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CLOSED FORM FORMULA FOR THE NUMBER OF RESTRICTED COMPOSITIONS

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Abstract

In this paper, compositions of a natural number are studied. The number of restricted compositions is given in a closed form, and some applications are presented.

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1. Introduction

Compositions and partitions of a natural number n frequently appear in research and in practical applications. Although the number of compositions or partitions, satisfying particular requirements, can be obtained from their generating functions, this is a serious drawback, since it requires symbolic computational facilities, exact computations, and because of computational complexity involved. In this paper, we present a closed form formula for the number of restricted compositions, and give some applications of the results.

Let us be more precise. The list of natural numbers t_i, which sum up to a natural number n, is *an integer composition of* n. The set of all such lists, where the ordering of the summands matters, is the set of all integer compositions of n. *The set of restricted integer compositions of* n is the subset of all compositions that satisfy some additional restrictions, e.g., on the number of summands, on the values of summands, ...Let $a, b, n \in \mathbb{N}$ with $a \le b \le n$. Let $C(n, a, b)$ denote the number of compositions of n, such that summands t_i are natural numbers, bounded as $a \le t_i \le b$, for all i. Furthermore, let $C(n, k, a, b)$ denote the number of those restricted compositions of n, where the number of summands is equal to k,

$$\sum_{i=1}^{k} t_i = n, \quad a \le t_j \le b, \quad j = 1, 2, \ldots, k.$$

Clearly,

$$C(n, a, b) = \sum_{k=\lceil \frac{n}{b} \rceil}^{\lfloor \frac{n}{a} \rfloor} C(n, k, a, b).$$

It is trivial to prove that $C(n) := C(n,1,n) = 2^{n-1}$ and $C(n,k) := C(n,k,1,n) = \binom{n-1}{k-1}$. There are also known formulas for the special cases

$$C(n,k,a,n) = \binom{n-ka+k-1}{k-1}, \quad C(n,a,n) = \sum_{k=1}^{\lfloor \frac{n}{a} \rfloor} C(n,k,a,n).$$

Also an obvious recursive relation for the general case

$$C(n,k,a,b) = \sum_{i=n-b}^{n-a} C(i,k-1,a,b)$$

is right at hand. Nevertheless, the generating functions are known for both $C(n,k,a,b)$ and $C(n,a,b)$. They are of the form ([2, 6])

$$\left(z^a \frac{1-z^{b-a+1}}{1-z}\right)^k \quad \text{and} \quad \frac{1}{1-z^a \frac{1-z^{b-a+1}}{1-z}}, \tag{1}$$

respectively. We are also interested in a closed-form formula for the number of compositions of n with more than one maximal (or minimal) element. We will denote them by $\mathrm{Max}(n)$ and $\mathrm{Min}(n)$, respectively. Again, there are the generating functions known for $C(n) - \mathrm{Max}(n)$ and $C(n) - \mathrm{Min}(n)$ and are of the form

$$(1-z)^2 \sum_{j=1}^{\infty} \left(\frac{z^j}{1-2z+z^{j+1}}\right)^2 \quad \text{and} \quad (1-z)^2 \sum_{j=1}^{\infty} \left(\frac{z^j}{1-z-z^j}\right)^2,$$

respectively ([6]).

It is quite easy to obtain closed form formulas at least for $C(n,1,b)$ and $C(n,a,b)$, $a > 1$. Namely, by (1)

$$\frac{1}{1-z\,\frac{1-z^b}{1-z}} = \sum_{n=0}^{\infty} C(n,1,b)\, z^n.$$

Since

$$\begin{aligned} \frac{1}{1-z\,\frac{1-z^b}{1-z}} &= \frac{1-z}{1-2\,z+z^{b+1}} = (1-z)\sum_{i=0}^{\infty}(2\,z - z^{b+1})^i \\ &= (1-z)\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j \binom{i}{j} 2^{i-j}\, z^{i-j}\, z^{j\,(b+1)}, \end{aligned}$$

the coefficient at z^n becomes

$$C(n,1,b) = g(n,b) - g(n-1,b),$$

where

$$g(n,b) := \sum_{\substack{i,j \\ i+j\,b=n}} (-1)^j \binom{i}{j} 2^{i-j}.$$

Similarly $C(n,a,b) = g(n,a,b) - g(n-1,a,b)$, $a > 1$, where

$$g(n,a,b) = \sum_{\substack{i,j,\ell \\ i+j(a-1)+\ell(b-a+1)=n}} (-1)^{\ell} \binom{i}{j}\binom{j}{\ell}.$$

But it seems that deriving an explicit formula for $C(n,k,a,b)$ is a far more difficult problem.

The paper is organized as follows. In Section 2 closed form formulae for the number of restricted compositions and restricted partitions are obtained. They are used as a basis for studying two related problems in Section 3. The paper is concluded by some examples in Section 4.

2. Restricted compositions

In this section, our aim is to find a combinatorial closed-form expression for $C(n,k,a,b)$.

T 2.1. *Let $a \le b \le n$ and $\left\lceil \frac{n}{b} \right\rceil \le k \le \left\lfloor \frac{n}{a} \right\rfloor$. To each composition of n assign a vector $\boldsymbol{i} = (i_2, i_3, \ldots, i_b)$, where i_j denotes the frequency of the number j in the composition. Moreover, let*

$$\alpha_j := n - k\,(j-1) - \sum_{\ell=j+1}^{b} (\ell - j + 1)\, i_\ell,\ \beta_j := k - \sum_{\ell=j+1}^{b} i_\ell\,,\ \gamma_j := \left\lfloor \frac{n - k - \sum_{\ell=j+1}^{b} (\ell-1)\, i_\ell}{j-1} \right\rfloor,$$

$j = 2, 3, \ldots, b$. Then

(a) $\displaystyle C(n,k,1,b) = \sum_{\substack{i_2=\alpha_2,\, i_3,\ldots,i_b \\ \max\{0,\alpha_j\} \le i_j \le \min\{\beta_j,\gamma_j\}}} \prod_{\ell=2}^{b} \binom{k - \sum_{j=2}^{\ell-1} i_j}{i_\ell}.$

(b) $C(n,k,a,b) = C(n-k(a-1), k, 1, b-(a-1))$.

(c) *If* $\dfrac{kb-n}{k-1} \in \mathbb{N}$ *and* $\dfrac{ka+(b-a)-n}{k-1} \in \mathbb{N}_0$, *then* $C(n,k,a,b) = \binom{n-ka+k-1}{k-1}$.

P . At first, note that the frequency of number 1 is $n - \sum_{\ell=2}^{b} \ell\, i_\ell$ and so the number of summands in the composition is

$$k(\boldsymbol{i}) := n - \sum_{\ell=2}^{b} (\ell-1) i_\ell. \tag{2}$$

Furthermore, there are exactly

$$\prod_{\ell=2}^{b} \binom{k(\boldsymbol{i}) - \sum_{j=2}^{\ell-1} i_j}{i_\ell}$$

different compositions with the same vector $\boldsymbol{i}$. Since the number of summands has to be k, the only admissible compositions are those with $k(\boldsymbol{i}) = k$. Therefore, the relations

$$\left(k - \sum_{\ell=j}^{b} i_\ell\right)(j-1) \geq n - \sum_{\ell=j}^{b} \ell\, i_\ell \geq k - \sum_{\ell=j}^{b} i_\ell \geq 0, \quad j = 2, 3, \ldots, b,$$

have to be satisfied. With the help of (2), we obtain the appropriate ranges for numbers i_j,

$$\max\{0, \alpha_j\} \leq i_j \leq \min\{\beta_j, \gamma_j\},\ 3 \leq j \leq b, \quad i_2 := \alpha_2 = \gamma_2.$$

The first formula is therefore proven. In order to show that an additional condition, which requires the summands in the composition to be at least a, does not increase the difficulty of the problem, let us define a function

$$f : \left\{(t_1, \ldots, t_k),\ \sum_{i=1}^{k} t_i = n,\ a \leq t_i \leq b\right\} \rightarrow$$

$$\rightarrow \left\{(s_1, \ldots, s_k),\ \sum_{i=1}^{k} s_i = n - k(a-1),\ 1 \leq s_i \leq b - (a-1)\right\},$$

$$f : (t_1, t_2, \ldots, t_k) \mapsto (t_1 - (a-1), t_2 - (a-1), \ldots, t_k - (a-1)),$$

which is clearly a bijection and thus $C(n, k, a, b) = C(n - k(a-1), k, 1, b - (a-1))$. To prove the last statement of the theorem, assume $C(n, k, a, b) = C(m, k, a_2, m)$. Then $m = n + k(m - b)$ and $a_2 = a + m - b$. Hence

$$m = \frac{kb - n}{k-1}, \quad a_2 = \frac{ka + (b-a) - n}{k-1}.$$

If $m \in \mathbb{N}$ and $a_2 \in \mathbb{N}_0$, then $C(m, k, a_2, m)$ is well defined and it follows

$$C(n, k, a, b) = C(m, k, a_2, m) = C(m - ka_2, k, 0, m - ka_2) = \binom{n - ka + k - 1}{k-1}.$$

□

This result can be used to derive some interesting properties of restricted compositions.

C 2.2. *The following formulae hold true:*

$$C(n, k, 1, 2) = \binom{k}{n-k}, \qquad C(n, 1, 2) = \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \binom{k}{n-k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k},$$

$$C(n, k, a, a+1) = \binom{k}{n-ka}, \qquad C(n, a, a+1) = \sum_{k=\lceil \frac{n}{a+1} \rceil}^{\lfloor \frac{n}{a} \rfloor} \binom{k}{n-ka}.$$

Suppose now that one is interested in restricted partitions. The list of natural numbers, which sum up to n and where the ordering of summands is not important, is the set of integer partitions of n. The partitions, where the number of summands is equal to k and where they are bounded between a and b, will be denoted by $P(n,k,a,b)$. The following corollary follows directly from Theorem 2.1.

C 2.3. *Let $1 \le a \le b \le n$ and $\left\lceil \frac{n}{b} \right\rceil \le k \le \left\lfloor \frac{n}{a} \right\rfloor$. To each partition of n assign a vector $\boldsymbol{i} = (i_2, i_3, \ldots, i_b)$, where i_j denotes the frequency of the number j in the partition. Moreover, let α_j, β_j and γ_j, $j = 2, 3, \ldots, b$, be as in Theorem 2.1. Then*

(a) $P(n,k,1,b) = \displaystyle\sum_{\substack{i_2=\alpha_2, i_3,\ldots,i_b \\ \max\{0,\alpha_j\}\le i_j \le \min\{\beta_j,\gamma_j\}}} 1,$

(b) $P(n,k,a,b) = P(n-k(a-1), k, 1, b-(a-1)).$

3. Two related problems

It is interesting to consider the problem of counting the compositions, where more than one maximal (or minimal) summand exists. An application will be given in the last section. Using Theorem 2.1, one can prove the following theorem.

T 3.1. *Let* $\mathrm{Max}(n)$ *denote the number of all compositions of n, such that there are at least two maximal summands, and let* $\mathrm{Min}(n)$ *denote the number of all compositions of n, such that there are at least two minimal summands. Then*

$$\mathrm{Max}(n) = 1 + \sum_{i=2}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{\nu_i=2}^{\left\lfloor \frac{n}{i} \right\rfloor} \sum_{k=\left\lceil \frac{n-i\nu_i}{i-1} \right\rceil}^{n-i\nu_i} \binom{k+\nu_i}{\nu_i} C(n-i\nu_i, k, 1, i-1),$$

$$\mathrm{Min}(n) = \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{\nu_i=2}^{\left\lfloor \frac{n}{i} \right\rfloor} \sum_{k=\mathrm{sign}(n-i\nu_i)}^{\left\lfloor \frac{n-i\nu_i}{i+1} \right\rfloor} \binom{k+\nu_i}{\nu_i} C(n-i\nu_i, k, i+1, n-i\nu_i).$$

P . Let us denote the value of maximal summands by i and the frequency of i in the composition by ν_i. If $i = 1$, then there is exactly one appropriate composition. Let now $i \in \{2, 3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\}$ and $\nu_i \in \{2, 3, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\}$. Consider now the summands, which are smaller than i, and denote the number of these summands by $k := k(i, \nu_i)$. Clearly $\left\lceil \frac{n-i\nu_i}{i-1} \right\rceil \le k \le n - i\nu_i$. Then there are $C(n-i\nu_i, k, 1, i-1)$ different possible compositions among them. But now the maximal summands could be arranged through the sequence of summands, which implies $\binom{k+\nu_i}{\nu_i}$ possibilities of where to set these ν_i maximal summands.

To prove the second formula, let i denote the value of minimal summands. Therefore $i \in \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\}$, and $\nu_i \in \{2, 3, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\}$. Let now k denote the number of summands which are greater than i. If $n - i\nu_i = 0$, then $k = 0$ and there is exactly one such composition. Suppose now $n - i\nu_i > 0$. If $\left\lfloor \frac{n-i\nu_i}{i+1} \right\rfloor = 0$, there is no appropriate composition, containing ν_i summands i, otherwise k can be any number between 1 and $\left\lfloor \frac{n-i\nu_i}{i+1} \right\rfloor$. Further, there are exactly $\binom{k+\nu_i}{\nu_i} C(n-i\nu_i, k, i+1, n-i\nu_i)$ compositions, containing ν_i summands i and k summands greater than i. □

T 1. Values Max(n) and Min(n) for $n \leq 13$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
Max(n)	0	0	0	1	3	8	17	36	72	144	286	569	1133
Min(n)	0	1	1	5	8	21	44	94	197	416	857	1766	3621

Let $\mathrm{Max}_C(n)$ ($\mathrm{Min}_C(n)$) denote the number of compositions of n, such that there is exactly one maximal (minimal) summand, respectively.

Since $\mathrm{Max}(n) + \mathrm{Max}_C(n) = C(n) = 2^{n-1}$ and $\mathrm{Min}(n) + \mathrm{Min}_C(n) = C(n)$, Max($n$) and Min($n$) can be computed also via $\mathrm{Max}_C(n)$ and $\mathrm{Min}_C(n)$.

C 3.2. *Let* Max(n) *and* Min(n) *be as in Theorem 3.1. Then*

$$\mathrm{Max}_C(n) = \sum_{i=2}^{n} \sum_{k=\left\lceil \frac{n-i}{i-1} \right\rceil}^{n-i} (k+1)\, C(n-i, k, 1, i-1),$$

$$\mathrm{Min}_C(n) = \sum_{i=1}^{n} \sum_{k=\mathrm{sign}(n-i)}^{\left\lfloor \frac{n-i}{i+1} \right\rfloor} (k+1)\, C(n-i, k, i+1, n-i).$$

P . The expressions can be obtained similarly as in the proof of Theorem 3.1. □

Although it seems easier to obtain Max(n) and Min(n) from $\mathrm{Max}_C(n)$ and $\mathrm{Min}_C(n)$, let us note, that the time complexity increases this way.

The next important question is the asymptotic behavior of Max(n) and Min(n) for large integers n. Numerical examples and Table 1 indicate the following conjecture.

C 3.3. *Let* Max(n) *and* Min(n) *be as in Theorem 3.1. Then*

$$\lim_{n\to\infty} \frac{\mathrm{Max}(n+1)}{\mathrm{Max}(n)} = \lim_{n\to\infty} \frac{\mathrm{Min}(n+1)}{\mathrm{Min}(n)} = 2.$$

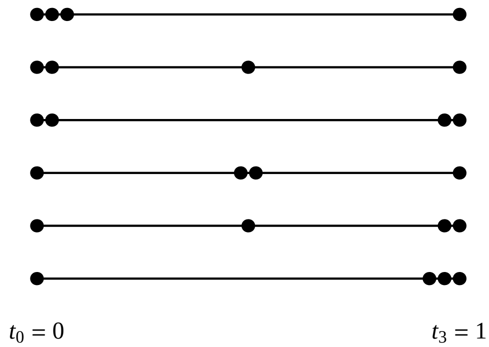
4. Examples

An interesting application of Max(n) arises in numerical analysis, in particular in asymptotic analysis of the geometric Lagrange interpolation problem by Pythagorean-hodograph (PH) curves ([1, 3]). Here the number of cases of the problem considered, that need to be studied, can be significantly reduced by knowing Max(n) in advance. More precisely, if the geometric interpolation (see [4], e.g.) by PH curves of degree n

is considered, the unknown interpolating parameters t_i, $i = 1, 2, \dots, n-1$, have to lie in

$$\mathcal{D} = \left\{ (t_i)_{i=1}^{n-1} \in \mathbb{R}^{n-1} \,|\, t_0 := 0 < t_1 < t_2 < \cdots < t_{n-1} < 1 =: t_n \right\}.$$

It turns out that the interpolation problem requires the analysis of a particular nonlinear system of equations involving the unknown t_i only at the boundary of $\mathcal{D}$. Quite clearly, if the point in $\mathbb{R}^{n-1}$ is to be on the boundary of $\mathcal{D}$, at least two consecutive t_i have to coincide (but not all of them, since $t_0 = 0$ and $t_n = 1$). Thus the number of cases considered is equal to $C(n+1) - 2 = 2^n - 2$ (see Figure 1, e.g.).



F 1. All possible cases for $n = 3$.

Some further observations reduce the problem only to the analysis of particular parts of the boundary. Let

$$\nu_i := \begin{cases} 0, & t_{i-1} \neq t_i, \\ \max\limits_{0 \le j \le i-1} \{ i - j \,|\, t_{\ell+1} = t_\ell,\ j \le \ell \le i-1 \}, & \text{otherwise}, \end{cases}$$

where $i = 1, 2, \dots, n$. It turns out that if the sequence $(\nu_i)_{i=1}^n$ has a unique maximum, the corresponding choice of parameters $(t_i)_{i=1}^{n-1}$ can be skipped in the analysis. But the number of sequences $(\nu_i)_{i=1}^n$ for which the maximum is not unique is precisely $\mathrm{Max}(n+1)$.

Let us conclude the paper with an another example. In high order parametric polynomial approximation of circular arcs ([5], e.g.), the coefficients of the optimal solution involve the number of restricted partitions of a natural number. Namely, the coefficients of the parametric polynomial approximant $\boldsymbol{p}(t) = (x(t), y(t))^T$, where

$$x(t) := \sum_{k=0}^{n} \alpha_k t^k, \quad y(t) := \sum_{k=0}^{n} \beta_k t^k,$$

are of the form

$$\alpha_k = \begin{cases} \displaystyle\sum_{j=0}^{k(n-k)} \widetilde{P}(j, k, n-k) \cos\left(\frac{k^2 \pi}{2n} + \frac{\pi}{n} j \right), & k \text{ is even}, \\ 0, & k \text{ is odd}, \end{cases}$$

and

$$\beta_k = \begin{cases} 0, & k \text{ is even}, \\ \displaystyle\sum_{j=0}^{k(n-k)} \widetilde{P}(j,k,n-k)\, \sin\left(\frac{k^2\pi}{2n} + \frac{\pi}{n}j\right), & k \text{ is odd}, \end{cases},$$

where $\widetilde{P}(n,k,b) := \sum_{\ell=1}^{k} P(n,\ell,1,b)$.

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