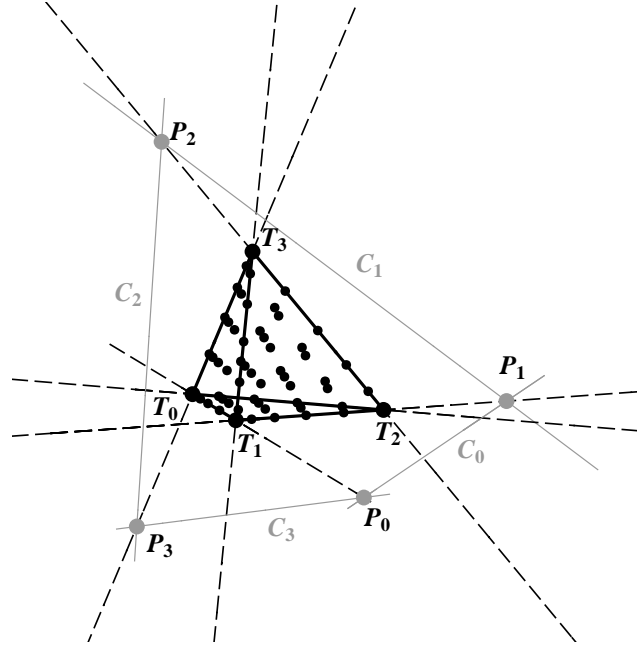


Fig. 1. A 4-pencil lattice with its control points  $\mathbf{P}_i$  and centers  $\mathbf{C}_i$  on a tetrahedron  $\langle \mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3 \rangle$ .

control points

$$\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_d, \quad \mathbf{P}_i \in \mathbb{R}^d,$$

where  $\mathbf{P}_i$  lies on the line through  $\mathbf{T}_i$  and  $\mathbf{T}_{i+1}$  outside of the segment  $\mathbf{T}_i\mathbf{T}_{i+1}$  (see Fig. 1). The center  $\mathbf{C}_i$  is then uniquely determined by a sequence of control points

$$\mathbf{P}_i, \mathbf{P}_{i+1}, \dots, \mathbf{P}_{i+d-2},$$

where

$$\{\mathbf{P}_{i+1}, \mathbf{P}_{i+2}, \dots, \mathbf{P}_{i+d-2}\} \subseteq \mathbf{C}_i \cap \mathbf{C}_{i+1}.$$

Here and throughout the paper, indices of points, centers, lattice parameters, etc., are assumed to be taken modulo  $d + 1$ . Wherever necessary, an emphasis on this assumption will be given explicitly by a function

$$m(i) := i \bmod (d + 1).$$

With  $d$  prescribed, indices considered belong to  $\mathbb{Z}_{d+1} := \{0, 1, \dots, d\} = m(\mathbb{Z})$ .

A natural bijective imbedding  $u : \mathbb{Z}_{d+1}^{r+1} \rightarrow \mathbb{N}_0^{r+1}$ , defined as

$$u \left( (i_j)_{j=0}^r \right) := \left( i_j + (d+1) \sum_{k=0}^{j-1} H(i_k - i_{k+1}) \right)_{j=0}^r,$$

where  $H(s)$ ,

$$H(s) := \begin{cases} 1, & s > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is the usual Heaviside step function, will significantly simplify further discussion. A graphical interpretation of this map (Fig. 2) explains also a term *winding number* of an index vector  $(i_j)_{j=0}^r$ , defined as

$$w \left( (i_j)_{j=0}^r \right) := \sum_{k=0}^{r-1} H(i_k - i_{k+1}) + H(i_r - i_0).$$

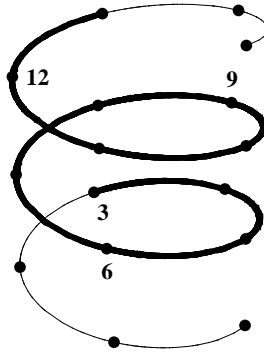


Fig. 2. Let  $d = 4$ ,  $\mathbf{i} = (3, 1, 4, 2)$  and  $r = 3$ . Then  $u(\mathbf{i}) = (3, 6, 9, 12)$  and  $w(\mathbf{i}) = 2$ .

The standard multiindex notation will be used. Let  $\boldsymbol{\gamma} = (\gamma_0, \dots, \gamma_d)$ ,  $\gamma_i \in \mathbb{N}_0$ , denote an index vector and let

$$|\boldsymbol{\gamma}| := \sum_{i=0}^d \gamma_i.$$

Further, let us shorten the notation with

$$[j]_\alpha := \sum_{i=0}^{j-1} \alpha^i = \begin{cases} j, & \alpha = 1, \\ \frac{1 - \alpha^j}{1 - \alpha}, & \text{otherwise,} \end{cases} \quad j \in \mathbb{N}_0.$$

In [5] the barycentric coordinates of a  $(d+1)$ -pencil lattice on a simplex  $\Delta = \langle \mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_d \rangle$  w.r.t.  $\Delta$  were determined by  $d+1$  free parameters

$\boldsymbol{\xi} = (\xi_0, \dots, \xi_d)$  as

$$B_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) = \frac{1}{D_{\boldsymbol{\gamma}, \boldsymbol{\xi}}} \left( \alpha^{n-\gamma_0} [\gamma_0]_{\alpha}, \xi_0 \alpha^{n-\gamma_0-\gamma_1} [\gamma_1]_{\alpha}, \xi_0 \xi_1 \alpha^{n-\gamma_0-\gamma_1-\gamma_2} [\gamma_2]_{\alpha}, \dots, \right. \\ \left. \xi_0 \xi_1 \cdots \xi_{d-1} [\gamma_d]_{\alpha} \right), \quad (1)$$

where

$$D_{\boldsymbol{\gamma}, \boldsymbol{\xi}} := \alpha^{n-\gamma_0} [\gamma_0]_{\alpha} + \xi_0 \alpha^{n-\gamma_0-\gamma_1} [\gamma_1]_{\alpha} + \dots + \xi_0 \xi_1 \cdots \xi_{d-1} [\gamma_d]_{\alpha},$$

and  $\boldsymbol{\gamma} \in \mathbb{N}_0^{d+1}$ ,  $|\boldsymbol{\gamma}| = n$ ,  $\alpha^n = \prod_{i=0}^d \xi_i$ .

### 3 Operations on $(d+1)$ -pencil lattices

In this section, necessary tools for extending a  $(d+1)$ -pencil lattice from a simplex to a simplicial partition will be provided. Note that they pave a way to an important part of numerical analysis, computer algorithms. Several theorems, which are closely related to each other, will be presented. The most important for the extension of a lattice from a simplex to a simplicial partition in the next section is Theorem 5 together with its corollaries. But the basis for all results in this section is the following assertion, which reveals a restriction of a lattice to a face of the simplex.

**Theorem 1** *Let a  $(d+1)$ -pencil lattice of order  $n$  on a  $d$ -simplex  $\Delta = \langle \mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_d \rangle$ , given in the barycentric form, be determined by the parameters  $\boldsymbol{\xi} = (\xi_0, \xi_1, \dots, \xi_d)$  as in (1). Let the indices*

$$\mathbf{i} = (i_0, i_1, \dots, i_r), \quad 0 \leq i_j \leq d, \quad \text{where } i_k \neq i_j \text{ if } k \neq j, \quad r \leq d, \quad w(\mathbf{i}) = 1,$$

*select an  $r$ -face  $\Delta' = \langle \mathbf{T}_{i_0}, \mathbf{T}_{i_1}, \dots, \mathbf{T}_{i_r} \rangle \subset \Delta$ . A restriction of the lattice to  $\Delta'$  is an  $(r+1)$ -pencil lattice on  $\Delta'$ , with the barycentric coordinates w.r.t.  $\Delta'$  determined by  $\boldsymbol{\xi}' = (\xi'_0, \xi'_1, \dots, \xi'_r)$ , where*

$$\xi'_j = \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)}, \quad j = 0, 1, \dots, r, \quad (2)$$

*and  $\boldsymbol{\ell} = (\ell_j)_{j=0}^{r+1} = u((i_0, i_1, \dots, i_r, i_0))$  (see Fig. 3).*

**PROOF.** The notation  $(v)_k$  will throughout the proof denote the  $k$ -th component of vector  $v$ . Let  $\boldsymbol{\gamma} \in \mathbb{N}_0^{r+1}$ ,  $|\boldsymbol{\gamma}| = n$ , be an index vector of a lattice point

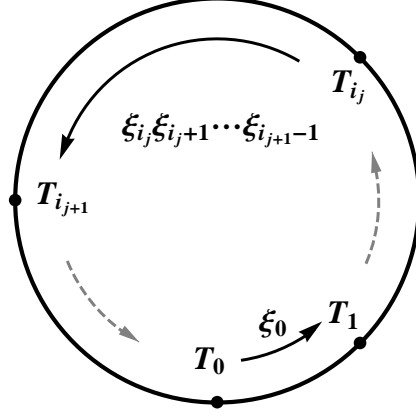


Fig. 3. Parameters that determine a lattice on an  $r$ -face  $\Delta' \subset \Delta$ .

generated by  $\xi'$  over  $\Delta'$ . The map  $\phi : \mathbb{N}_0^{r+1} \rightarrow \mathbb{N}_0^{d+1}$ ,

$$(\phi(\gamma))_{k+1} = \begin{cases} \gamma_j, & i_j = k, 0 \leq j \leq r \\ 0, & \text{otherwise} \end{cases}, \quad k = 0, 1, \dots, d,$$

gives a relation between the index vectors of a particular point expressed in both lattices. Thus

$$(B_{\phi(\gamma)}(\xi))_{k+1} = 0, \quad k \neq i_j, 0 \leq j \leq r,$$

and one has to verify

$$(B_{\phi(\gamma)}(\xi))_{i_j+1} = (B_{\gamma}(\xi'))_{j+1}, \quad j = 0, 1, \dots, r, \quad (3)$$

only. Let  $\alpha^n = \prod_{k=0}^d \xi_k$ , and  $\alpha'^n = \prod_{k=0}^r \xi'_k$ . Note that

$$[(\phi(\gamma))_{i_j+1}]_{\alpha} = [\gamma_j]_{\alpha},$$

so by (1)  $D_{\phi(\gamma), \xi} \cdot (B_{\phi(\gamma)}(\xi))_{i_j+1}$  simplifies to

$$\left( \prod_{k=0}^{i_j-1} \xi_k \right) \alpha^{n - \sum_{t=0}^{i_j} (\phi(\gamma))_{t+1}} [\gamma_j]_{\alpha}. \quad (4)$$

Suppose the relations (2) hold. Then

$$\alpha'^n = \prod_{j=0}^r \xi'_j = \prod_{j=0}^r \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)}.$$

But the assertion  $w(\mathbf{i}) = 1$  implies the existence of a precisely one  $s$ ,  $0 \leq s \leq r$ , such that

$$0 \leq \underbrace{i_{s+1} < i_{s+2} < \cdots < i_r}_{r-s} < \underbrace{i_0 < i_1 < \cdots < i_s}_{s+1} \leq d,$$

and

$$\prod_{\substack{j=0 \\ j \neq s}}^r \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)} = \left( \prod_{k=\ell_0}^{\ell_s-1} \xi_{m(k)} \right) \left( \prod_{k=\ell_{s+1}}^{\ell_{r+1}-1} \xi_{m(k)} \right) = \prod_{k=i_{s+1}}^{i_s-1} \xi_k,$$

with

$$\prod_{k=\ell_s}^{\ell_{s+1}-1} \xi_{m(k)} = \left( \prod_{k=i_s}^d \xi_k \right) \left( \prod_{k=0}^{i_{s+1}-1} \xi_k \right).$$

Therefore  $\alpha' = \alpha$ . Similarly,

$$\prod_{k=0}^{j-1} \xi'_k = \prod_{k=\ell_0}^{\ell_j-1} \xi_{m(k)},$$

and so

$$D_{\gamma, \xi'} \cdot (B_{\gamma}(\xi'))_{j+1} = \left( \prod_{k=\ell_0}^{\ell_j-1} \xi_{m(k)} \right) \alpha^{n - \sum_{t=0}^j \gamma t} [\gamma_j]_{\alpha}. \quad (5)$$

Note that (3) follows from (1) if the quotient of the expressions (4) and (5) does not depend on  $j$ . A brief look on (4) at  $j = 0$  reveals this quotient as

$$c = \left( \prod_{k=0}^{i_0-1} \xi_k \right) \alpha^{-\sum_{t=0}^{i_0-1} (\phi(\gamma))_{t+1}}.$$

Indeed, the constant  $c$  is a quotient of (4) and (5) if

$$\frac{1}{c} \cdot \left( \prod_{k=0}^{i_j-1} \xi_k \right) \alpha^{n - \sum_{t=0}^{i_j} (\phi(\gamma))_{t+1}} = \left( \prod_{k=\ell_0}^{\ell_j-1} \xi_{m(k)} \right) \alpha^{n - \sum_{t=0}^j \gamma t} \quad (6)$$

for  $0 \leq j \leq r$ . To begin with, suppose that  $0 \leq j \leq s$ . Then  $i_k = \ell_k$ ,  $0 \leq k \leq j$ ,  $i_0 < i_1 < \cdots < i_j$ , and the left hand side of the equation (6) simplifies to

$$\left( \prod_{k=i_0}^{i_j-1} \xi_k \right) \alpha^{n - \sum_{t=i_0}^{i_j} (\phi(\gamma))_{t+1}} = \left( \prod_{k=\ell_0}^{\ell_j-1} \xi_{m(k)} \right) \alpha^{n - \sum_{t=0}^j \gamma t},$$

as required. Now let  $j > s$ . Thus  $i_j < i_0$  and the left hand side of (6) simplifies to

$$\left( \prod_{k=i_j}^{i_0-1} \xi_k^{-1} \right) \alpha^{n + \sum_{t=i_j+1}^{i_0-1} (\phi(\gamma))_{t+1}}.$$

Since

$$\left( \prod_{k=i_j}^{i_0-1} \xi_k^{-1} \right) \alpha^n = \left( \prod_{k=0}^{i_j-1} \xi_k \right) \left( \prod_{k=i_0}^d \xi_k \right) = \prod_{k=\ell_0}^{\ell_j-1} \xi_{m(k)}$$

and

$$\sum_{t=i_j+1}^{i_0-1} (\phi(\gamma))_{t+1} = n - \sum_{t=i_0}^d (\phi(\gamma))_{t+1} - \sum_{t=0}^{i_j} (\phi(\gamma))_{t+1} = n - \sum_{t=0}^j \gamma_t,$$

the proof is completed.  $\square$

Let us apply Theorem 1 in a particularly simple example: a restriction of a lattice to a line segment  $\Delta' = \langle \mathbf{T}_{i_0}, \mathbf{T}_{i_1} \rangle$ . Quite clearly,  $w((i_0, i_1)) = 1$ . Thus

$$\boldsymbol{\xi}' = (\xi'_0, \xi'_1) = \left( \xi'_0, \frac{\alpha^n}{\xi'_0} \right) = \left( \prod_{k=\ell_0}^{\ell_1-1} \xi_{m(k)}, \prod_{k=\ell_1}^{\ell_2-1} \xi_{m(k)} \right) \quad (7)$$

and

$$\xi'_0 = \begin{cases} \prod_{k=i_0}^{i_1-1} \xi_k, & i_0 < i_1, \\ \alpha^n \prod_{k=i_1}^{i_0-1} \xi_k^{-1}, & i_0 > i_1. \end{cases} \quad (8)$$

By (1), the barycentric coordinates of the lattice points on  $\Delta'$  are

$$\left( \frac{[n]_\alpha - [n - \gamma_0]_\alpha}{[n]_\alpha - [n - \gamma_0]_\alpha + [n - \gamma_0]_\alpha \xi'_0}, \frac{[n - \gamma_0]_\alpha \xi'_0}{[n]_\alpha - [n - \gamma_0]_\alpha + [n - \gamma_0]_\alpha \xi'_0} \right), \quad \gamma_0 = n, n-1, \dots, 0, \quad (9)$$

as already obtained in [4]. However, if the lattice points (9) are prescribed, the corresponding  $\xi'_0, \xi'_1$ , and  $\alpha = \sqrt[n]{\xi'_0 \xi'_1}$  are not unique, even for  $n \geq 3$  ([4, Theorem 2]). In the latter case, there are precisely two pairs of parameters,

$$(\xi'_0, \xi'_1), \quad \left( \frac{1}{\xi'_1}, \frac{1}{\xi'_0} \right), \quad (10)$$

that generate the same lattice points (9). This is straightforward to deduce from identities

$$\frac{1}{\alpha^{2n-1-\gamma_0}} ([n]_\alpha - [n - \gamma_0]_\alpha) = [n]_{\frac{1}{\alpha}} - [n - \gamma_0]_{\frac{1}{\alpha}}, \\ \frac{1}{\alpha^{2n-1-\gamma_0}} [n - \gamma_0]_\alpha = \frac{1}{\alpha^n} [n - \gamma_0]_{\frac{1}{\alpha}}.$$

Now let us extend the example to line segments of an edge cycle

$$\langle \mathbf{T}_{i_k}, \mathbf{T}_{i_{k+1}} \rangle, \quad k = 0, 1, \dots, r, \quad i_{r+1} = i_0,$$

with  $\mathbf{i} = (i_k)_{k=0}^r$ , and  $(\ell_k)_{k=0}^{r+1} = (u(i_0, i_1, \dots, i_r, i_0))$ . Let  $(\xi'_{0,k}, \xi'_{1,k})$  denote parameters of the restriction of the lattice to  $\langle \mathbf{T}_{i_k}, \mathbf{T}_{i_{k+1}} \rangle$ . From (7) and (8)



one obtains

$$\prod_{k=0}^r \xi'_{0,k} = \prod_{k=0}^r \prod_{t=\ell_k}^{\ell_{k+1}-1} \xi_{m(t)} = \prod_{t=\ell_0}^{\ell_{r+1}-1} \xi_{m(t)} = \alpha^{n \cdot w(\mathbf{i})}, \quad (11)$$

that gives the value  $\alpha$  in terms of parameters  $\xi'_{0,k}$  only. Consider the lattice points at a particular edge  $\langle \mathbf{T}_{i_k}, \mathbf{T}_{i_{k+1}} \rangle$ . By (10) they could be generated as a restriction of at most two different classes of lattices, the one with

$$\alpha = \sqrt[n]{\xi'_{0,k} \xi'_{1,k}},$$

or the additional one, having

$$\alpha = \frac{1}{\sqrt[n]{\xi'_{0,k} \xi'_{1,k}}}.$$

In order to explore the second possibility further, let  $\tau_k, 0 < \tau_k < 1$ , be the first barycentric coordinate of a lattice point given by (9) on  $\langle \mathbf{T}_{i_k}, \mathbf{T}_{i_{k+1}} \rangle$  at  $\gamma_0 = n - 1$ . Such a lattice point exists for any  $n \geq 2$ . Then

$$\xi'_{0,k} = \xi'_{0,k}(\alpha) := \frac{1 - \tau_k}{\tau_k} ([n]_\alpha - 1),$$

and (11) simplifies to

$$\prod_{k=0}^r \xi'_{0,k}(\alpha) = \left( \prod_{k=0}^r \frac{1 - \tau_k}{\tau_k} \right) ([n]_\alpha - 1)^{r+1} = \alpha^{n \cdot w(\mathbf{i})}.$$

The equation

$$f(\rho) := [n]_\rho - 1 - c \rho^{\frac{n \cdot w(\mathbf{i})}{r+1}} = 0, \quad c := \left( \prod_{k=0}^r \frac{1 - \tau_k}{\tau_k} \right)^{-\frac{1}{r+1}} > 0,$$

has at least one positive solution,  $\rho = \alpha$ , by the assumption. But  $f$  is a polynomial in  $\sqrt[r+1]{\rho}$ , and the Descartes's rule of signs shows that there are at most two zeros of  $f$  in  $(0, \infty)$ . If there are two, then by the observation for a particular edge the zeros are necessarily  $\rho$  and  $1/\rho$ , and an elimination of  $c$  from

$$f(\rho) = 0, \quad f\left(\frac{1}{\rho}\right) = 0,$$

yields

$$\frac{[n]_\rho - 1}{\rho^{\frac{n \cdot w(\mathbf{i})}{r+1}}} = \frac{[n]_{1/\rho} - 1}{\rho^{-\frac{n \cdot w(\mathbf{i})}{r+1}}}. \quad (12)$$

However,

$$[n]_\rho - 1 = \rho [n - 1]_\rho, \quad [n]_{1/\rho} - 1 = \rho^{-(n-1)} [n - 1]_\rho,$$

and (12) reduces to

$$\rho^{n\left(\frac{2w(\mathbf{i})}{r+1}-1\right)} - 1 = 0,$$

that can only be satisfied for a positive  $\rho$ ,  $\rho \neq 1$ , iff  $w(\mathbf{i}) = \frac{r+1}{2}$ . Thus we obtain the following observation. Suppose the restriction of a  $(d+1)$ -pencil lattice of order  $n$  with a barycentric representation determined by parameters  $\xi = (\xi_0, \xi_1, \dots, \xi_d)$  is known for some edge of a simplex. By (9) we can then determine whether the corresponding  $\alpha = \sqrt[n]{\prod_{k=0}^d \xi_k} = 1$  or  $\alpha \neq 1$ . If  $\alpha \neq 1$  (thus  $\alpha \neq 1/\alpha$ ) there could be two classes of lattices, having the same restriction to this edge. The following theorem shows that in this case the restriction to a particular edge cycle has to be known, in order to determine the corresponding  $\alpha$  uniquely.

**Theorem 2** *Let the barycentric representation of a  $(d+1)$ -pencil lattice of order  $n$  on a  $d$ -simplex  $\Delta = \langle \mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_d \rangle$  be given by the parameters  $\xi = (\xi_0, \xi_1, \dots, \xi_d)$  and let  $\prod_{k=0}^d \xi_k \neq 1$ . A restriction of the lattice to a cycle*

$$\langle \mathbf{T}_{i_k}, \mathbf{T}_{i_{k+1}} \rangle, \quad k = 0, 1, \dots, r, \quad i_{r+1} := i_0, \quad \mathbf{i} = (i_k)_{k=0}^r,$$

*determines the corresponding*

$$\alpha = \sqrt[n]{\prod_{k=0}^d \xi_k}$$

*uniquely iff  $w(\mathbf{i}) \neq \frac{r+1}{2}$ .*

It is obvious that a  $(d+1)$ -pencil lattice on  $\Delta$  is determined if restrictions to all its edges are known. But only particular  $d+1$  edges are actually needed (see Fig. 4), as proves the following theorem. For simplicity, let  $\mathcal{G}(S)$  denote a graph induced by vertices and edges of a simplicial complex determined by  $S$ . Here  $S$  is a union of some arbitrarily dimensional faces of a simplex. Moreover, the subgraph  $\mathcal{G}(S_1)$  spans the graph  $\mathcal{G}(S)$  if the sets of vertices of both graphs coincide.

**Theorem 3** *A  $(d+1)$ -pencil lattice on  $\Delta = \langle \mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_d \rangle$  with parameters  $\xi = (\xi_0, \xi_1, \dots, \xi_d)$  is uniquely determined by restrictions to distinct edges*

$$e_k = \langle \mathbf{T}_{i_k}, \mathbf{T}_{j_k} \rangle, \quad k = 0, 1, \dots, d,$$

*iff the graph  $g := \mathcal{G}\left(\bigcup_{k=0}^d e_k\right)$  spans the graph  $\mathcal{G}(\Delta)$  and*

- (a)  $\prod_{k=0}^d \xi_k = 1$  or
- (b)  $g$  contains a cycle

$$e_{t_q} = \langle \mathbf{T}_{i_{t_q}}, \mathbf{T}_{j_{t_q}} \rangle, \quad q = 0, 1, \dots, r,$$

with  $i_{t_{q+1}} = j_{t_q}$ ,  $q = 0, 1, \dots, r-1$ ,  $j_{t_r} = i_{t_0}$ , such that

$$w \left( \left( i_{t_q} \right)_{q=0}^r \right) \neq \frac{r+1}{2}. \quad (13)$$

**PROOF.** If  $g$  does not span  $\mathcal{G}(\Delta)$ , one can find a vertex  $\mathbf{T}_t \in \mathcal{G}(\Delta)$  such that  $\{e_k\}_{k=0}^d \subset \Delta' = \langle \mathbf{T}_0, \dots, \mathbf{T}_{t-1}, \mathbf{T}_{t+1}, \dots, \mathbf{T}_d \rangle$ . Let the lattice on  $\Delta$  be given by  $\boldsymbol{\xi} = (\xi_k)_{k=0}^d$ . By Theorem 1, its restriction to  $\Delta'$  is determined by parameters  $(\xi_0, \dots, \xi_{t-2}, \xi_{t-1}\xi_t, \dots, \xi_d)$ . That makes impossible to recover both  $\xi_{t-1}$  and  $\xi_t$ , since only the product  $\xi_{t-1}\xi_t$  is pinned down. Suppose now that  $g$  spans  $\mathcal{G}(\Delta)$ . Let  $e' \in \mathcal{G}(\Delta)$  be any edge such that  $e' \notin g$ . Then there exists a cycle in  $\mathcal{G}(\cup_{k=0}^d e_k \cup e')$  that contains  $e'$ . The restriction of the lattice to  $e'$  is determined by (11) iff  $\alpha^n = \prod_{k=0}^d \xi_k$  is known. But the latter is assured by the assumptions (a) or (b) and Theorem 2. Thus a restriction of the lattice to any edge is determined, and restrictions to the edges  $\langle \mathbf{T}_k, \mathbf{T}_{k+1} \rangle$ ,  $k = 0, 1, \dots, d$ , yield parameters  $\boldsymbol{\xi}$ . The proof is completed.  $\square$

Note that this result covers also the smallest cycle, i.e.,  $\langle \mathbf{T}_i, \mathbf{T}_j \rangle$ ,  $\langle \mathbf{T}_j, \mathbf{T}_i \rangle$ .

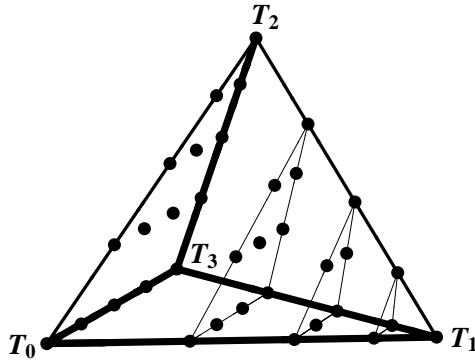


Fig. 4. A restriction to  $d+1$  edges that uniquely determines the lattice on the simplex.

The assumption (13) in Theorem 3 is clearly used to determine the product  $\alpha^n$  uniquely. But if this product is known, Theorem 3 simplifies to the following corollary that needs no additional proof.

**Corollary 4** *Suppose that the product*

$$\alpha^n = \prod_{k=0}^d \xi_k,$$

*that corresponds to the barycentric representation of a  $(d+1)$ -pencil lattice with parameters  $\boldsymbol{\xi}$  on  $\Delta = \langle \mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_d \rangle$ , is known. The lattice is determined*

by restrictions to distinct edges

$$e_k = \langle \mathbf{T}_{i_k}, \mathbf{T}_{j_k} \rangle, \quad k = 1, 2, \dots, d,$$

iff the graph  $g := \mathcal{G} \left( \bigcup_{k=1}^d e_k \right)$  spans the graph  $\mathcal{G}(\Delta)$ .

Now we turn our attention to a relation between two  $(d + 1)$ -pencil lattices of order  $n$  that share a common face. Since this face is a simplex too, the first step is to determine when two lattices defined over the same simplex are equivalent, i.e., they have the same lattice points. As expected, the choice of centers is inherent to equivalent lattices.

**Theorem 5** *Let  $\Delta$  be a given simplex, with vertices ordered as*

$$\Delta = \langle \mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_d \rangle, \quad (14)$$

and reordered according to an index vector  $\mathbf{i} = (i_0, i_1, \dots, i_d)$  as

$$\Delta' = \langle \mathbf{T}'_0, \mathbf{T}'_1, \dots, \mathbf{T}'_d \rangle = \langle \mathbf{T}_{i_0}, \mathbf{T}_{i_1}, \dots, \mathbf{T}_{i_d} \rangle. \quad (15)$$

Suppose that on the simplices  $\Delta$  and  $\Delta'$  there are given two  $(d + 1)$ -pencil lattices of order  $n$ , with barycentric coordinates determined by parameters  $\boldsymbol{\xi} = (\xi_0, \xi_1, \dots, \xi_d)$ , and  $\boldsymbol{\xi}' = (\xi'_0, \xi'_1, \dots, \xi'_d)$  w.r.t. the vertex sequences (14) and (15), respectively. Both lattices share the same lattice points iff one of the following possibilities holds:

- (a)  $w(\mathbf{i}) = 1$ , and  $\xi'_j = \xi_{i_j}$ ,  $j = 0, 1, \dots, d$ ;
- (b)  $w(\mathbf{i}) = d$ , and  $\xi'_j = \frac{1}{\xi_{i_{j+1}}}$ ,  $j = 0, 1, \dots, d$ ;
- (c)  $1 < w(\mathbf{i}) < d$ , and  $\prod_{j=0}^d \xi_j = 1$ ,

$$\xi'_j = \begin{cases} \prod_{k=i_j}^{i_{j+1}-1} \xi_k, & i_j < i_{j+1}, \\ \prod_{k=i_{j+1}}^{i_j-1} \frac{1}{\xi_k}, & i_j > i_{j+1}, \end{cases} \quad j = 0, 1, \dots, d.$$

**PROOF.** It is straightforward to verify the assertion if  $d = 1$ . Suppose now that  $d > 1$ . Then there is a 3-cycle along the edges of the simplices  $\Delta$  and  $\Delta'$ . So, by Theorem 2, the products  $\alpha^n = \prod_{j=0}^d \xi_j$  and  $\alpha'^n = \prod_{j=0}^d \xi'_j$  are determined uniquely. Let us consider a restriction of the lattice determined by  $\boldsymbol{\xi}$  to  $\langle \mathbf{T}_{i_j}, \mathbf{T}_{i_{j+1}} \rangle$ , and let us denote  $\boldsymbol{\ell} = (\ell_j)_{j=0}^{d+1} = u((i_0, i_1, \dots, i_d, i_0))$ . The lattice points of both lattices should coincide. Theorem 1 and the relation (10) reveal

two possible choices,

$$\xi'_j = \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)} \quad (16)$$

if  $\alpha' = \alpha$ , and

$$\xi'_j = \frac{1}{\alpha^n} \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)} \quad (17)$$

if  $\alpha' = \frac{1}{\alpha}$ . Of course,  $\xi'_j$  can always be determined from (16) or (17). However, the relation between  $\alpha$  and  $\alpha'$  should not be violated. Let us multiply these equations for all possible  $j$  together. From (16) we obtain

$$\prod_{j=0}^d \xi'_j = \alpha'^n = \alpha^n = \prod_{j=0}^d \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)} = \alpha^{n \cdot w(\mathbf{i})}.$$

This relation could only be satisfied if  $w(\mathbf{i}) = 1$  (the assertion (a)), or  $\alpha = \alpha' = 1$ . Similarly,

$$\prod_{j=0}^d \xi'_j = \alpha'^n = \frac{1}{\alpha^n} = \prod_{j=0}^d \frac{1}{\alpha^n} \prod_{k=\ell_j}^{\ell_{j+1}-1} \xi_{m(k)} = \alpha^{n \cdot (w(\mathbf{i}) - (d+1))}$$

confirms (b). If  $1 < w(\mathbf{i}) < d$ , only the possibility  $\alpha = \alpha' = 1$  is left, and a brief look on (8) completes the necessary part of the proof. But if either one of the possibilities (a), (b) or (c) holds, the lattices agree on all edges of  $\Delta$ , i.e.,  $\langle \mathbf{T}'_j, \mathbf{T}'_k \rangle = \langle \mathbf{T}_{i_j}, \mathbf{T}_{i_k} \rangle$ ,  $j < k$ , and therefore on the whole simplex.  $\square$

If  $\alpha = \alpha' = 1$ , both lattices can coincide for any winding number of the index vector  $\mathbf{i}$ . But consecutively a restriction on lattice parameters is obtained. Theorem 5 clearly suggests how a lattice known at some face should be extended to a whole simplex if one is not prepared to loose a degree of freedom with the assumption  $\alpha = 1$ .

**Corollary 6** *Let  $\Delta = \langle \mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_r \rangle$  be a given face, with the lattice determined by  $\boldsymbol{\xi} = (\xi_0, \xi_1, \dots, \xi_r)$ . The lattice can be extended to*

$$\Delta' = \langle \mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_i, \mathbf{T}', \mathbf{T}_{i+1}, \dots, \mathbf{T}_r \rangle \subset \mathbb{R}^{r+1}$$

*by parameters*

$$\boldsymbol{\xi}' = \left( \xi_0, \xi_1, \dots, \xi_{i-1}, \eta, \frac{\xi_i}{\eta}, \xi_{i+1}, \dots, \xi_r \right),$$

*where  $\eta > 0$  is an additional free parameter.*

Now consider two  $(d+1)$ -pencil lattices of order  $n$  that share a lattice on a common face of simplices (Fig. 5). By combining Theorem 1 and Theorem 5 one obtains the following corollary.

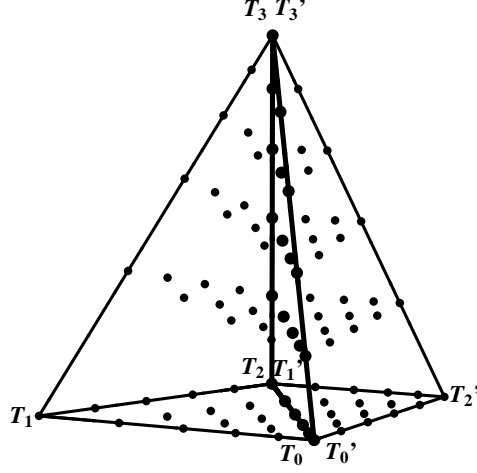


Fig. 5. Matching of two lattices for  $d = 3$  on the common facet of simplices.

**Corollary 7** *Let*

$$\Delta = \langle \mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_d \rangle, \quad \Delta' = \langle \mathbf{T}'_0, \mathbf{T}'_1, \dots, \mathbf{T}'_d \rangle$$

*be given simplices, and let the lattices be determined by parameters*

$$\boldsymbol{\xi} = (\xi_0, \xi_1, \dots, \xi_d), \quad \boldsymbol{\xi}' = (\xi'_0, \xi'_1, \dots, \xi'_d),$$

*respectively. Suppose that*

$$\langle \mathbf{T}_{i_0}, \mathbf{T}_{i_1}, \dots, \mathbf{T}_{i_r} \rangle = \langle \mathbf{T}'_{i'_0}, \mathbf{T}'_{i'_1}, \dots, \mathbf{T}'_{i'_r} \rangle, \quad 1 \leq r \leq d,$$

*$0 \leq i_0 < i_1 < \dots < i_r \leq d$ , is a common  $r$ -face of  $\Delta$  and  $\Delta'$ , with corresponding vertices*

$$\mathbf{T}_{i_k} = \mathbf{T}'_{i'_k}, \quad k = 0, 1, \dots, r.$$

*Let  $(\ell_0, \dots, \ell_{r+1}) = u((i_0, \dots, i_r, i_0))$  and  $(\ell'_0, \dots, \ell'_{r+1}) = u((i'_0, \dots, i'_r, i'_0))$ . If  $\alpha^n = \prod_{i=0}^d \xi_i \neq 1$ , the lattices agree at the common  $r$ -face iff one of the following possibilities holds:*

(a)  $w(\mathbf{i}') = 1$  and

$$\prod_{t=\ell_k}^{\ell_{k+1}-1} \xi_{m(t)} = \prod_{t=\ell'_k}^{\ell'_{k+1}-1} \xi'_{m(t)}, \quad k = 0, 1, \dots, r;$$

(b)  $w(\mathbf{i}') = r$  and

$$\prod_{t=\ell_k}^{\ell_{k+1}-1} \xi_{m(t)} = \alpha^n \prod_{t=\ell'_k}^{\ell'_{k+1}-1} \xi'_{m(t)}, \quad k = 0, 1, \dots, r.$$

## 4 Lattice on a simplicial partition

We are now able to extend a  $(d + 1)$ -pencil lattice from a simplex to a simplicial partition. Of course, this extension should be such that any pair of simplices that share a common face should share the lattice restriction to that face too. The following theorem and the corresponding proof provide an explicit approach for the construction of the extended  $(d + 1)$ -pencil lattice over a simplicial partition. This leads to an efficient computer algorithm for the design of a lattice. The simplest case,  $d = 2$ , has already been discussed in [4].

**Theorem 8** *Let  $\mathcal{T} = \{\Delta_i\}_{i \geq 0}$  be a regular simplicial partition in  $\mathbb{R}^d$  with  $V \geq d + 1$  vertices*

$$\mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_{V-1}. \quad (18)$$

*Then there exists a  $(d + 1)$ -pencil lattice on  $\mathcal{T}$  with precisely  $V$  degrees of freedom.*

Recall that a simplicial partition in  $\mathbb{R}^d$  is regular if every pair of adjacent simplices has an  $r$ -face in common,  $r \in \{0, 1, \dots, d - 1\}$ .

**PROOF.** For any simplex  $\Delta \in \mathcal{T}$ , let us order the points similarly as in (18), i.e.,

$$\Delta = \langle \mathbf{T}_{i_0}, \mathbf{T}_{i_1}, \dots, \mathbf{T}_{i_d} \rangle, \quad 0 \leq i_0 < i_1 < \dots < i_d \leq V - 1,$$

and let us choose the local barycentric representation of the lattice on each of the simplices accordingly. Note that this choice of local lattice control points assures that any pair of simplices  $\Delta, \Delta' \in \mathcal{T}$ ,

$$\Delta = \langle \mathbf{T}_{i_0}, \mathbf{T}_{i_1}, \dots, \mathbf{T}_{i_d} \rangle, \quad \Delta' = \langle \mathbf{T}_{i'_0}, \mathbf{T}_{i'_1}, \dots, \mathbf{T}_{i'_d} \rangle,$$

with a common  $r$ -face, denoted in  $\Delta$  as

$$\langle \mathbf{T}_{i_{j_0}}, \mathbf{T}_{i_{j_1}}, \dots, \mathbf{T}_{i_{j_r}} \rangle, \quad i_{j_0} < i_{j_1} < \dots < i_{j_r},$$

and corresponding vertices in  $\Delta'$  given by

$$\mathbf{T}_{i'_{j'_k}} = \mathbf{T}_{i_{j_k}}, \quad k = 0, 1, \dots, r,$$

satisfies

$$w((i_{j_0}, i_{j_1}, \dots, i_{j_r})) = w((i'_{j'_0}, i'_{j'_1}, \dots, i'_{j'_r})) = 1. \quad (19)$$

The proof proceeds by the induction on the number of simplices in a simplicial partition  $\mathcal{T}' \subset \mathcal{T}$ , with an additional assertion that a product of local barycentric lattice parameters for each simplex considered is equal to the same constant  $\alpha^n$ . Since  $\mathcal{T}$  is regular, we may, without loss of generality, assume that  $\mathcal{T}'$  grows from a single simplex to  $\mathcal{T}$  in such a way that each simplex added has  $f$ ,  $1 \leq f \leq d$ ,  $(d - 1)$ -faces in common with simplices in the instantaneous

partition  $\mathcal{T}'$ . If  $\mathcal{T}' = \{\Delta\}$ , then by (1) the lattice has  $d + 1$  free parameters  $\boldsymbol{\xi} = (\xi_i)_{i=0}^d$ , defining  $\alpha^n = \prod_{i=0}^d \xi_i$ . The number of degrees of freedom clearly equals the number of vertices of  $\mathcal{T}'$ . Thus the assertion holds true. Suppose now that it holds true for  $\mathcal{T}'$ , and let us show that it holds also for

$$\mathcal{T}' \cup \{\Delta'\}, \quad \Delta' = \langle \mathbf{T}_{i'_0}, \mathbf{T}_{i'_1}, \dots, \mathbf{T}_{i'_d} \rangle \notin \mathcal{T}'.$$

Let the local barycentric lattice representation on  $\Delta'$  depend on parameters  $\boldsymbol{\xi}' = (\xi'_i)_{i=0}^d$ , and let  $\{F_1, F_2, \dots, F_f\}$  be the set of all distinct  $(d - 1)$ -faces of  $\Delta'$  that are shared with simplices in  $\mathcal{T}'$ . Since (19) holds, Corollary 7(a) confirms that the lattice can be extended from the common face  $F_1$  to  $\Delta'$  provided particular  $d$  relations concerning  $\boldsymbol{\xi}'$  are satisfied. With an index  $r$  uniquely determined by  $\mathbf{T}_{i'_r} \in \Delta' \setminus F_1$ , these relations determine  $d$  values

$$\left( \xi'_0, \xi'_1, \dots, \xi'_{r-2}, \xi'_{r-1} \xi'_r, \xi'_{r+1}, \xi'_{r+2}, \dots, \xi'_d \right),$$

and assure  $\prod_{i=0}^d \xi'_i = \alpha^n$ . If  $f = 1$ ,  $\mathbf{T}_{i'_r} \notin \mathcal{T}'$ . So  $\mathcal{T}' \rightarrow \mathcal{T}' \cup \{\Delta'\}$  brings up precisely one additional vertex as well as one additional free parameter, and the induction step in the case  $f = 1$  is concluded. Let now  $2 \leq f \leq d$ . The number of vertices in  $\mathcal{T}' \cup \{\Delta'\}$  is equal to the number of vertices in  $\mathcal{T}'$ . At least one of the edges  $\langle \mathbf{T}_{i'_{r-1}}, \mathbf{T}_{i'_r} \rangle$ , and  $\langle \mathbf{T}_{i'_r}, \mathbf{T}_{i'_{r+1}} \rangle$  belongs to  $F_2$ . Let us denote it by  $e$ . Since  $\alpha$  has already been determined, a restriction of the lattice to the edge  $e$  determines the last free parameter in  $\boldsymbol{\xi}'$  uniquely. Note that the lattice given by  $\boldsymbol{\xi}'$  by the construction agrees with any lattice on  $F_2$ , inherited from  $\mathcal{T}'$ , on  $F_1 \cap F_2$  and  $e$ . But  $\mathcal{G}((F_1 \cap F_2) \cup e)$  spans  $\mathcal{G}(F_2)$ , so by Corollary 4 both lattices have to coincide on all of  $F_2$ . Similarly,  $\mathcal{G}((F_1 \cap F_j) \cup (F_2 \cap F_j))$  spans  $\mathcal{G}(F_j)$  for any  $j$ ,  $3 \leq j \leq d$ , and the lattice given by  $\boldsymbol{\xi}'$  agrees with inherited lattice on any  $F_j$ . The induction step in the case  $f > 1$  is concluded too, and the proof is completed.  $\square$

## 5 Example

In this section an example for the case  $d = 3$  is given, which illustrates the results from previous sections. Here  $\Delta = \langle \mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3 \rangle$  is a tetrahedron. Let us observe an example of a star ([7]) with  $2m - 2$ ,  $m \geq 3$ , tetrahedrons, where  $m$  and  $m - 2$  tetrahedrons are glued together in such a way, that they share a common edge, respectively (see Fig. 6). This example also covers the minimal possible star in  $\mathbb{R}^3$  with 4 tetrahedrons ( $m = 3$ ). Our aim is to explicitly express  $(d + 1)(2m - 2) = 8(m - 1)$  parameters  $\xi_j^{(i)} > 0$ ,  $j = 0, \dots, 3$ ,  $i = 1, \dots, 2m - 2$ , with  $V = m + 2$  independent free parameters that determine the lattice on this simplicial partition with  $V$  vertices and  $2m - 2$  tetrahedrons. Here  $\xi_j^{(i)}$  is the parameter that determines the control point  $\mathbf{P}_j^{(i)}$  of a lattice on



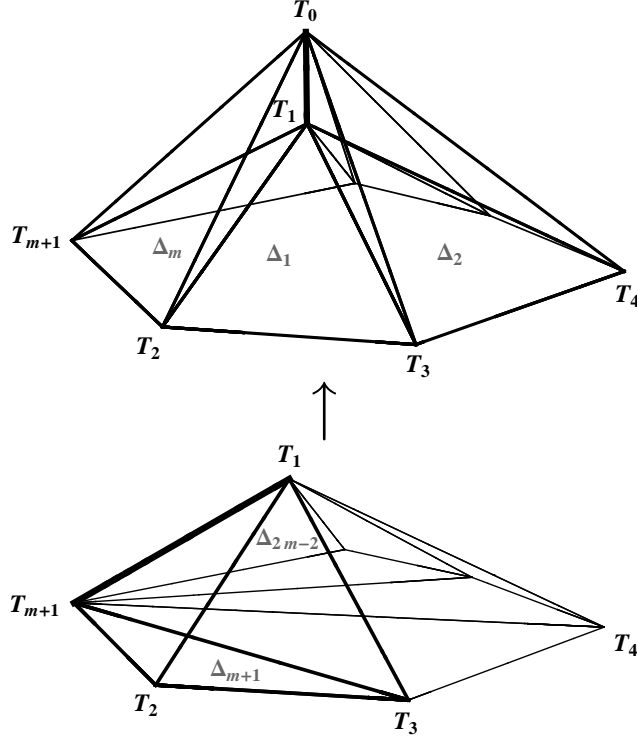


Fig. 6. The star with  $2m - 2$  tetrahedrons, where  $m$  and  $m - 2$  tetrahedrons have a common edge, respectively.

the  $i$ -th tetrahedron  $\Delta_i$  (Fig. 6). Let us label the vertices of the simplicial partition with  $\mathbf{T}_i$ ,  $i = 0, 1, \dots, m + 1$ , (Fig. 6) and let us denote the simplices as  $\Delta_i := \langle \mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_{i+1}, \mathbf{T}_{i+2} \rangle$ ,  $i = 1, \dots, m$ ,  $\mathbf{T}_{m+2} := \mathbf{T}_2$ , and  $\Delta_i := \langle \mathbf{T}_1, \mathbf{T}_{i-m+1}, \mathbf{T}_{i-m+2}, \mathbf{T}_{m+1} \rangle$ ,  $i = m + 1, \dots, 2m - 2$  (Fig. 6). The construction in the proof of Theorem 8 gives us the relations between the parameters  $\xi_j^{(i)}$  so that the lattice points on all common faces of the star agree. Let us consider how the parameters for the lattices on  $\Delta_1$  and  $\Delta_2$  are related. Since  $w((0, 1, 3)) = w((0, 1, 2)) = 1$ , the lattice on  $\Delta_2$  is by Corollary 7 determined with parameters  $\xi_j^{(1)}$ ,  $j = 0, 1, 2, 3$ , and the additional one  $\xi_2^{(2)}$  as

$$\xi_0^{(2)} = \xi_0^{(1)}, \quad \xi_1^{(2)} = \xi_1^{(1)} \xi_2^{(1)}, \quad \xi_2^{(2)} = \xi_2^{(2)}, \quad \xi_3^{(2)} = \frac{\xi_3^{(1)}}{\xi_2^{(2)}}.$$

Using a similar approach for all simplices  $\Delta_i$ , all parameters can be expressed by  $V$  parameters

$$\xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}, \xi_2^{(2)}, \xi_2^{(3)}, \dots, \xi_2^{(m-1)}$$

as

$$\xi_0^{(i)} = \xi_0^{(1)}, \quad \xi_1^{(i)} = \xi_1^{(1)} \xi_2^{(1)} \prod_{j=2}^{i-1} \xi_2^{(j)}, \quad \xi_2^{(i)} = \xi_2^{(i)}, \quad \xi_3^{(i)} = \frac{\xi_3^{(1)}}{\prod_{j=2}^i \xi_2^{(j)}},$$

for  $i = 2, 3, \dots, m - 1$ , and

$$\begin{aligned}\xi_0^{(m)} &= \xi_0^{(1)}, \quad \xi_1^{(m)} = \xi_1^{(1)}, \quad \xi_2^{(m)} = \xi_2^{(1)} \prod_{j=2}^{m-1} \xi_2^{(j)}, \quad \xi_3^{(m)} = \frac{\xi_3^{(1)}}{\prod_{j=2}^{m-1} \xi_2^{(j)}}, \\ \xi_0^{(m+1)} &= \xi_1^{(1)}, \quad \xi_1^{(m+1)} = \xi_2^{(1)}, \quad \xi_2^{(m+1)} = \prod_{j=2}^{m-1} \xi_2^{(j)}, \quad \xi_3^{(m+1)} = \frac{\xi_0^{(1)} \xi_3^{(1)}}{\prod_{j=2}^{m-1} \xi_2^{(j)}}, \\ \xi_0^{(m+i)} &= \xi_1^{(1)} \xi_2^{(1)} \prod_{j=2}^{i-1} \xi_2^{(j)}, \quad \xi_1^{(m+i)} = \xi_2^{(i)}, \quad \xi_2^{(m+i)} = \prod_{j=i+1}^{m-1} \xi_2^{(j)}, \quad \xi_3^{(m+i)} = \frac{\xi_0^{(1)} \xi_3^{(1)}}{\prod_{j=2}^{m-1} \xi_2^{(j)}},\end{aligned}$$

for  $i = 2, 3, \dots, m - 2$ .

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