Curvature variation minimizing cubic Hermite interpolants

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Abstract

In this paper, planar parametric Hermite cubic interpolants with small curvature variation are studied. By minimization of an appropriate approximate functional, it is shown that a unique solution of the interpolation problem exists, and has a nice geometric interpretation. The best solution of such a problem is a quadratic geometric interpolant. The optimal approximation order 4 of the solution is confirmed. The approach is combined with strain energy minimization in order to obtain G^1 cubic interpolatory spline.

Key words: geometric Hermite interpolation, minimization, curvature.

1. Introduction

In [2] the author studied Hermite interpolation by planar rational cubic Bézier curves. A particular attention was paid to a specific class of such objects that reproduce circle arcs, probably the most often used curves in practice. Since in such applications the change in the curvature is more important than its magnitude, a new measure of the quality of the shape of the curve that involves the derivative of its curvature was introduced. For circle arcs, the value of such a functional is zero, but more importantly, such an approach is geometric in contrast to a usual variational one.

A good polynomial approximation of curves is important since in some applications only parametric polynomial objects are appropriate. This is especially true for geometric interpolation schemes, where not much is known

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on rational case (see [4, 6] e.g.). It is well known that geometric interpolation is independent of parameterization and provides interpolants with high approximation order and pleasant shape. There are several papers dealing with this topic (see [5] and the references therein, e.g.).

In this paper, cubic parametric Hermite polynomial planar interpolatory curves with small curvature variation are studied. By minimization of an appropriate approximate functional, it is shown that a unique solution exists and can be constructed in a purely geometric way. Furthermore, asymptotically the best solution of such a problem is a quadratic geometric interpolant. The optimal approximation order of the solution is confirmed. This approach can be combined with strain energy minimization in order to obtain G^1 cubic interpolatory spline.

2. Main problem

Suppose that we are given two distinct points $\boldsymbol{p}_0, \boldsymbol{p}_1 \in \mathbb{R}^2$ together with the corresponding tangent directions $\boldsymbol{d}_0, \boldsymbol{d}_1 \in \mathbb{R}^2$, $\|\boldsymbol{d}_0\|_2 = \|\boldsymbol{d}_1\|_2 = 1$. We are looking for a solution of the standard Hermite interpolation problem

$$\boldsymbol{b}(0) = \boldsymbol{p}_0, \quad \boldsymbol{b}(1) = \boldsymbol{p}_1, \quad \boldsymbol{b}'(0) = \alpha_0\,\boldsymbol{d}_0, \quad \boldsymbol{b}'(1) = \alpha_1\,\boldsymbol{d}_1, \tag{1}$$

where $\boldsymbol{b} : [0, 1] \to \mathbb{R}^2$ is a cubic parametric polynomial curve and α_0, α_1 are positive reals. As it is very well known, this problem has a unique solution, provided α_0 and α_1 are given in advance. They are usually used as free shape parameters, since they influence the shape of the curve. There are several approaches on how to choose suitable parameters α_0 and α_1. In most cases, minimization of an appropriate functional is applied (approximate strain energy, e.g., [7, 8, 10, 11]). As it was proposed in [2], one can model a curve by minimizing its curvature derivative. Then it is natural to minimize

$$\int_0^1 (\kappa'(t))^2 \, dt \tag{2}$$

over all possible choices of α_0 and α_1. Here κ denotes the curvature of the curve $\boldsymbol{b}$. Since these two parameters are involved in (2) in a highly nonlinear way, one may consider to linearize it. A similar idea as for the minimization of the bending energy will be used ([7]). Namely, if the assumption, that a particular parameterization is close to the arc-length one, is used, then (2)

is approximately

$$\int_0^1 \left((\boldsymbol{b}'(t) \times \boldsymbol{b}''(t))' \right)^2 \, dt = \int_0^1 (\boldsymbol{b}'(t) \times \boldsymbol{b}'''(t))^2 \, dt. \tag{3}$$

Here, and in the rest of the paper, $\boldsymbol{x} \times \boldsymbol{y} := x_1 y_2 - x_2 y_1$ denotes the planar vector product.

If $\boldsymbol{b}$ is written in the Bézier form, i.e.,

$$\boldsymbol{b}(t) = \sum_{j=0}^3 \boldsymbol{b}_j \, B_j^3(t),$$

where $\boldsymbol{b}_j$ are Bézier control points and B_j^3 are cubic Bernstein polynomials, then the assumptions (1) simplify $\boldsymbol{b}$ to

$$\boldsymbol{b}(t) = \boldsymbol{p}_0 \, B_0^3(t) + \left(\boldsymbol{p}_0 + \frac{1}{3} \alpha_0 \, \boldsymbol{d}_0 \right) B_1^3(t) + \left(\boldsymbol{p}_1 - \frac{1}{3} \alpha_1 \, \boldsymbol{d}_1 \right) B_2^3(t) + \boldsymbol{p}_1 \, B_3^3(t). \tag{4}$$

Using known facts about derivatives of Bézier curves,

$$\boldsymbol{b}'(t) = \alpha_0 \, \boldsymbol{d}_0 \, B_0^2(t) + (3 \, \Delta \boldsymbol{p}_0 - \alpha_0 \, \boldsymbol{d}_0 - \alpha_1 \, \boldsymbol{d}_1) \, B_1^2(t) + \alpha_1 \, \boldsymbol{d}_1 \, B_2^2(t),$$
$$\boldsymbol{b}'''(t) = 6 \, (\alpha_0 \, \boldsymbol{d}_0 + \alpha_1 \, \boldsymbol{d}_1 - 2 \, \Delta \boldsymbol{p}_0),$$

where $\Delta \boldsymbol{p}_0 := \boldsymbol{p}_1 - \boldsymbol{p}_0$, the functional (3) becomes

$$\begin{aligned}
f(\alpha_0, \alpha_1) &:= \int_0^1 (\boldsymbol{b}'(t) \times \boldsymbol{b}'''(t))^2 \, dt \\
&= 12 \left(\alpha_0^2 \, \alpha_1^2 \, (\boldsymbol{d}_0 \times \boldsymbol{d}_1)^2 + 4 \, \alpha_0^2 \, (\boldsymbol{d}_0 \times \Delta \boldsymbol{p}_0)^2 + 4 \, \alpha_1^2 \, (\boldsymbol{d}_1 \times \Delta \boldsymbol{p}_0)^2 \right. \\
&\quad - 2 \, \alpha_0^2 \, \alpha_1 \, (\boldsymbol{d}_0 \times \boldsymbol{d}_1) \, (\boldsymbol{d}_0 \times \Delta \boldsymbol{p}_0) + 2 \, \alpha_0 \, \alpha_1^2 \, (\boldsymbol{d}_0 \times \boldsymbol{d}_1) \, (\boldsymbol{d}_1 \times \Delta \boldsymbol{p}_0) \\
&\quad \left. + 4 \, \alpha_0 \, \alpha_1 \, (\boldsymbol{d}_0 \times \Delta \boldsymbol{p}_0) \, (\boldsymbol{d}_1 \times \Delta \boldsymbol{p}_0) \right). \tag{5}
\end{aligned}$$

Let us simplify the notation by introducing

$$c_{0,1} := \boldsymbol{d}_0 \times \boldsymbol{d}_1, \quad c_j := \boldsymbol{d}_j \times \Delta \boldsymbol{p}_0, \quad j = 0, 1,$$

and let $\mathcal{D} := \{(\alpha_0, \alpha_1) \in \mathbb{R}^2 | \, \alpha_0 > 0, \, \alpha_1 > 0\}$. Now the extrema of the above functional are given as solutions of the normal system

$$f_1(\alpha_0, \alpha_1) := \alpha_0 \, \alpha_1^2 \, c_{0,1}^2 + 4 \, \alpha_0 \, c_0^2 - 2 \, \alpha_0 \, \alpha_1 \, c_{0,1} \, c_0 + \alpha_1^2 \, c_{0,1} \, c_1 + 2 \, \alpha_1 c_0 \, c_1 = 0,$$
$$f_2(\alpha_0, \alpha_1) := \alpha_0^2 \, \alpha_1 \, c_{0,1}^2 + 4 \, \alpha_1 \, c_1^2 + 2 \, \alpha_0 \, \alpha_1 \, c_{0,1} \, c_1 - \alpha_0^2 \, c_{0,1} \, c_0 + 2 \, \alpha_0 \, c_0 \, c_1 = 0.$$
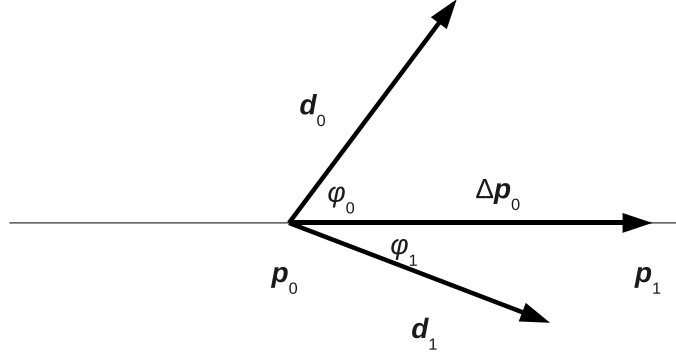
Figure 1: Admissible tangent directions.

The above nonlinear system can be solved analytically by using resultants. Namely, the resultant

$$
\begin{aligned}
\mathrm{Res}(f_1, f_2, \alpha_1) &= \\
&= 3\, c_0^2 \left(\alpha_0^2\, c_{0,1}^2 + 2\, \alpha_0\, c_{0,1}\, c_1 + 8\, c_1^2 \right) \left(c_{0,1}\, \alpha_0 + c_1 \right) \left(c_{0,1}\, \alpha_0 + 2\, c_1 \right) \alpha_0 = 0
\end{aligned}
$$

implies three possible admissible solutions for α_0, i.e.,

$$
\alpha_0 = 0, \quad \alpha_0 = -\frac{c_1}{c_{0,1}}, \quad \alpha_0 = -2\frac{c_1}{c_{0,1}}.
$$

It is straightforward to see that the corresponding solutions for α_1 are

$$
\alpha_1 = 0, \quad \alpha_1 = \frac{c_0}{c_{0,1}}, \quad \alpha_1 = 2\frac{c_0}{c_{0,1}}.
$$

The first solution does not lie in $\mathcal{D}$. The cylindrical decomposition of the inequalities $f(\alpha_0, \alpha_1) > 0$, $\alpha_0 > 0$, $\alpha_1 > 0$, reveals $\alpha_0 > 0$, $\alpha_1 > 0$ and

$$
\left(\frac{\alpha_0 c_0}{\alpha_1} + c_1 \neq 0 \text{ or } c_{0,1} + \frac{c_1}{\alpha_0} + \sqrt{-3\left(\frac{\alpha_0 c_0 + \alpha_1 c_1}{\alpha_0 \alpha_1} \right)^2} \neq \frac{c_0}{\alpha_1} \right).
$$

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The last possibility can be clearly omitted, and a simple computation yields

$$f(\alpha_0, -\frac{\alpha_0 c_0}{c_1}) = \frac{12\,\alpha_0^2 c_0^2 (\alpha_0 c_{0,1} + 2\,c_1)^2}{c_1^2}.$$

Thus $f \geq 0$ on $\mathcal{D}$, and $f(\alpha_0, \alpha_1) = 0$ at the only global minimum in $\mathcal{D}$,

$$\alpha_0 = -2\frac{c_1}{c_{0,1}}, \quad \alpha_1 = 2\frac{c_0}{c_{0,1}}. \tag{6}$$

It lies in $\mathcal{D}$ if and only if $c_0\,c_{0,1} > 0$ and $c_1\,c_{0,1} < 0$. Thus we have proved the following theorem.

Theorem 1. *Let $c_0\,c_{0,1} > 0$ and $c_1\,c_{0,1} < 0$. Then the functional* (3) *has a unique global minimum in $\mathcal{D}$ and it is reached at* (6).

The assumptions of Thm. 1 have a simple geometric interpretation. Let $\varphi_0 := \angle(\boldsymbol{d}_0, \Delta\boldsymbol{p}_0)$ and $\varphi_1 := \angle(\boldsymbol{d}_1, \Delta\boldsymbol{p}_0)$, where $\angle(\boldsymbol{a}, \boldsymbol{b})$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Then the assumptions of the theorem are fulfilled if and only if the vectors $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ point to opposite sides of the line with the directional vector $\Delta\boldsymbol{p}_0$ and $\varphi_0 + \varphi_1 < \pi$ (see Fig. 1).

Examples of curves with a minimal curvature variation are shown in Fig. 2, together with their curvature plots.

3. Asymptotic analysis

A natural question arises: which is asymptotically the best cubic curvature variation minimizing curve?

Suppose that data are sampled from a regular convex planar curve $\boldsymbol{g}$, $\boldsymbol{g} : [0, h] \rightarrow \mathbb{R}^2$. Since $\boldsymbol{g}$ is regular at $\boldsymbol{g}(0)$, one can further assume that

$$\boldsymbol{g}(x) = \begin{pmatrix} x \\ y(x) \end{pmatrix}, \tag{7}$$

where

$$y(x) := \sum_{j=2}^{\infty} g_j\, x^j.$$

Note that this can be achieved by using a particular reparameterization on the first component of the curve, translation and rotation. Since $\boldsymbol{g}$ is convex,
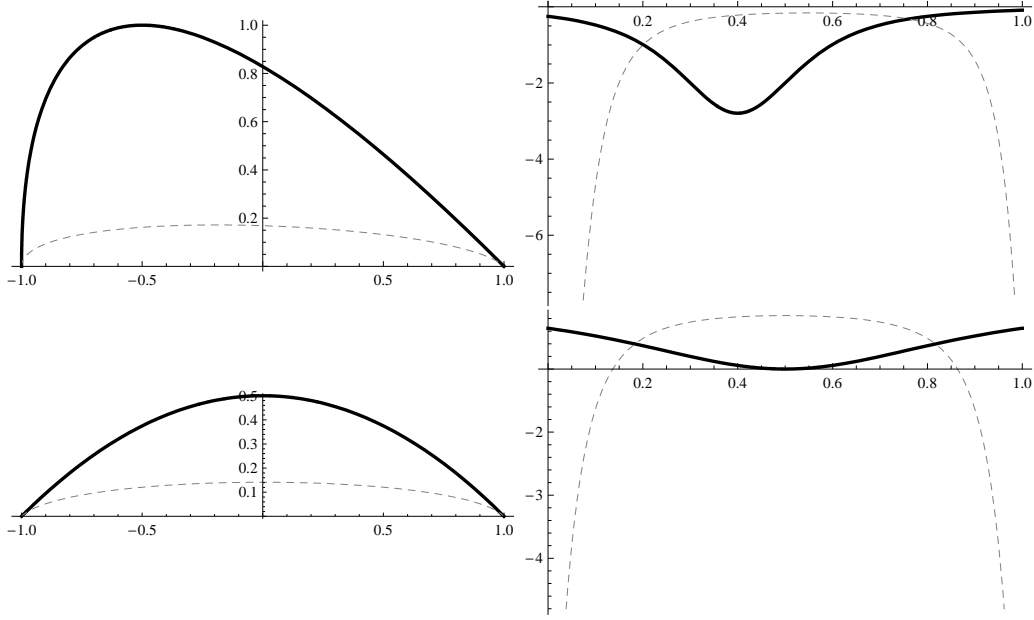
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Figure 2: Left: Variation minimizing curves (black solid) and particular cubic interpolants from [2] (gray dashed) for two sets of data. Right: Corresponding curvature plots.

$g_2 \neq 0$. Now the data become

$$
\begin{aligned}
\boldsymbol{p}_0 &= \boldsymbol{g}(0) = (0,0)^T, \\
\boldsymbol{p}_1 &= \boldsymbol{g}(h) = (h, y(h))^T, \\
\boldsymbol{d}_0 &= \boldsymbol{g}'(0) = (1,0)^T, \\
\boldsymbol{d}_1 &= \boldsymbol{g}'(h) = \frac{1}{\sqrt{1 + (y'(h))^2}} \, (1, y'(h))^T, \\
\Delta \boldsymbol{p}_0 &= \boldsymbol{g}(h) - \boldsymbol{g}(0) = (h, y(h))^T.
\end{aligned}
\tag{8}
$$

Note that here and throughout this section the "dash" notation will denote a derivative with respect to the particular new parameter, introduced in (7). Consequently,

$$
\alpha_0 = -2\frac{y(h) - h\,y'(h)}{y'(h)} = h + \frac{g_3}{2\,g_2}h^2 + \frac{4\,g_2\,g_4 - 3\,g_3^2}{4\,g_2^2}h^3 + \mathcal{O}(h^4),
$$

$$
\alpha_1 = 2\sqrt{1 + y'(h)^2}\,\frac{y(h)}{y'(h)} = h - \frac{g_3}{2\,g_2}h^2 + \left(2g_2^2 - \frac{4\,g_2\,g_4 - 3\,g_3^2}{4\,g_2^2}\right)h^3 + \mathcal{O}(h^4).
$$

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We can now follow [1] and [5] to establish that the approximation of $\boldsymbol{g}$ by a polynomial curve $\boldsymbol{b}$, given by (4) and (8), is fourth order accurate. It is enough to show that $\Delta \boldsymbol{b}_j = h\,\boldsymbol{v}_j + \mathcal{O}(h^2)$, $\boldsymbol{v}_j \neq \boldsymbol{0}$, $j = 0, 1, 2$, $\Delta^2 \boldsymbol{b}_j = \mathcal{O}(h^2)$, $j = 0, 1$ and $\Delta^3 \boldsymbol{b}_0 = \mathcal{O}(h^3)$. But a straightforward computation reveals

$$\Delta \boldsymbol{b}_j = \begin{pmatrix} 1 \\ \frac{1}{3} \\ 0 \end{pmatrix} h + \mathcal{O}(h^2), \quad j = 0, 1, 2,$$

$$\Delta^2 \boldsymbol{b}_j = \begin{pmatrix} -\dfrac{g_3}{6\,g_2} \\ \dfrac{g_2}{3} \end{pmatrix} h^2 + \mathcal{O}(h^3), \quad j = 0, 1,$$

$$\Delta^3 \boldsymbol{b}_0 = \boldsymbol{0}.$$

Note that the last equation follows from $\Delta^3 \boldsymbol{b}_0 = -2\Delta \boldsymbol{p}_0 + \alpha_0 \boldsymbol{d}_0 + \alpha_1 \boldsymbol{d}_1 = \boldsymbol{0}$. But this is precisely the relation $\boldsymbol{b}_3 = \boldsymbol{b}_0 + 3(\boldsymbol{b}_2 - \boldsymbol{b}_1)$, that guarantees that the leading coefficient of a cubic Bézier curve is zero. This yields the following theorem.

Theorem 2. *The best cubic curvature variation minimizing curve is a quadratic geometric interpolant with an optimal approximation order* 4.

For more details on geometric interpolation and asymptotic analysis consider [9, 1, 5].

4. Generalization

Since curvature variation has an important influence on the shape of a curve, a natural idea arises to combine the considered functional (3) with the strain energy minimization in order to obtain better curves. Furthermore, thus also a spline interpolation problem can be considered.

Let us define the problem more precisely. Suppose that data points

$$\boldsymbol{p}_j \in \mathbb{R}^2, \quad j = 0, 1, \ldots, n,$$

with $\boldsymbol{p}_j \neq \boldsymbol{p}_{j+1}$ and associated interpolation parameters

$$t_j \in \mathbb{R}, \quad j = 0, 1, \ldots, n, \quad t_0 < t_1 < \cdots < t_n,$$

are given. We will assume that the interpolation parameters are prescribed (usually they are derived from data points, e.g., by the centripetal, chord

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length or α-parameterization, see [3]). Our goal is to find a G^1 continuous parametric spline curve $\boldsymbol{s} : [t_0, t_n] \to \mathbb{R}^2$ such that

$$
\begin{aligned}
\boldsymbol{s}_i &:= \boldsymbol{s}|_{[t_{i-1}, t_i]} \in \mathbb{P}_3, \quad i = 1, 2, \ldots, n, \\
\boldsymbol{s}_i(t_k) &= \boldsymbol{p}_k, \quad k = i - 1, i, \quad i = 1, 2, \ldots, n, \\
\boldsymbol{s}_i'(t_k) &= \alpha_{i,k-i+1} \boldsymbol{d}_k, \quad k = i - 1, i, \quad i = 1, 2, \ldots, n,
\end{aligned}
$$

where $\alpha_{i,k-i+1} > 0$ are unknown scalars, $\boldsymbol{d}_k$ are normalized tangent direction vectors, and $\mathbb{P}_3$ is the space of planar parametric polynomials of degree ≤ 3.

To shorten the notation, let us define $\boldsymbol{\alpha} := (\alpha_{i,k-i+1})_{i=1,k=i-1}^{n,i} \in \mathbb{R}^{2n}$, and

$$
c_{01,i} := \boldsymbol{d}_{i-1} \times \boldsymbol{d}_i, \quad c_{0,i} := \boldsymbol{d}_{i-1} \times \Delta\boldsymbol{p}_{i-1}, \quad c_{1,i} := \boldsymbol{d}_i \times \Delta\boldsymbol{p}_{i-1}.
$$

In the paper [7], the approximate strain energy minimization approach for such a problem was analyzed in detail, and a closed form solution with nice geometrical properties was found. Combining the strain energy and the curvature variation gives the functional

$$
\varphi_T(\boldsymbol{\alpha}) := w_1 \int_{t_0}^{t_n} \|\kappa(t)\|^2 \, dt + w_2 \int_{t_0}^{t_n} (\kappa'(t))^2 \, dt.
$$

Note that appropriate weights w_1 and w_2 need to be added since the summands differ in magnitudes. The strain energy is practically never analyzed in such a form, and besides, a minimization of such a functional would result in a highly nonlinear problem.

Thus the idea is to apply a combination of functional approximation approaches from [7] and the preceding sections. This results in the functional

$$
\varphi_{SC}(\boldsymbol{\alpha}) := w_1 \int_{t_0}^{t_n} \|\boldsymbol{s}''(t)\|^2 \, dt + w_2 \int_{t_0}^{t_n} (\boldsymbol{s}'(t) \times \boldsymbol{s}'''(t))^2 \, dt. \tag{9}
$$

Note that by [7] the problem can be studied locally. Therefore the combined functional (9) can be rewritten as

$$
\varphi_{SC}(\boldsymbol{\alpha}) = \sum_{i=1}^{n} \varphi_{SC,i}(\alpha_{i,0}, \alpha_{i,1}),
$$

where

$$
\varphi_{SC,i}(\alpha_{i,0}, \alpha_{i,1}) := w_1 \int_{t_{i-1}}^{t_i} \|\boldsymbol{s}_i''(t)\|^2 \, dt + w_2 \int_{t_{i-1}}^{t_i} (\boldsymbol{s}_i'(t) \times \boldsymbol{s}_i'''(t))^2 \, dt, \tag{10}
$$

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and $\boldsymbol{s}_i$ denotes the polynomial segment of $\boldsymbol{s}$ on $[t_{i-1}, t_i]$.

But the problem is still too hard to tackle, thus by [7] another approximation is applied to the first summand in (10) to obtain

$$\int_{t_{i-1}}^{t_i} \|\boldsymbol{s}_i''(t)\|^2 \, dt \approx \frac{2}{\Delta t_{i-1}} \psi_i(\boldsymbol{\alpha}),$$

where

$$\psi_i(\boldsymbol{\alpha}) := \left\| \frac{1}{\Delta t_{i-1}} \Delta \boldsymbol{p}_{i-1} - \alpha_{i,0}\, \boldsymbol{d}_{i-1} \right\|^2 + \left\| \alpha_{i,1}\, \boldsymbol{d}_i - \frac{1}{\Delta t_{i-1}} \Delta \boldsymbol{p}_{i-1} \right\|^2.$$

Note that a reparameterization is needed in the second summand in (10),

$$\int_{t_{i-1}}^{t_i} \left(\boldsymbol{s}_i'(t) \times \boldsymbol{s}_i'''(t) \right)^2 \, dt = \left(\frac{1}{\Delta t_{i-1}} \right)^7 \int_0^1 \left(\boldsymbol{s}_i'(u) \times \boldsymbol{s}_i'''(u) \right)^2 \, du.$$

Therefore, by applying the explicit forms of functionals in [7, Thm. 2] and (5), we obtain the final form of the combined functional

$$
\begin{aligned}
\varphi_{SC,i}(\alpha_{i,0}, \alpha_{i,1}) = {} & \frac{2\, w_1}{(\Delta t_{i-1})^3} \left((\Delta t_{i-1})^2 \left(\alpha_{i,0}^2 + \alpha_{i,1}^2 \right) \right. \\
& \left. - 2\, \Delta t_{i-1}(\alpha_{i,0}\, \boldsymbol{d}_{i-1} + \alpha_{i,1}\, \boldsymbol{d}_i)\Delta \boldsymbol{p}_{i-1} + 2\, \|\Delta \boldsymbol{p}_{i-1}\|^2 \right) \\
& + 12\, w_2\, \frac{1}{(\Delta t_{i-1})^7} \left(\alpha_{i,0}^2 \alpha_{i,1}^2 c_{01,i}^2 + 4\, \alpha_{i,0}^2 c_{0,i}^2 + 4\, \alpha_{i,1}^2 c_{1,i}^2 \right. \\
& \left. - 2\, \alpha_{i,0}^2 \alpha_{i,1} c_{01,i} c_{0,i} + 2\, \alpha_{i,0} \alpha_{i,1}^2 c_{01,i} c_{1,i} + 4\, \alpha_{i,0} \alpha_{i,1} c_{0,i} c_{1,i} \right). \quad (11)
\end{aligned}
$$

Recall that the first and the second summand of the combined functional (11) have unique minima, that can be given in explicit forms, when considered separately. But the combined functional is much more complicated, and the problem of finding its minima is much harder. Fortunately, the homotopy method paves the way to the solution. By using the linear homotopy $\Gamma_1 + \lambda \Gamma_2$, $\lambda \in [0, 1]$, where Γ_1 and Γ_2 denote the first and the second summand of the functional $\varphi_{SC,i}$, respectively, we can follow the solution from the explicit minimum of Γ_1 ([7, Thm. 2]) to the minimum of the combined functional (11).

5. Examples

Let us conclude the paper by some examples. Take a unit circle first and sample 3 points $\boldsymbol{p}_0 = (1, 0)^\top$, $\boldsymbol{p}_1 = (0, 1)^\top$, $\boldsymbol{p}_2 = (-1, 0)^\top$, and the corresponding tangents $\boldsymbol{d}_0 = (0, 1)^\top$, $\boldsymbol{d}_1 = (-1, 0)^\top$, $\boldsymbol{d}_2 = (0, -1)^\top$. In Fig. 3

the circle arc (dashed) and the interpolant, obtained by curvature variation minimization (gray, thick) are shown, together with the corresponding curvatures. In Fig. 4, steps in obtaining the solution of the combined method by homotopy are demonstrated. The homotopy starts at the solution, obtained by strain energy minimization, and follows it until the solution of the combined method is obtained. The final curve is indistinguishable from the circle arc. The equidistant parameterization is used.

The curvature variation minimizing spline is in fact parabolic. But it gives a more pleasant approximation than the strain energy minimizing spline. The combined method with weights $w_1 = w_2 = 1$ yields a similar result, but the curve is cubic and gives a better approximation than the curvature variation minimizing spline.
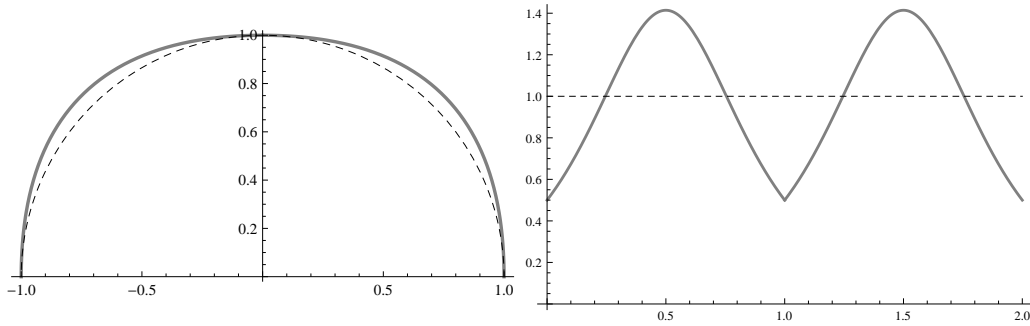

Figure 3: Approximation of a circle arc (dashed) by curvature variation minimizing curve (solid) together with the corresponding curvature plots (right).
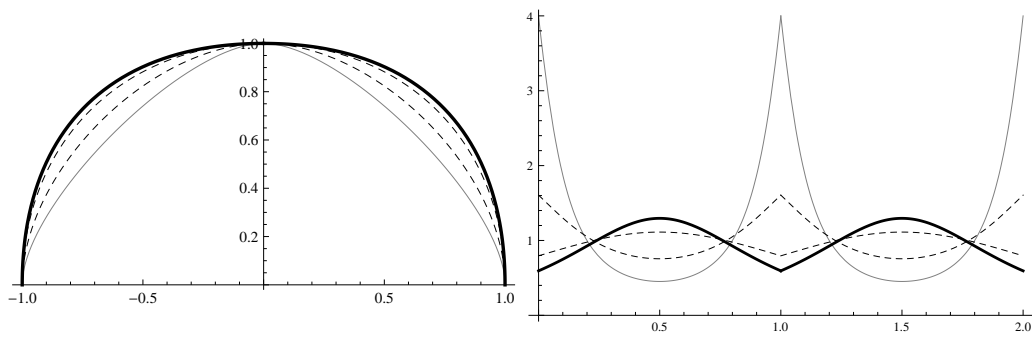
Figure 4: Geometric interpolation of data, sampled from a circle, by using a combined method. The homotopy starts at a solution, obtained by strain energy minimization (solid gray), and proceeds through two solutions at parameter values $\lambda \in \{0.2, 0.4\}$ (dashed) to the final solution (thick black). In the right-hand side figure, the corresponding curvature plots are shown.

As the final example, let us consider the data taken from a nonconvex curve

$$\boldsymbol{f}(t) = \left(t^3 - t + 1\right) \sin t, \; t \cos t\right)^T, \quad t \in [0, 1]. \tag{12}$$

Since the curve is not convex, the optimal solution might not exist, and in this case one has to divide a curve into two (or more) pieces. It is easy to see that (12) has an inflection point at the parameter value $t_0 \approx 0.3678$. Fig. 5 shows geometric interpolants and its curvature profile for the case where the curve $\boldsymbol{f}$ has been divided into two pieces at the inflection point $\boldsymbol{f}(t_0)$ while Fig. 6 shows similar geometric interpolants for the case where $\boldsymbol{f}$ has been divided at $\boldsymbol{f}(0.48)$. Note that in the later case the existence conditions from Thm. 1 are still satisfied.
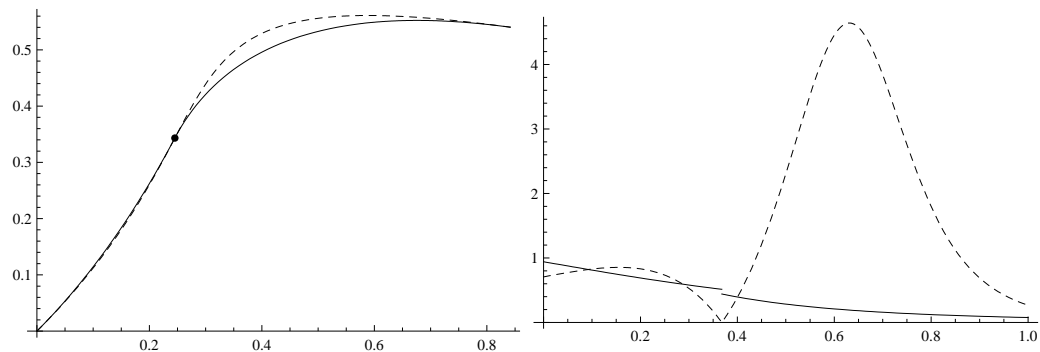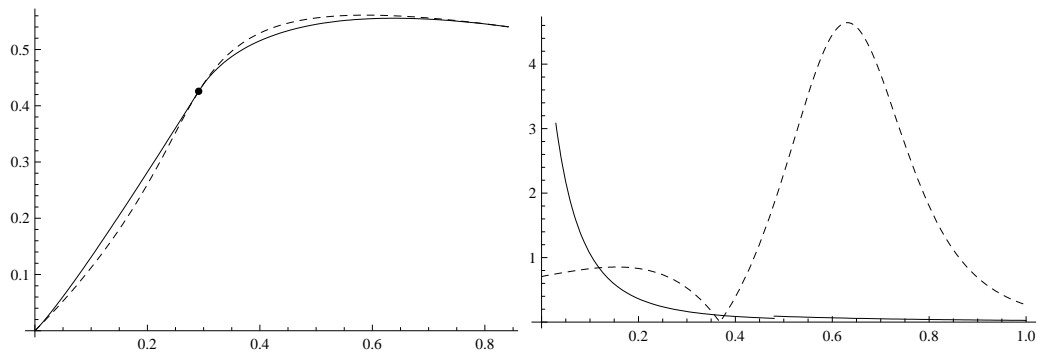

Figure 5: Geometric interpolants (solid) of nonconvex curve (12) (dashed) in the case when the curve has been divided into two parts at the point $\boldsymbol{f}(t_0)$ (left) and its corresponding curvature profile (right).

Figure 6: Geometric interpolants (solid) of nonconvex curve (12) (dashed) in the case when the curve has been divided into two parts at the point $\boldsymbol{f}(0.48)$ (left) and its corresponding curvature profile (right).

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