ON GEOMETRIC INTERPOLATION BY POLYNOMIAL CURVES

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Abstract. In this paper, geometric interpolation by parametric polynomial curves is considered. Discussion is focused on the case where the number of interpolated points is equal to $d+2$, and d denotes the degree of the interpolating polynomial curve. The interpolation takes place in $\mathbb{R}^d$. Even though the problem is nonlinear, simple necessary and sufficient conditions for existence of the solution are stated. These conditions are entirely geometric, and do not depend on the asymptotic analysis. Furthermore, they provide an efficient and stable way to the numeric solution of the problem.

Key words. polynomial curve, geometric interpolation, existence, uniqueness, approximation order

AMS subject classifications. 65D05, 65D07

1. Introduction.

Let a sequence of data points

$$\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_{r+1} \in \mathbb{R}^d, \ \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1},$$

be given. A parametric curve interpolates these points in the geometric sense if the parameter values at which it passes through the points $\boldsymbol{T}_i$ are not prescribed in advance. In the limiting case of the geometric interpolation if two consecutive points coincide, this scheme leads to the interpolation of a point, and a tangent direction at the same parameter value. Further, threefold interpolation at a point requires also the curvature to be known there, etc. The threefold interpolation by cubics in the plane can be traced back to [3], the paper that initiated the study of geometric interpolation. In order to make the proofs of the results simple, only distinct points $\boldsymbol{T}_i$ will be considered in this paper. The extension to the osculatory case will appear elsewhere.

The disadvantage of the geometric approach is obvious. Namely, the problem of finding the interpolatory curve is nonlinear, so the questions of existence, uniqueness, and computation of the solution arise.

However, there are important advantages too. Free parameter values at which the points $\boldsymbol{T}_i$ are interpolated may raise the approximation order. This fact has been observed in [3], and in many of the subsequent papers. As a bound for the polynomial geometric interpolation, it has been conjectured in [5] that a parametric polynomial curve of degree n in d dimensional Euclidean space can, in general, interpolate

$$r+2 = n+1+\left\lfloor \frac{n-1}{d-1} \right\rfloor$$

points in $\mathbb{R}^d$, and reach the same approximation order. The conjecture has been proved only for a few particular cases. But perhaps the most important bonus of all is that the geometric approach provides the basis for the G^m continuous spline schemes where the interpolants do not depend on the local parametrisation. This is an important and often-required property in the CAGD applications.

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Suppose now that the interpolatory curve is a parametric polynomial curve

$$\boldsymbol{B}_n : [a, b] \to \mathbb{R}^d$$

of degree n. Since linear reparametrisation does not change the degree of the polynomial, the assumption $a := 0$ and $b := 1$ can be made. Thus the construction of $\boldsymbol{B}_n$ requires to determine t_i,

$$t_0 := 0 < t_1 < t_2 < \ldots < t_r < t_{r+1} := 1, \tag{1.1}$$

such that

$$\boldsymbol{B}_n(t_i) = \boldsymbol{T}_i, \; i = 0, 1, \ldots, r+1. \tag{1.2}$$

This is the nonlinear part of the problem. Once t_i are known, it is straightforward to obtain the curve $\boldsymbol{B}_n$ in any of the well-known forms such as Bézier, Newton or Lagrange.

In order to keep the number of free parameters equal to the number of the unknowns, a certain Diophantine equation has to be satisfied ([4]). The case

$$n = r = d \tag{1.3}$$

turns out to be the simplest to handle ([6]). Nevertheless, few results can be found in the literature without the assumption that the data points are sufficiently dense and taken from some smooth underlying curve. In the plane case ($d = 2$), some results are included in [10, 8, 9], and in the space case ($d = 3$) in [4].

As it will be shown below, the case $n = r = d$ can be worked out without any asymptotic assumptions. Perhaps the methods applied here could be used to study more complicated problems too, such as those outlined in [11], and even the spline case ([12]).

Let us assume (1.3) throughout the paper, and simplify the notation of $\boldsymbol{B}_n$ to

$$\boldsymbol{B} := \boldsymbol{B}_n = \boldsymbol{B}_d.$$

The equations that determine the unknowns t_i in this particular case will be worked out in the next section.

The key role in the paper is played by the matrix of data differences

$$\Delta T := \left(\Delta \boldsymbol{T}_i\right)_{i=0}^{d}, \qquad \Delta \boldsymbol{T}_i := \boldsymbol{T}_{i+1} - \boldsymbol{T}_i, \tag{1.4}$$

and by the signs of its minors

$$D_i := \det \left(\Delta \boldsymbol{T}_j\right)_{\substack{j=0\\ j\neq i}}^{d} \tag{1.5}$$

i.e., the signs of the volumes of the d-simplexes, spanned by the vectors

$$\Delta \boldsymbol{T}_0, \Delta \boldsymbol{T}_1, \ldots, \Delta \boldsymbol{T}_{i-1}, \Delta \boldsymbol{T}_{i+1}, \ldots, \Delta \boldsymbol{T}_d.$$

If the vectors $\Delta \boldsymbol{T}_i$ do not belong to a proper subspace of $\mathbb{R}^d$, and are not lying on a polynomial curve of degree $< d$, the matrix ΔT is of full rank, and the following conclusion can be made.

THEOREM 1.1. *Suppose* rank $\Delta T = d$. *Then the interpolating curve* $\boldsymbol{B}$ *exists if and only if the minors* D_i *are all of the same sign. If* $\boldsymbol{B}$ *exists it is regular, and the parameter values* $\boldsymbol{t} := (t_i)_{i=1}^d$ *are determined uniquely.*

In the plane case, the signs of the D_i can be identified by certain angles, as has been already observed in [10, 8]. More generally, if the data are convex in the discrete sense, one has

$$\operatorname{sign}(D_0) \;=\; \operatorname{sign}(D_d)\,.$$

The additional requirements of Theorem 1.1 simply guarantee that the data stay convex under the translations

$$\boldsymbol{T}_j \to \boldsymbol{T}_j - \Delta \boldsymbol{T}_i, \quad j = i+1, i+2, \ldots, d, \quad i = 1, 2, \ldots, d-1,$$

i.e., they are not too twisted.

Let $\mathrm{S}^- : \mathbb{R}^{d+1} \to \{0, 1, \ldots, d\}$ denote the number of strong sign changes in $\boldsymbol{x} \in \mathbb{R}^{d+1}$. Then Theorem 1.1 actually requires that the kernel of ΔT is spanned by

$$\boldsymbol{x} = \left((-1)^i D_i\right)_{i=0}^d,$$

with

$$\mathrm{S}^-(\boldsymbol{x}) = d.$$

This observation can be extended to the case of deficient rank $\Delta T < d$, but then the uniqueness or the regularity of a solution can not be expected. Still the following fact can be established.

THEOREM 1.2. *Let* rank $\Delta T < d$. *An interpolating curve* $\boldsymbol{B}$ *of degree* $\leq d$ *can be found if and only if there exists* $\boldsymbol{x} \in \ker \Delta T$ *such that* $\mathrm{S}^-(\boldsymbol{x}) = d$.

In the setup of Theorem 1.2, all regular $\boldsymbol{B}$ will return the same interpolatory curve, considered as a set of points. But the speed of moving along the curve will be different. The additional free parameters should be used to decrease the degree of the interpolating curve if possible. If the obtained lower-degree curve is unique, the proof of Theorem 1.1 can be repeated and the conclusion that it is regular can also be made. Reduction of the degree is not always possible. As an example take a cubic curve that interpolates five points in $\mathbb{R}^3$. If the data are lying on a plane, a cubic is still needed as a quadratic curve can interpolate four planar points in general.

Although the problem of determining the unknowns t_i is nonlinear, there is an efficient and stable way to the numerical solution, given as the following result.

THEOREM 1.3. *Suppose that the requirements of Theorem 1.1 are satisfied. The continuation method [1] will always compute the numerical solution.*

Practical evidence shows that the best way is to start the continuation method as one-step method. This step has to be reduced only if the solution lies near the boundary of (1.1).

2. The equations. Under the assumption (1.3) the system (1.2) can be rewritten as

$$\boldsymbol{B}(t_i) = \boldsymbol{T}_i, \quad i = 0, 1, \ldots, d+1,$$

where the unknowns are (vector) coefficients of the polynomial curve $\boldsymbol{B}$, and scalars $(t_i)_{i=1}^d$ that have to satisfy (1.1). But the divided difference on arbitrary $d+2$ points maps a polynomial of degree $\leq d$ to zero, so

$$[t_0, t_1, \ldots, t_{d+1}]\boldsymbol{B} = \boldsymbol{0},$$

and $[t_0, t_1, \ldots, t_{d+1}]$ should map the data $\boldsymbol{T}_i$ to zero too. Since t_i are required to be different, this fact can be written as

$$\sum_{i=0}^{d+1} \frac{1}{\dot\omega(t_i)} \boldsymbol{T}_i = \boldsymbol{0}, \quad \omega(t) := \prod_{i=0}^{d+1} (t - t_i), \tag{2.1}$$

i.e., d scalar equations for d scalar unknowns $t_1, t_2, \ldots, t_d$. The equations (2.1) are the only nonlinear part on the way to the interpolatory curve $\boldsymbol{B}$, and one can efficiently solve them by the continuation method as stated in Theorem 1.3. The final construction of $\boldsymbol{B}$ then follows the function case and is straightforward.

3. The proofs. The assertions in the introduction seem quite simple, but the proofs will take several steps. Here is a brief outline:

1. The system (2.1) will be transformed in a form more suitable for the analysis of the existence and the uniqueness.
2. It will be shown that the existence of a unique solution of the system (2.1) implies that D_i should all be of the same sign (with Lemma 3.1 as a part of the proof).
3. Lemma 3.2 will establish the fact that any solution of (2.1) that satisfies (1.1) should be simple, and Lemma 3.3 will assure that any such solution could not be arbitrarily close to the boundary.
4. A proof that the system (2.1) has a unique solution for particular data will be outlined.
5. The convex homotopy will help to carry over the conclusions from the particular to the general case in order to complete the proof of Theorem 1.1.
6. The proofs of Theorems 1.2 and 1.3 will complete the section.

As to the first step, let us recall that $[t_0, t_1, \ldots, t_{d+1}]1 = 0$. So the system (2.1) can be rewritten as

$$\sum_{i=0}^{d+1} \frac{1}{\dot\omega(t_i)} (\boldsymbol{T}_i - \boldsymbol{T}_0) = \sum_{i=1}^{d+1} \frac{1}{\dot\omega(t_i)} (\boldsymbol{T}_i - \boldsymbol{T}_0) = \boldsymbol{0}, \tag{3.1}$$

or

$$(\boldsymbol{T}_i - \boldsymbol{T}_0)_{i=1}^{d+1} \boldsymbol{\omega}, \tag{3.2}$$

where

$$\boldsymbol{\omega} := \left(\frac{1}{\dot\omega(t_i)} \right)_{i=1}^{d+1}. \tag{3.3}$$

By inserting $I = U^{-1}U$ between the two factors in (3.2), where

$$U := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \in \mathbb{R}^{d+1,d+1}, \quad U^{-1} = \begin{pmatrix} 1 & -1 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix},$$

the equations (3.1) become

$$\Delta T \, \boldsymbol{\omega}_{\Sigma} = \boldsymbol{0}, \tag{3.4}$$

with

$$\Delta T = (\boldsymbol{T}_i - \boldsymbol{T}_0)_{i=1}^{d+1} U^{-1} \in \mathbb{R}^{d,d+1}$$

defined by (1.4), and

$$\boldsymbol{\omega}_{\Sigma} := U\boldsymbol{\omega} = \left(\sum_{j=i}^{d+1} \frac{1}{\dot{\omega}(t_j)} \right)_{i=1}^{d+1}. \tag{3.5}$$

If at least one of the determinants D_i defined in (1.5) is different from zero, then ΔT is of full rank d, and the kernel of ΔT is spanned by the vector

$$\left((-1)^{d+1-i} D_i\right)_{i=0}^{d}.$$

Since $\boldsymbol{\omega}_{\Sigma}$ should be proportional to it, the nonlinear system (3.4) becomes

$$\alpha \sum_{j=i}^{d+1} \frac{1}{\dot{\omega}(t_j)} = (-1)^{d+1-i} D_{i-1}, \quad i = 1, 2, \ldots, d+1, \tag{3.6}$$

i.e., $d+1$ scalar equations for $d+1$ unknowns $\alpha, t_1, t_2, \ldots, t_d$.

The form of the system (3.6) is suitable to proceed with the second step of the proofs. Let $t_0 := 0 < t_1 < \cdots < t_d < t_{d+1} := 1$, and $\alpha \neq 0$ be a unique solution of the system (3.6). Then

$$D_d = \alpha \frac{1}{\dot{\omega}(t_{d+1})} \neq 0,$$

and $\operatorname{sign}(D_d) = \operatorname{sign}(\alpha)$. Thus

$$\operatorname{sign}(D_{i-1}) = \operatorname{sign}(\alpha), \quad i = 1, 2, \ldots, d,$$

if and only if $\mathrm{S}^-(\boldsymbol{\omega}_{\Sigma}) = d$. This fact will be established with the help of the following lemma.

Lemma 3.1. *Let p_i, $1 \le i \le d$, be the interpolating polynomial of degree $\le d+1$ that interpolates the data*

$$p_i(t_j) = \begin{cases} 0, & j = 0, 1, \ldots, i-1, \\ 1, & j = i, i+1, \ldots, d+1 \end{cases}$$

at $d+2$ distinct points $t_0 < t_1 < \cdots < t_{d+1}$. Then p_i is of degree $d+1$, and the sign of its leading coefficient is equal to $(-1)^{d+1-i}$.

Proof. The interpolating conditions imply $p_i \neq \text{const}$, thus $p_i' \neq 0$. By Rolle's theorem, p_i' has at least $i-1$ zeros on (t_0, t_{i-1}) and at least $d-i+1$ zeros on (t_i, t_{d+1}), i.e., at least d zeros on (t_0, t_{d+1}). Since p_i' does not vanish identically, the degree of p_i' is d, and the degree of p_i is $d+1$. Note that p_i' must be increasing on (t_{i-1}, t_i), and $\operatorname{sign}(p_i'(t_i)) = 1$. But then $\operatorname{sign}(p_i'(t_{d+1})) = (-1)^{d+1-i}$. Since the leading coefficient of p_i has to be of the same sign as $p_i'(t_{d+1})$, the lemma has been proved. □

Let p_i be the polynomial studied in Lemma 3.1. Its leading coefficient is equal to the divided difference

$$[t_0, t_1, \ldots, t_{d+1}]p_i = \sum_{j=0}^{d+1} \frac{p_i(t_j)}{\dot{\omega}(t_j)} = \sum_{j=i}^{d+1} \frac{1}{\dot{\omega}(t_j)},$$

and the fact

$$\operatorname{sign}\left(\sum_{j=i}^{d+1}\frac{1}{\dot\omega(t_j)}\right)=(-1)^{d+1-i}$$

is confirmed by the conclusion of Lemma 3.1. The first part of the proof of Theorem 1.1 is complete.

Let us continue with the step three of the proofs. If two consecutive equations in (3.6) are subtracted, the system reads

$$\frac{\alpha}{\dot\omega(t_i)}=(-1)^{d+1-i}(D_{i-1}+D_i),\quad i=1,2,\ldots,d+1,\quad D_{d+1}:=0. \tag{3.7}$$

A solution of the system (3.7) will be simple if the Jacobian J at that point is non-singular. A straightforward computation gives J as

$$J:=J(\boldsymbol{t},\alpha)=\operatorname{diag}\left(\frac{1}{\dot\omega(t_i)}\right)_{i=1}^{d+1}A, \tag{3.8}$$

with $A:=(a_{ij})_{i,j=1}^{d+1}$, and

$$a_{ij}=\begin{cases}\dfrac{\alpha}{t_i-t_j}, & i\neq j,\ j<d+1,\\ \displaystyle\sum_{\substack{k=0\\k\neq i}}^{d+1}\frac{\alpha}{t_k-t_i}, & i=j,\ j<d+1,\\ 1, & j=d+1.\end{cases}$$

The suggestions in [7] will help us to prove the following lemma.

Lemma 3.2. *The determinant of the matrix A is given as*

$$\det A=d!\,\alpha^d\,(t_0-t_{d+1})\frac{1}{\dot\omega(t_0)}.$$

Proof. By definition, $\det A$ is a sum of terms of the form

$$const\prod_{i\neq j}\frac{1}{t_i-t_j}, \tag{3.9}$$

where the total degree of the denominator, viewed as a polynomial in the variables

$$t_\ell,\quad \ell=0,1,\ldots,d+1,$$

is d, but for some terms *const* could be zero. The terms involving

$$\frac{1}{t_i-t_j}\ \text{ or }\ \frac{1}{(t_i-t_j)^2},\quad i,j=1,2,\ldots,d+1,\ i\neq j, \tag{3.10}$$

could not take part in (3.9). To see this, observe that for fixed $i\neq j$, $0\le\ i,j\le d$, only the elements

$$\begin{pmatrix}a_{ii}&a_{ij}\\a_{ji}&a_{jj}\end{pmatrix}=\alpha\begin{pmatrix}\dfrac{1}{t_j-t_i}&\dfrac{1}{t_i-t_j}\\ \dfrac{1}{t_j-t_i}&\dfrac{1}{t_i-t_j}\end{pmatrix}+\text{other terms}$$

in the matrix A are involved. So the contribution of (3.10) to $\det A$ is computed as the determinant of the matrix A where all the other elements in the rows i and j and in columns i and j are set to zero. But then all the 2×2 minors obtained from the rows i and j vanish identically, and the Laplace expansion shows that this determinant is equal to zero. A similar argument works for $i = d+1$, $j = 0$, too. But then only the d possible divisors $t_0 - t_i$, $i = 1, 2, \ldots, d$, are left, and $\det A$ has to be of the form

$$\det A = \alpha^d \frac{c}{\prod_{i=1}^{d} (t_0 - t_i)} = \alpha^d (t_0 - t_{d+1}) \frac{c}{\dot{\omega}(t_0)},$$

where c is a constant independent of t_i. Since

$$c = \frac{1}{\alpha^d (t_0 - t_{d+1})} \det \left(\operatorname{diag}(t_0 - t_i)_{i=1}^{d+1} A \right),$$

the sequence of limits $t_1 \to t_0, t_2 \to t_0, \ldots, t_d \to t_0$ simplifies c to

$$c = \frac{1}{t_0 - t_{d+1}} \det \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & d & 0 \\ -1 & -1 & \cdots & -1 & t_0 - t_{d+1} \end{pmatrix} = d!,$$

and the lemma is proved. □

Lemma 3.3. *Let $D_i \neq 0$ be all of the same sign. Then a constant $c > 0$, depending on the data D_i only can be found, such that for any solution of (3.7) that satisfies (1.1), the relations*

$$t_{i+1} - t_i \geq c > 0, \quad i = 0, 1, \ldots, d,$$

must hold.

Proof. Without loss of generality, one can assume that $\operatorname{sign}(\alpha) = \operatorname{sign}(D_i) > 0$. If $\alpha \geq \epsilon_0 > 0$ for some constant ϵ_0, clearly $t_{i+1} - t_i \geq \text{const} > 0$, since otherwise the left hand side of (3.7) would be unbounded. Thus t_i can approach each other only if $\alpha \to 0$. The last equation in (3.7) then implies $\dot{\omega}(t_{d+1}) \to 0$, and consequently $t_d \to t_{d+1} = 1$. Since $[t_0, t_1, \ldots, t_{d+1}]1 = 0$, summing all equations in (3.7) yields

$$\frac{\alpha}{\dot{\omega}(t_0)} = (-1)^{d+1} D_0,$$

which implies $\dot{\omega}(t_0) \to 0$ too, and further $t_1 \to t_0 = 0$. Thus at least two t_i stay separated by a constant. Suppose that ℓ, $1 \leq \ell \leq d$, is the smallest index such that t_ℓ, $t_{\ell+1}$ are separated, i.e., $t_\ell \to t_0$, but $t_{\ell+1} - t_\ell \geq \text{const} > 0$. Then

$$\frac{1}{t_i - t_j} = \frac{1}{t_0 - t_j} (1 + \mathcal{O}(t_i - t_0)), \quad i \leq \ell < j,$$

and

$$\frac{1}{\dot{\omega}(t_i)} = \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^{\ell} (t_i - t_j)} \frac{1}{\prod_{j=\ell+1}^{d+1} (t_0 - t_j)} (1 + \sum_{i=0}^{\ell} \mathcal{O}(t_i - t_0)), \quad i \leq \ell. \tag{3.11}$$

Let

$$w := \prod_{j=\ell+1}^{d+1} (t_j - t_0) \ \geq (t_{\ell+1} - t_\ell)^{d+1-\ell} \geq \mathrm{const}^{d+1-\ell} > 0.$$

By inserting (3.11) into (3.7), multiplied by w, one computes

$$\frac{\alpha}{\prod\limits_{\substack{j=0\\ j\neq i}}^{\ell} (t_i - t_j)} = (-1)^{\ell-i} w(D_{i-1} + D_i) + \text{ higher} - \text{order terms}, \ i = 0, 1, \ldots, \ell,$$

and the summing of these equations yields

$$\sum_{i=0}^{\ell} \frac{\alpha}{\prod\limits_{\substack{j=0\\ j\neq i}}^{\ell} (t_i - t_j)} = [t_0, t_1, \ldots, t_\ell]\alpha = wD_\ell + \text{ higher} - \text{order terms}, \tag{3.12}$$

a contradiction, since $[t_0, t_1, \ldots, t_\ell]\alpha = 0$ for $\ell \geq 1$, and $D_\ell \neq 0$ for $1 \leq \ell \leq d$. ◻

The fourth step of the proofs considers a particular set of data points, taken on the polynomial curve $\boldsymbol{f}(t) := (t^k)_{k=1}^d$ as

$$T_i^* := \boldsymbol{f}(\eta_i), \quad i = 0, 1, \ldots, d+1,$$

where

$$\eta_0 := 0 < \eta_1 < \cdots < \eta_d < \eta_{d+1} := 1. \tag{3.13}$$

Note that the corresponding determinants

$$D_i^* := \det{(\Delta T_i^*)}_{\substack{j=0\\ j\neq i}}^{d} \tag{3.14}$$

could be computed as

$$D_i^* = d! \int_{\eta_0}^{\eta_1} dx_1 \int_{\eta_1}^{\eta_2} dx_2 \ldots \int_{\eta_{i-1}}^{\eta_i} dx_i \int_{\eta_{i+1}}^{\eta_{i+2}} dx_{i+1} \ldots \int_{\eta_d}^{\eta_{d+1}} V(x_1, x_2, \ldots, x_d) dx_d,$$

where

$$V(x_1, x_2, \ldots, x_d) = \prod_{j>i} (x_j - x_i)$$

is the Vandermonde determinant. This implies $D_i^* > 0$, and rank $\Delta T^* = d$, since η_i are ordered by (3.13). The necessary conditions of Theorem 1.1 are met, and one of the solutions of (2.1) for the particular data is obviously

$$t_i = \eta_i, \quad i = 1, \ldots, d.$$

In order to complete the proof of Theorem 1.1 for these data, it must be shown that this is the only solution. The system in its basic form (2.1) is

$$\sum_{i=0}^{d+1} \frac{\eta_i^\ell}{\dot\omega(t_i)} = 0, \quad \ell = 1, 2, \ldots, d, \tag{3.15}$$

and the identity $[t_0, t_1, \ldots, t_{d+1}]1 = 0$ can always be added. But then (3.15) is reduced to the fact that the vector

$$\left(\frac{1}{\dot{\omega}(t_i)}\right)_{i=0}^{d+1} \tag{3.16}$$

should span the kernel of the matrix

$$\left(\eta_j^i\right)_{i=0;j=0}^{d;d+1}.$$

A straightforward computation shows that (3.16) should be proportional to the vector of the same structure, but with all t_i being replaced by η_i. So t_i and η_i are equivalent, and one can simplify further discussion by exchanging the role of the unknowns and the parameters. Thus suppose t_i to be known, and η_i to be determined.

The equations (3.15) imply that the values η_i must be equal to the values $p(t_i)$ of some polynomial p of degree $\leq d$, and

$$[t_0, t_1, \ldots, t_{d+1}]p^\ell = 0, \quad \ell = 1, 2, \ldots, d. \tag{3.17}$$

It is easy to see that (3.17) does not, in general, determine the polynomial p uniquely, even for small d. Take $d = 3$, and equidistant partition $t_i = \frac{i}{4}$. Then the divided difference $[t_0, t_1, t_2, t_3, t_4]$ obviously maps to zero the powers t^ℓ, $\ell = 1, 2, 3$, but also p^ℓ, where

$$p(t) := \frac{1}{3}t(16 - 45t + 32t^2).$$

However, this p does not produce $\eta_i = p(t_i)$ in the order as required in (3.13) since it is not monotone on $[0, 1]$.

Let us proceed to show that for a particular choice of t_i the solution of (3.17) that satisfies (3.13) is unique. Let $0 < h \ll 1$, and

$$t_i = \frac{i}{d}h, \ i = 1, 2, \ldots, d.$$

Note $p(0) = 0, \ p(1) = 1$. Thus p can be expressed as follows

$$p(t) = \sum_{i=1}^{d} c_i t^i, \ c_d := 1 - \sum_{i=1}^{d-1} c_i,$$

and the first equation of (3.17) is satisfied automatically. Let us recall that the divided difference can also be written as

$$\oint_{\partial\Omega} \frac{f(z)}{\omega(z)} dz = \sum_{i=0}^{d+1} \operatorname{Res}\left(\frac{f}{\omega}; t_i\right) = [t_0, t_1, \ldots, t_{d+1}]f, \ t_i \in I,$$

if f is analytical on the set $\Omega \subset \mathbb{C}, \ I \subset \Omega$. Here, $\operatorname{Res}(g; z)$ denotes the residuum of g at z. Thus (3.17) can be written as

$$\sum_{i=0}^{d+1} \operatorname{Res}\left(\frac{p^\ell}{\omega}; t_i\right) = 0, \quad \ell = 2, 3, \ldots, d. \tag{3.18}$$

The fraction $\dfrac{p^\ell}{\omega}$ has only isolated singularities in $\mathbb{C}^*$, therefore

$$\sum_{i=0}^{d+1} \operatorname{Res}\left(\frac{p^\ell}{\omega}; t_i\right) + \operatorname{Res}\left(\frac{p^\ell}{\omega}; \infty\right) = 0,\ \ell = 2, 3, \ldots, d,$$

and the system (3.18) is simplified to

$$\operatorname{Res}\left(\frac{p^\ell}{\omega}; \infty\right) = 0, \quad \ell = 2, 3, \ldots, d. \tag{3.19}$$

The rational function $\dfrac{1}{\omega}$ expands at ∞ as

$$\frac{1}{\omega(z)} = \frac{1}{z^{d+2}} + \sum_{i=d+3}^{\infty} \frac{1}{z^i}\left(\frac{d+1}{2}h + \mathcal{O}(h^2)\right).$$

Also,

$$p^\ell(z) = \sum_{k=\ell}^{\ell\, d} z^k \sum_{i_1+i_2+\ldots+i_\ell=k} c_{i_1} c_{i_2} \ldots c_{i_\ell}. \tag{3.20}$$

In (3.20), only the terms with $k \geq d+1$ will contribute to the residue. Since $d+1 > \ell$, the system (3.19) reads

$$\sum_{k=d+1}^{\ell\, d} \sum_{i_1+i_2+\ldots+i_\ell=k} c_{i_1} c_{i_2} \ldots c_{i_\ell} + \mathcal{O}(h) = 0,\ \ell = 2, 3, \ldots, d. \tag{3.21}$$

But $p^\ell(1) = 1$, and (3.20) simplifies (3.21) to

$$1 - \sum_{k=l}^{d} \sum_{i_1+i_2+\ldots+i_\ell=k} c_{i_1} c_{i_2} \ldots c_{i_\ell} + \mathcal{O}(h) = 0,\ \ell = d, d-1, \ldots, 2. \tag{3.22}$$

First let us consider (3.22) when $h \to 0$. Then the first two equations read

$$1 - c_1^d = 0, \tag{3.23}$$

and

$$1 - c_1^{d-1} - d c_1^{d-2} c_2 = 0, \tag{3.24}$$

and the rest as

$$1 - c_1^\ell - \ell c_1^{\ell-1} c_{d-\ell+1} + g_\ell(c_1, c_2, \ldots, c_{d-\ell}) = 0,\ \ell = d-2, d-3, \ldots, 2. \tag{3.25}$$

Equation (3.23) implies that $c_1 = 1$ is the only real solution. This is true also for even d, because $c_1 = -1$ implies that p is not monotone. But then (3.24) implies $c_2 = 0$, and (3.25) $c_i = 0$, $i = 3, 4, \ldots, d-1$. A brief look at (3.22) reveals that $g_\ell(c_1, c_2, \ldots, c_{d-\ell})$ involves products that include at least two c_i, with $2 \leq i \leq d - \ell$.

So the lower triangular nonlinear system (3.23), (3.24), and (3.25) has nonsingular Jacobian at the limit point $h = 0$, and the limit solution is

$$(c_1, c_2, \ldots, c_{d-1}) = (1, 0, \ldots, 0).$$

Thus, by the implicit function theorem, there exists $h_0 > 0$, such that for all h, $0 \leq h \leq h_0$, there is a unique monotone solution p of the system (3.17), i.e., $p(t) = t$, independently of h. Consequently the system (3.6) has a unique solution (3.13). Note that this does not contradict the conclusion of Lemma 3.3 since there the data were constants, but here they are moving towards the boundary together with the solution.

Consider now the general case, the step five of the proofs. Without loss of generality, one may assume that D_i are all positive. Let us join the particular data D_i^*, defined in (3.14), and the general data D_i with a convex homotopy,

$$D_i(\lambda) := (1-\lambda) D_i^* + \lambda D_i > 0, \quad \lambda \in [0,1].$$

Let

$$\boldsymbol{H}(\boldsymbol{t}, \alpha; \lambda) := \left(\frac{\alpha}{\dot{\omega}(t_i)}\right)_{i=1}^{d+1} - \Big((-1)^{d+1-i}\big(D_{i-1}(\lambda) + D_i(\lambda)\big)\Big)_{i=1}^{d+1}, \quad \lambda \in [0,1],$$

so that the system (3.7) is simplified to

$$\boldsymbol{H}(\boldsymbol{t}, \alpha; \lambda) = \boldsymbol{0}.$$

For each fixed $\lambda \in [0,1]$ the requirements of Lemma 3.3 are met, so $\boldsymbol{H}$ has no zero arbitrarily close to the boundary. But the interval $[0,1]$ is compact, and the data $D_i(\lambda)$ depend continuously on λ. Thus the term $D_\ell(\lambda)$ in (3.12) can be bounded independently of λ,

$$|D_\ell(\lambda)| \geq \inf_{\lambda \in [0,1]} |D_\ell(\lambda)| = \min_{\lambda \in [0,1]} |D_\ell(\lambda)| = \text{const}_\ell \geq \text{const} > 0,$$

and the contradiction that proves the Lemma 3.3 holds uniformly. So a compact set

$$\mathcal{D} \subset \{\boldsymbol{t} \,|\, t_0 < t_1 < \cdots < t_d < t_{d+1}\} \times \{\alpha \,|\, 0 \leq \alpha < \infty)\},$$

has to exist, such that $\boldsymbol{H}$ does not vanish at the boundary of $\mathcal{D}$ for any $\lambda \in [0,1]$. But then Brouwer's degree ([2, pp.52-53]) of $\boldsymbol{H}$ is invariant for $\lambda \in [0,1]$ on $\mathcal{D}$. In $\boldsymbol{H}$, only the data depend on λ, and a brief look at the homotopy reveals that its Jacobian is simply $J(\boldsymbol{t}, \alpha)$, as given in (3.8). This simplifies Brouwer's degree to

$$\sum_{(\boldsymbol{t},\alpha) \in \mathcal{D},\, \boldsymbol{H}(\boldsymbol{t},\alpha;\lambda)=0} \operatorname{sign}\big(\det J(\boldsymbol{t}, \alpha)\big).$$

But by Lemma 3.2, $\det J$ vanishes nowhere in $\mathcal{D}$, and Brouwer's degree is further simplified to

$$\pm \# \{(\boldsymbol{t}, \alpha) \,|\, (\boldsymbol{t}, \alpha) \in \mathcal{D},\, \boldsymbol{H}(\boldsymbol{t}, \alpha; \lambda) = 0\},$$

so it provides the exact count of zeroes in $\mathcal{D}$. But the particular problem $\boldsymbol{H}(\boldsymbol{t}, \alpha; 0) = 0$ has a unique solution, so have all $\boldsymbol{H}(\boldsymbol{t}, \alpha; \lambda)$.

In order to complete the proof of Theorem 1.1, it remains to show that $\boldsymbol{B}$, based upon $\boldsymbol{t}$ that we have just determined, is a regular curve.

Note that $\boldsymbol{B}$ can also be written as

$$\boldsymbol{B} = \sum_{j=0}^{d+1} \boldsymbol{T}_j \ell_j, \; \ell_j(t) := \frac{\omega(t)}{(t-t_j)\dot{\omega}(t_j)}.$$

If $\boldsymbol{B}$ is not regular, then

$$\dot{\boldsymbol{B}}(\tilde{t}) = 0 = \Delta T U (\dot{\ell}_i(\tilde{t}))_{i=1}^{d+1}$$

for some $\tilde{t} \in [0,1]$. Since $\ker \Delta T$ is spanned by $\boldsymbol{\omega}_{\Sigma} = U\boldsymbol{\omega}$, given in (3.5) and (3.3), the vector $\left(\ell_i(\tilde{t})\right)_{i=1}^{d+1}$ should be proportional to $\boldsymbol{\omega}$. But then

$$\dot{\omega}(t_i)\dot{\ell}_i(\tilde{t}) = \left(\frac{\dot{\omega}(\tilde{t})}{\tilde{t}-t_i} - \frac{\omega(\tilde{t})}{(\tilde{t}-t_i)^2}\right) = \text{const}$$

for all $t_i \neq \tilde{t}$, which implies that at least two of t_i are equal, a contradiction that confirms the regularity of the interpolating curve. The proof of Theorem 1.1 is complete.

Theorem 1.3 follows from Lemma 3.2. The continuation method ([1]) always leads to the solution if the Jacobian of the system is globally nonsingular.

Let us finally prove Theorem 1.2. If the interpolating polynomial $\boldsymbol{B}$ exists, then the corresponding $\boldsymbol{\omega}_{\Sigma} \in \ker \Delta T$, as defined in (3.5), clearly satisfies $\mathrm{S}^-(\boldsymbol{\omega}_{\Sigma}) = d$. On the other hand, if $\boldsymbol{x} = (x_i)_{i=0}^{d} \in \ker \Delta T$ can be found such that $\mathrm{S}^-(\boldsymbol{x}) = d$, than x_i may replace the right hand side $(-1)^{d+1-i} D_i$ in (3.6). The existence part of Theorem 1.1 still carries through, and Theorem 1.2 is proved.

Let us illustrate the last proof by a simple example. Let data be given on a line in a plane,

$$\boldsymbol{T}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{T}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \boldsymbol{T}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \boldsymbol{T}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.26}$$

Then

$$\Delta T = \frac{1}{6}\begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

and $\operatorname{rank} \Delta T = 1$. Furthermore, the vector $\boldsymbol{x} \in \ker \Delta T$ such that $\mathrm{S}^-(\boldsymbol{x}) = d = 2$ is given as a parametric family

$$\boldsymbol{x} = \boldsymbol{x}(\mu) := (\mu, -3-2\mu, 1), \; \mu > 0.$$

For such an $\boldsymbol{x}$, the system (3.6) has the solution

$$t_1 = t_1(\mu) := \frac{1}{1-\mu}\left(1 - \sqrt{\frac{2\mu(\mu+2)}{3(\mu+1)}}\right), \; t_2 = t_2(\mu) := t_1(\mu) + \sqrt{\frac{\mu}{6(\mu+1)(\mu+2)}},$$

and the data (3.26) are interpolated by a quadratically parametrized line

$$\boldsymbol{B} = \phi \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \phi(t) := \frac{(1-2t_2^2)t - (1-2t_2)t^2}{2t_2(1-t_2)}.$$

Further more, the curve $\boldsymbol{B}$ is regular iff

$$1 - \frac{1}{\sqrt{2}} \leq t_2 \leq \frac{1}{\sqrt{2}}, \quad \left(\sqrt{3}-1\right)\left(\sqrt{2}-1\right) \leq \mu \leq \left(\sqrt{2}+1\right)\left(2+\sqrt{6}\right).$$

There is only one free parameter to decrease the degree of $\boldsymbol{B}$, and $t_2 = t_2(2) = \frac{1}{2}$ reduces the parametrisation to the simplest case $\phi(t) = t$. This parametrisation is regular since it is a unique solution of degree one of the interpolation problem. This concludes the proofs.

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