Implicitization and Inversion of Planar Parametric Rational Curves

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Abstract

Planar parametric rational curves are a basic tool in Computer Aided Geometric Design (CAGD). Among others, there are two important geometric problems which have to be solved in CAGD. Given a planar parametric rational curve:

1) Find an implicit equation of the curve (*implicitization*).
2) Find the parameter value(s) corresponding to the coordinates of a point known to lie on the curve (*inversion*).

The solution of these two problems will be discussed in this paper. Three different methods for the implicitization will be presented and the problem of inversion will be solved using the Bezout resultant.

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1 Introduction

Rational parametric polynomial curves are widely used in Computer Aided Geometric Design (CAGD). It is easy to deal with them, they have an easy representation in computer memory and good computational aspects (an easy evaluation at a given parameter, they are degree preserving under linear reparametrization, only basic arithmetic operations have to be used to obtain a derivative, integral,...). A class of planar curves from this family is of great importance in CAGD but also in other branches of computer science (mobile robots e.g.). So it is very important to solve some frequent problems appearing when such curves are used. Although parametric representation of curves is usually better than any other, there is an important problem when an implicit representation of the curve is desired. Namely, if one has to check if a given point $T(x, y)$ is on a particular curve then an implicit representation of the curve implies an easy answer. The problem of converting a parametric representation into an implicit one is known as the problem of *implicitization*. Another problem appearing frequently is, how to determine the parameter value(s) of a point on a parametric curve. This problem is known as *inversion*.

So let

$$x(t) = \frac{X(t)}{W(t)} \quad \text{and} \quad y(t) = \frac{Y(t)}{W(t)}$$

be a planar parametric rational curve (parametric curve in further text), i.e., X, Y and W are polynomials. We shall assume that the polynomials X, Y and W do not have a common root. Note that x and y have the same denominator. This is a little restrictive but usually required in CAGD. The problem of implicitization requires to find an implicit form of the parametric curve, i.e., $F(x, y) = 0$. On the other hand, for a given point $T(x_0, y_0)$ known to lie on a parametric curve, the problem of inversion looks for a parameter value(s) t_0 for which $x(t_0) = x_0$ and $y(t_0) = y_0$.

As mentioned before, an implicit representation of a parametric curve is used, when one has to check if a given point $T(x_0, y_0)$ is or is not on the curve. It is of course a purely theoretical question. In practice, this is checked under some considerable tolerance (machine precision,...). The inversion might be used when one has to find a parameter of the located point (a mobile robot e.g.) on a particular parametric curve.

Both problems mentioned above are closely connected with the problem of finding common roots of two scalar polynomials which will be discussed in the next section.

2 Common roots of two polynomials

The problem of finding common roots of two scalar polynomials has been solved long before computers have become an important part of (numerical) analysis. Nowadays, one could use a direct way of solving this problem in two steps. First, all roots of the first polynomial are computed and then each of them is checked if it is also a root of the second polynomial. Using a computer, this can be formally done for polynomials of relatively high degree, since algorithms for solving algebraic equations are available. But in the sense of numerical analysis the problem of finding *all* the roots of a given polynomial might be unstable (for the polynomials having roots close together e.g.). Instead, we can use an old method known as *resultant* of the polynomials.

Let

$$\begin{aligned} p(t) &= a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0, \\ q(t) &= b_n t^n + b_{n-1} t^{n-1} + \cdots + b_0 \end{aligned}$$

be two scalar polynomials, where at least one of a_m or b_n is nonzero. Let us define the *Sylvester resultant* as the determinant of the $(m+n) \times (m+n)$ matrix

$$S(p,q) = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} & a_m & 0 & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{m-2} & a_{m-1} & a_m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & a_2 & a_3 & \cdots & a_m \\ b_0 & b_1 & b_2 & \cdots & b_{n-1} & b_n & 0 & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} & b_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}.$$

The following theorem gives a very well known result which can be found in [1]:

Theorem 2.1 *Polynomials p and q have a common root if and only if*

$$\det S(p,q) = 0.$$

Since the dimension of the Sylvester resultant (i.e. the dimension of the matrix $S(p,q)$) is $m+n$, we sometimes use the Bezout resultant, defined by $B(p,q) = [c_{ij}]_{i,j=1}^{m}$, where the coefficients c_{ij} satisfy the relation

$$\frac{p(t)\,q(u) - p(u)\,q(t)}{t-u} = \sum_{i,j=1}^{m} c_{ij} t^{i-1}\,u^{j-1}.$$

Here the assumption that the degree $p = m \geq n$ has been made quietly. A similar result as for the Sylvester resultant holds for the Bezout resultant (see [3] or [1]):

Theorem 2.2 *Polynomials p and q have a common root if and only if*

$$\det B(p,q) = 0.$$

Example 2.1 *Let $p(t) = t^2 - 3\,t + 2$ and $q(t) = 2\,t^2 + 4\,t - 6$. An easy "computation" gives*

$$S(p,q) = \begin{bmatrix} 2 & -3 & 1 & 0 \\ 0 & 2 & -3 & 1 \\ 2 & 4 & -6 & 0 \\ 0 & 2 & 4 & -6 \end{bmatrix}.$$

Since $\det S(p,q) = 0$, polynomials p and q has a common root. Indeed, $p(1) = q(1) = 0$. Similarly,

$$B(p,q) = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix},$$

which obviously has zero determinant and implies the existence of the common root of p and q.

3 Implicitization and inversion

Resultants can be used to solve the problem of implicitization and inversion as we shall show now. Let $x(t) = X(t)/W(t)$, $y(t) = Y(t)/W(t)$ be a parametric curve, where X, Y and W do not have a common factor. A straightforward way to obtain the implicit equation is to write it in the form

$$a_1\,x^m + a_2\,x^{m-1}\,y + \cdots + a_{\binom{m+2}{2}} = 0, \tag{1}$$

since we can assume that m is the maximal degree of X, Y and W. It implies a homogeneous system of $2\,m+1$ equations for $(m+2)(m+1)/2$ unknowns and a_i can be always determined. But this usually requires a lot of calculations.

To avoid this problem, let us write

$$\begin{aligned} p_x(t) &= x\,W(t) - X(t), \\ q_y(t) &= y\,W(t) - Y(t). \end{aligned}$$

where x and y are the coordinates of the point on the curve. Then, by Theorem 2.1 (or Theorem 2.2), the implicit equation of the curve is $\det S(p_x, q_y) = 0$ (or $\det B(p_x, q_y) = 0$). Since X, Y and W do not have a common factor, none of these determinants can vanish identically. The same resultants can be used to check if a given point $T(x_0, y_0)$ is on a curve. We simply compute one of the resultants and compare it with zero (generally to machine precision). Both resultants have some advantages and disadvantages. The Sylvester resultant has degree $m+n$, but can be easily derived from the coefficients of the polynomials. On the other hand the Bezout resultant has degree m, it is symmetric, but its coefficients are more complicated to derive.

Example 3.1 *Let the parametric curve be given by*

$$x(t) = \frac{2\,t - t^2}{1 - t + t^2}, \quad y(t) = \frac{1 - t^2}{1 - t + t^2}.$$

Then

$$\begin{aligned} p_x(t) &= x\,(1 - t + t^2) - 2\,t + t^2 = x - (x+2)\,t + (x+1)\,t^2, \\ q_y(t) &= y\,(1 - t + t^2) - 1 + t^2 = y - 1 - y\,t + (y+1)\,t^2. \end{aligned}$$

The Sylvester resultant of p_x and q_y becomes

$$\begin{vmatrix} x & -x-2 & x+1 & 0 \\ 0 & x & -x-2 & x+1 \\ y-1 & -y & y+1 & 0 \\ 0 & y-1 & -y & y+1 \end{vmatrix} = 3\,x^2 - 3x\,y + 3\,y^2 - 3 = 0,$$

and the Bezout one is

$$\begin{vmatrix} x - 2\,y + 2 & -2\,x + y - 1 \\ -2\,x + y - 1 & x + y + 2 \end{vmatrix} = -3\,x^2 + 3\,x\,y - 3\,y^2 + 3 = 0.$$

Both of them imply the implict equation of the curve

$$x^2 - x\,y + y^2 - 1 = 0.$$

The problem of inversion is easily solved using the matrix $B(p_x, q_y)$.

Theorem 3.1 *If $T(x_0, y_0)$ is a simple point on the curve, then the parameter t_0 corresponding to a given point is*

$$t_0 = \frac{\text{cofactor}\, B_{1,2}(p_{x_0}, q_{y_0})}{\text{cofactor}\, B_{1,1}(p_{x_0}, q_{y_0})}.$$

If the point is not simple, the parameters can still be found using the same matrix, but we shall omit this case.

Example 3.2 *If we look at the same curve as in Example 3.1, then by the previous theorem the parameter, which belongs to a particular point $T(x_0, y_0)$ on the curve, can be expressed as*

$$t_0 = -\frac{-2\,x_0 + y_0 - 1}{x_0 + y_0 + 2}.$$

4 The method of moving lines

This is a relatively new method (see [5] for a list of selected papers), which combines the idea of unknown coefficients (1) and resultants. The moving line of degree d is defined as

$$\sum_{i=0}^{d}(A_i\,X + B_i\,Y + C_i\,W)\,t^i = 0.$$

Here X, Y and W are the homogeneous coordinates of the line. Since its coefficients vary with the parameter t, the line is moving in the plane. A moving line is said to follow the parametric curve, if

$$\sum_{i=0}^{d}(A_i\,X(t) + B_i\,Y(t) + C_i\,W(t))\,t^i \equiv 0.$$

The results which will follow hold for curves of even degree. They can be generalized to curves of odd degree too, but this is beyond the scope of this paper. So let the degree of the curve be $2\,m$. Then, every moving line of degree m following a curve of degree $2\,m$ has $3\,(m+1) = 3\,m+3$ unknown coefficients. Since the total degree of the moving line is $m + 2m = 3m$, we obtain a homogeneous system of $3\,m+1$ linear equations for $3\,m+3$ unknown coefficients. This implies that there are always at least two linearly independent moving lines of degree m which follow a curve of degree $2\,m$. The implicit equation of the curve can be found by using the following theorem.

Theorem 4.1 *If a parametric curve has even degree $2\,m$, and p_1, p_2 are two linearly independent moving lines following the curve and there are no moving lines of degree $< m$ that follow the curve, then $\det S(p_1, p_2) = 0$ is an implicit equation of the curve.*

We shall conclude with an example.

Example 4.1 *Let the curve be defined as in Example 3.1. The coefficients of two moving lines of degree 1 that follow the curve can be obtained by solving the system of equations*

$$\begin{aligned} B_0 + C_0 &= 0 \\ 2\,A_0 - C_0 + B_1 + C_1 &= 0 \\ -A_0 - B_0 + C_0 + 2\,A_1 - C_1 &= 0 \\ -A_1 - B_1 + C1 &= 0. \end{aligned}$$

The two-parametric family of solutions is obtained and we can choose two linearly independent solutions as

$$\begin{aligned} (-X + Y - W) + t\,(X + W) &= 0 \\ (-Y + W) + t\,(-X + Y) &= 0. \end{aligned}$$

The Sylvester resultant of these lines is

$$\begin{vmatrix} -X + Y - W & X + W \\ -Y + W & -X + Y \end{vmatrix}$$

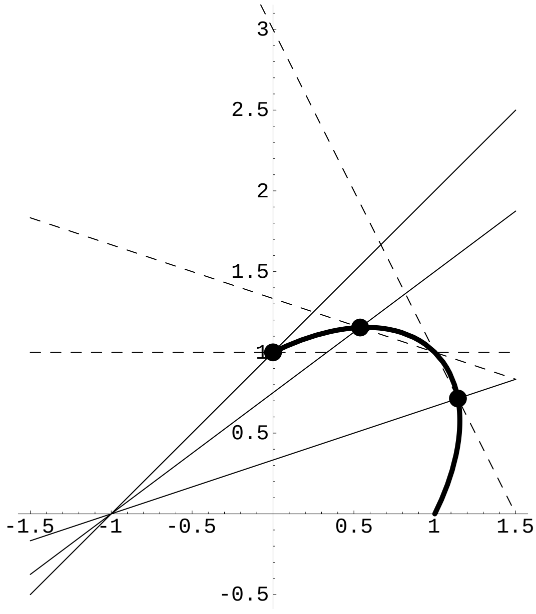

Figure 1: A set of moving lines from our example at the parameter values $t = 0$, $t = 1/4$ and $t = 2/3$. The plain line always represents the first moving line, and the dashed one the second moving line. Their intersections are points on the curve they are following. The bold segment is the set of intersections of the moving lines for the parameter values $t \in [0, 1]$.

and, since there are no moving lines of degree 0 *that follow the curve,*

$$X^2 - X\,Y + Y^2 - W^2 = 0$$

is an implicit equation of the curve. After dividing it by W^2 *we get*

$$x^2 - x\,y + y^2 - 1 = 0$$

which is already a known result.

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