

# Lagrange geometric interpolation by rational spatial cubic Bézier curves

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## Abstract

In the paper, the Lagrange geometric interpolation by spatial rational cubic Bézier curves is studied. It is shown that under some natural conditions the solution of the interpolation problem exists and is unique. Furthermore, it is given in a simple closed form which makes it attractive for practical applications. Asymptotic analysis confirms the expected approximation order, i.e., order six. Numerical examples pave the way for a promising nonlinear geometric subdivision scheme.

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## 1. Introduction

Geometric interpolation by parametric curves is a modern research topic dealing with interpolation of geometric data (points, tangent directions, etc.), independently of parameterization. In comparison to classical interpolation schemes this brings additional shape parameters and consequently a higher approximation order. Also the shape of the interpolant is more pleasant since the parameters that need to be prescribed in classical schemes are here chosen automatically.

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The first rigorous analysis of a particular geometric interpolation scheme goes back to [1], which was further refined in [2]. Later, Höllig and Koch stated in [3] a general conjecture on polynomial geometric interpolation asserting that these interpolants could, in general, interpolate much more data than their classical counterparts of the same degree. However, the analysis of geometric interpolation schemes is a challenging task since it involves analysis of systems of nonlinear equations. Several results on existence, uniqueness, geometric conditions for solvability, and algorithms for construction are known (see [4], [5], [6], [7], [8] and the references therein). While planar polynomial geometric interpolation is a well researched topic, not much is known for interpolation in higher dimensional spaces  $\mathbb{R}^d$ . Obviously the rational geometric interpolation is a even more challenging topic and even less results are known. A nice survey on known facts is, e.g., in [9]. Some recent results are mainly dealing with the planar case and can be found in [10], [11], [12], [13], while the spatial case is still to be investigated.

A first step, analysis of geometric Hermite interpolation of spatial data, was done in [14]. A general interpolation scheme for Hermite  $G^{n-1}$  interpolation by a rational curve of degree  $n$  was derived, and cubic and quartic cases were studied in detail.

In practice, the Lagrange interpolation turns more useful since only data points are interpolated, and no additional higher order approximation data is needed. Of course, the problem is much harder than the Hermite one. In this paper Lagrange geometric interpolation of six data points by spatial rational cubic Bézier curves is tackled. It is shown that under some natural conditions the solution exists and is unique. The solution can be written in a closed form. The asymptotic analysis carries through, and it confirms the expected approximation order six. The numerical examples are used as a cornerstone for a new nonlinear geometric subdivision scheme that could be used for particular sets of the initial data. The evidence supplied suggests its nice properties.

The outline of the paper is as follows. In the next section, the Lagrange interpolation problem is stated. It is analysed in the third section, where the solution of the problem is revealed. In the fourth section, asymptotic analysis is carried through, and the optimal approximation order is confirmed. The paper is concluded by numerical examples, based on a new subdivision scheme, and by an appendix, where a geometric proof of a similar result for the unordered data is given.

## 2. Geometric interpolation problem

Suppose that data points

$$\mathbf{P}_\ell \in \mathbb{R}^3, \ell = 0, 1, \dots, 5, \quad \mathbf{P}_{\ell+1} \neq \mathbf{P}_\ell, \quad (1)$$

are prescribed. The goal is to determine a rational spatial cubic Bézier curve  $\mathbf{r} : [0, 1] \rightarrow \mathbb{R}^3$  that interpolates the data points at parameter values

$$0 =: t_0 < t_1 < t_2 < t_3 < t_4 < t_5 := 1, \quad \mathbf{t} := (t_\ell)_{\ell=0}^5, \quad (2)$$

with  $t_1, t_2, t_3, t_4$  yet to be determined, i.e.,

$$\mathbf{r}(t_\ell) = \mathbf{P}_\ell, \quad \ell = 0, 1, \dots, 5. \quad (3)$$

A rational cubic Bézier curve can be written as

$$\mathbf{r} = \frac{1}{\pi} \mathbf{p} = \sum_{i=0}^3 \mathbf{b}_i r_i, \quad \mathbf{p} := \sum_{i=0}^3 w_i \mathbf{b}_i B_i^3, \quad \pi := \sum_{i=0}^3 w_i B_i^3, \quad r_i := \frac{1}{\pi} w_i B_i^3, \quad (4)$$

where  $\mathbf{b}_i \in \mathbb{R}^3$  are the control points,  $w_i$  are the weights, and  $B_i^3(t) := \binom{3}{i} t^i (1-t)^{3-i}$  are the cubic Bernstein basis polynomials. By using the normalized form of the curve (see [15]), we can without loss of generality assume that  $w_0 = w_3 = 1$  and thus

$$\pi(t_0) = 1, \quad \pi(t_5) = 1. \quad (5)$$

The interpolation conditions (3) give 18 scalar equations that should determine 18 unknowns, i.e., the components of control points  $\mathbf{b}_i$ ,  $i = 0, 1, 2, 3$ , the weights  $w_1$  and  $w_2$ , and the parameters  $t_1, t_2, t_3, t_4$ . Quite clearly, due to the end point interpolation property,

$$\mathbf{r}(t_0) = \mathbf{b}_0 = \mathbf{P}_0, \quad \mathbf{r}(t_5) = \mathbf{b}_3 = \mathbf{P}_5.$$

If the scalar unknowns  $t_\ell$  and  $w_i$  have already been determined, it is a straightforward linear task to compute  $\mathbf{b}_1$  and  $\mathbf{b}_2$  from (4).

### 3. Equations

Let us reduce the interpolation conditions (3) to equations that determine the scalar unknowns only. Let  $[\tau_\ell, \tau_{\ell+1}, \dots, \tau_{\ell+k}]$  denote the  $k$ -th order divided difference based upon the knots  $\tau_\ell, \tau_{\ell+1}, \dots, \tau_{\ell+k}$ . Since  $\mathbf{p} = \pi \mathbf{r}$ , introduced in (4), is a cubic curve, the divided differences  $[t_j, t_{j+1}, \dots, t_{j+4}]$ ,  $j = 0, 1$ , map it to  $\mathbf{0}$ . However, the parameters (2) should be distinct, and the closed form divided difference representation can be applied (see e.g., [16]). This gives two vector equations for six scalar unknowns

$$[t_j, t_{j+1}, \dots, t_{j+4}] (\pi \mathbf{r}) = \sum_{\ell=j}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)} \mathbf{r}(t_\ell) = \sum_{\ell=j}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)} \mathbf{P}_\ell = \mathbf{0}, \quad j = 0, 1, \quad (6)$$

where

$$\omega_j(t) := \prod_{\ell=j}^{j+4} (t - t_\ell), \quad j = 0, 1. \quad (7)$$

Since  $\pi$  is a cubic polynomial, its divided difference at five points vanishes, thus

$$\mathbf{P}_0 \sum_{\ell=j}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)} = \mathbf{0}, \quad j = 0, 1, \quad (8)$$

and we may further simplify (6) to

$$\sum_{\ell=1}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)} (\mathbf{P}_\ell - \mathbf{P}_0) = \sum_{\ell=1}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)} \sum_{i=0}^{\ell-1} \Delta \mathbf{P}_i = \sum_{i=0}^{j+3} \Delta \mathbf{P}_i \sum_{\ell=i+1}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)} = \mathbf{0},$$

$j = 0, 1$ , with  $\Delta \mathbf{P}_i := \mathbf{P}_{i+1} - \mathbf{P}_i$ . If  $i = 0$  and  $j = 1$ , the term

$$\sum_{\ell=i+1}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)}$$

vanishes, and we obtain a compact form of the equations, i.e.,

$$\sum_{i=j}^{j+3} \Delta \mathbf{P}_i \sum_{\ell=i+1}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)} = \mathbf{0}, \quad j = 0, 1. \quad (9)$$

But (9) simply shows that for  $j = 0, 1$ , vectors

$$\begin{aligned} \mathbf{k}_0 &:= \left( \sum_{\ell=j+1}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)}, \sum_{\ell=j+2}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)}, \sum_{\ell=j+3}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)}, \frac{\pi(t_{j+4})}{\dot{\omega}_j(t_{j+4})}, 0 \right)^T, \\ \mathbf{k}_1 &:= \left( 0, \sum_{\ell=j+1}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)}, \sum_{\ell=j+2}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)}, \sum_{\ell=j+3}^{j+4} \frac{\pi(t_\ell)}{\dot{\omega}_j(t_\ell)}, \frac{\pi(t_{j+4})}{\dot{\omega}_j(t_{j+4})} \right)^T \end{aligned} \quad (10)$$

belong to the kernel of the data difference matrix

$$\Delta \mathbf{P} := (\Delta \mathbf{P}_i)_{i=0}^4 \in \mathbb{R}^{3,5}. \quad (11)$$

From (5), (7) and (8) we observe

$$\sum_{\ell=1}^4 \frac{\pi(t_\ell)}{\dot{\omega}_0(t_\ell)} = -\frac{\pi(t_0)}{\dot{\omega}_0(t_0)} = -\frac{1}{\dot{\omega}_0(t_0)} < 0, \quad \frac{\pi(t_5)}{\dot{\omega}_1(t_5)} = \frac{1}{\dot{\omega}_1(t_5)} > 0. \quad (12)$$

Note that the kernel  $\ker \Delta \mathbf{P}$  is at least two-dimensional. But if the interpolation problem (3) has a solution then by (10) and (12) there should exist vectors

$$\zeta_j := (\zeta_{j,i})_{i=0}^4, \quad \zeta_j \in \ker \Delta \mathbf{P}, \quad j = 0, 1, \quad (13)$$

with

$$\zeta_{0,0} = 1, \quad \zeta_{0,4} = 0, \quad \zeta_{1,0} = 0, \quad \zeta_{1,4} = 1, \quad (14)$$

that satisfy

$$\mathbf{k}_0 = -\frac{1}{\dot{\omega}_0(t_0)} \zeta_0, \quad \mathbf{k}_1 = \frac{1}{\dot{\omega}_1(t_5)} \zeta_1.$$

Componentwise these equations are

$$\begin{aligned} \sum_{\ell=i+1}^4 \frac{\pi(t_\ell)}{\dot{\omega}_0(t_\ell)} &= -\frac{1}{\dot{\omega}_0(t_0)} \zeta_{0,i}, \quad i = 0, 1, 2, 3, \\ \sum_{\ell=i+1}^5 \frac{\pi(t_\ell)}{\dot{\omega}_1(t_\ell)} &= -\sum_{\ell=1}^i \frac{\pi(t_\ell)}{\dot{\omega}_1(t_\ell)} = \frac{1}{\dot{\omega}_1(t_5)} \zeta_{1,i}, \quad i = 1, 2, 3, 4. \end{aligned} \quad (15)$$

Let  $\Delta \zeta_{j,i-1} := \zeta_{j,i} - \zeta_{j,i-1}$ . By subtracting consecutive equations in (15) we derive

$$\frac{\pi(t_i)}{\dot{\omega}_0(t_i)} = \frac{1}{\dot{\omega}_0(t_0)} \Delta \zeta_{0,i-1}, \quad \frac{\pi(t_i)}{\dot{\omega}_1(t_i)} = -\frac{1}{\dot{\omega}_1(t_5)} \Delta \zeta_{1,i-1}, \quad i = 1, 2, 3, 4. \quad (16)$$

Moreover,

$$\begin{aligned} \text{sign } \dot{\omega}_0(t_i) &= (-1)^i, \quad i = 0, 1, \dots, 3, \quad \text{sign } \dot{\omega}_0(t_4) = \text{sign } \dot{\omega}_0(t_5) = 1, \\ \text{sign } \dot{\omega}_1(t_0) &= \text{sign } \dot{\omega}_1(t_1) = 1, \quad \text{sign } \dot{\omega}_1(t_i) = (-1)^{i-1}, \quad i = 2, 3, \dots, 5. \end{aligned} \quad (17)$$

Let us define

$$\delta_{i-1} := \frac{\Delta \zeta_{0,i-1}}{\Delta \zeta_{1,i-1}}, \quad i = 1, 2, 3, 4, \quad \boldsymbol{\delta} := (\delta_i)_{i=0}^3.$$

If we eliminate  $\pi(t_i), i = 1, 2, 3, 4$ , from (16), we obtain a system of four equations

$$\frac{\dot{\omega}_1(t_5)}{\dot{\omega}_0(t_0)} \delta_{i-1} + \frac{\dot{\omega}_1(t_i)}{\dot{\omega}_0(t_i)} = 0, \quad i = 1, 2, 3, 4, \quad (18)$$

for the unknown parameters  $t_i$ . By (17) and (18),  $\delta_{i-1} > 0$ ,  $i = 1, 2, 3, 4$ . After multiplying the  $i$ -th equation by a common nonzero factor  $(t_i - t_0)/(t_5 - t_i)$  and subtracting the consecutive equations, we obtain

$$\frac{t_5 - t_2}{t_2 - t_0} \delta_0 = \frac{t_5 - t_1}{t_1 - t_0} \delta_1, \quad \frac{t_5 - t_3}{t_3 - t_0} \delta_1 = \frac{t_5 - t_2}{t_2 - t_0} \delta_2, \quad \frac{t_5 - t_4}{t_4 - t_0} \delta_2 = \frac{t_5 - t_3}{t_3 - t_0} \delta_3,$$

which allows one to express  $t_2, t_3, t_4$  in terms of  $t_1$ . But then the last equation in (18) simplifies to

$$\frac{\delta_1 \delta_2 \delta_3 (t_5 - t_1)^3}{\delta_0^2 (t_1 - t_0)^3} = 1,$$

and we end up with a simple closed form solution

$$t_i = \frac{\sqrt[3]{\delta_0 \delta_1 \delta_2 \delta_3}}{\delta_{i-1} + \sqrt[3]{\delta_0 \delta_1 \delta_2 \delta_3}}, \quad i = 1, 2, 3, 4. \quad (19)$$

Since  $\delta_i$  should be positive, the parameters  $t_i$  satisfy (2) if and only if

$$\delta_0 > \delta_1 > \delta_2 > \delta_3 > 0. \quad (20)$$

Once the parameter values  $t_i$  have been determined by (19), there are several ways to compute the weights  $w_i$  from (5) and (16). Since

$$w_1 = \frac{1}{3} \dot{\pi}(0) + 1, \quad w_2 = -\frac{1}{3} \dot{\pi}(1) + 1, \quad (21)$$

we may express the cubic polynomial  $\pi$  as an interpolating polynomial, based upon pairs

$$(t_0, \pi(t_0)), (t_2, \pi(t_2)), (t_3, \pi(t_3)), (t_4, \pi(t_4))$$

from the first set of equations in (16), and

$$(t_1, \pi(t_1)), (t_2, \pi(t_2)), (t_3, \pi(t_3)), (t_5, \pi(t_5))$$

from the second one. Then (21) gives the weights

$$\begin{aligned} w_1 &= \frac{-\delta_0 (\Delta\zeta_{0,0} + 1) - \delta_1 (\Delta\zeta_{0,1} + 1) - \delta_2 (\Delta\zeta_{0,2} + 1) - \delta_3 (\Delta\zeta_{0,3} + 1)}{3\sqrt[3]{\delta_0\delta_1\delta_2\delta_3}}, \\ w_2 &= \frac{1}{3}\sqrt[3]{\delta_0\delta_1\delta_2\delta_3} \left( \frac{\Delta\zeta_{1,0} - 1}{\delta_0} + \frac{\Delta\zeta_{1,1} - 1}{\delta_1} + \frac{\Delta\zeta_{1,2} - 1}{\delta_2} + \frac{\Delta\zeta_{1,3} - 1}{\delta_3} \right). \end{aligned} \quad (22)$$

Let us summarize the discussion.

**Theorem 1.** *Suppose that there exist kernel vectors  $\zeta_j := (\zeta_{j,i})_{i=0}^4$  of the data difference matrix  $\Delta\mathbf{P}$ , defined in (11), that satisfy (14) and (20). Then there exists a cubic rational Bézier curve that interpolates the data (1) at parameters (19), with weights provided by (22).*

**Remark 1.** As it was pointed out by a referee, the existence and uniqueness of the solution of the problem of geometric interpolation of six unordered spatial points by a rational cubic is well known in projective and algebraic geometry. We include a simple proof of this result, also kindly provided by the same referee, in the appendix since it is hard to find it in the literature. However, the interpolation problem considered in our paper differs significantly, since the order of points to be interpolated is prescribed.

If the data (1) are not planar, we can elaborate the assertion of Theorem 1 a little bit further. Let us define determinants

$$D_{ijk} := \det(\Delta\mathbf{P}_i, \Delta\mathbf{P}_j, \Delta\mathbf{P}_k), \quad i, j, k \in \{0, 1, \dots, 4\}.$$

**Corollary 2.** *Suppose that  $D_{123} \neq 0$ . Then the cubic rational Bézier interpolating curve exists if*

$$\frac{D_{023} + D_{123}}{D_{234}} > \frac{D_{013} + D_{023}}{D_{134} + D_{234}} > \frac{D_{012} + D_{013}}{D_{124} + D_{134}} > \frac{D_{012}}{D_{123} + D_{124}}. \quad (23)$$

*If  $D_{ijk} = 0$  for some  $0 \leq i < j < k \leq 4$ ,  $k - i < 4$ , the interpolation problem may have a solution only if the data are planar.*

PROOF. If  $D_{123} \neq 0$ , the kernel vectors (13) are by the Cramer's rule

$$\boldsymbol{\zeta}_0 = \left( 1, -\frac{D_{023}}{D_{123}}, \frac{D_{013}}{D_{123}}, -\frac{D_{012}}{D_{123}}, 0 \right), \quad \boldsymbol{\zeta}_1 = \left( 0, -\frac{D_{234}}{D_{123}}, \frac{D_{134}}{D_{123}}, -\frac{D_{124}}{D_{123}}, 1 \right). \quad (24)$$

Further,

$$\delta_0 = \frac{D_{023} + D_{123}}{D_{234}}, \quad \delta_1 = \frac{D_{013} + D_{023}}{D_{134} + D_{234}}, \quad \delta_2 = \frac{D_{012} + D_{013}}{D_{124} + D_{134}}, \quad \delta_3 = \frac{D_{012}}{D_{123} + D_{124}},$$

and the requirement (20) follows from (23). If  $D_{ijk} = 0$  for some  $0 \leq i < j < k \leq 4$ ,  $k - i < 4$ , the vectors  $\Delta \mathbf{P}_i$ ,  $\Delta \mathbf{P}_j$ , and  $\Delta \mathbf{P}_k$  are coplanar, and the kernel vectors  $\boldsymbol{\zeta}_j$  that satisfy (14) do not exist unless  $\Delta \mathbf{P}_\ell$ ,  $\ell \in \{0, 1, \dots, 4\} \setminus \{i, j, k\}$ , belong to the same plane.

Algebraic conditions (23) have a simple geometric interpretation that may be used for the computation of the unknown scalar parameters too. Let

$$V_{ijk}^s := (-1)^s \det(\mathbf{P}_i - \mathbf{P}_s, \mathbf{P}_j - \mathbf{P}_s, \mathbf{P}_k - \mathbf{P}_s), \quad s = 0, 5, \quad i, j, k \in \{1, 2, 3, 4\}.$$

Geometrically,  $\frac{1}{6}V_{ijk}^s$  is the signed volume of the tetrahedron spanned by the vectors  $\mathbf{P}_r - \mathbf{P}_s$ ,  $r = i, j, k$ . It is then easy to verify that

$$D_{123} = (-1)^s (-V_{123}^s + V_{124}^s - V_{134}^s + V_{234}^s), \quad s = 0, 5. \quad (25)$$

Further, with  $\Delta \boldsymbol{\zeta}_i := (\Delta \zeta_{i,j})_{j=0}^3$ ,  $i = 0, 1$ ,

$$\begin{aligned} \Delta \boldsymbol{\zeta}_0 &= \frac{1}{D_{123}} (-V_{234}^0, V_{134}^0, -V_{124}^0, V_{123}^0), \\ \Delta \boldsymbol{\zeta}_1 &= \frac{1}{D_{123}} (-V_{234}^5, V_{134}^5, -V_{124}^5, V_{123}^5), \end{aligned} \quad (26)$$

and consequently

$$\delta_0 = \frac{V_{234}^0}{V_{234}^5}, \quad \delta_1 = \frac{V_{134}^0}{V_{134}^5}, \quad \delta_2 = \frac{V_{124}^0}{V_{124}^5}, \quad \delta_3 = \frac{V_{123}^0}{V_{123}^5}. \quad (27)$$

So one obtains an admissible solution only if the sequence  $\boldsymbol{\delta}$  of volume quotients is monotonically decreasing but positive (Fig. 1). In order to investigate the background of this property let us assume that the data  $\mathbf{P}_\ell$  are



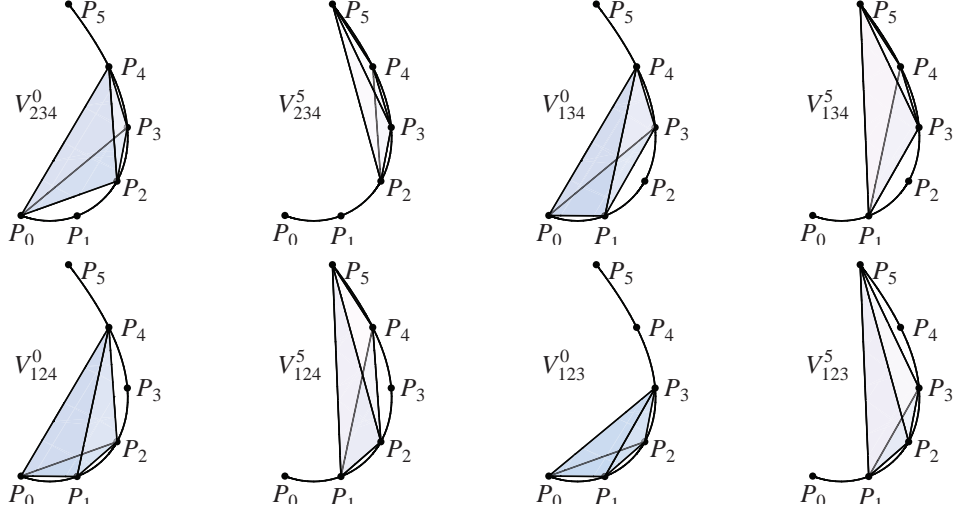


Figure 1: An example of the tetrahedrons that determine volumes  $V_{ijk}^0, V_{ijk}^5$  and the corresponding quotients. The sequence  $\delta$ , obtained in the computational background, is  $(2.83512, 1.1692, 0.556325, 0.212169)$ .

sampled from an analytic parametric curve  $\mathbf{f} : [a, b] \rightarrow \mathbb{R}^3$  at parameter values

$$\eta_0 < \eta_1 < \dots < \eta_5, \quad \eta_\ell \in [a, b], \quad \boldsymbol{\eta} := (\eta_\ell)_{\ell=0}^5.$$

Then the volumes  $V_{ijk}^s$  could also be written as

$$V_{ijk}^s = (-1)^s \mathcal{V}(\eta_s, \eta_i, \eta_j, \eta_k) \det([\eta_s, \eta_i] \mathbf{f}, [\eta_s, \eta_i, \eta_j] \mathbf{f}, [\eta_s, \eta_i, \eta_j, \eta_k] \mathbf{f}), \quad (28)$$

where

$$\mathcal{V}(u_1, u_2, u_3, u_4) = \prod_{j>i} (u_j - u_i) \quad (29)$$

denotes the Vandermonde determinant, based upon values  $u_1, u_2, u_3, u_4$ . The  $\det$  factor in (28) could be viewed as a (unnormalised) discrete torsion, depending on the parameter values involved. For  $\ell = 1, 2, 3, 4$ ,  $(i, j, k) = (r)_{r=1, r \neq \ell}^4$ , this gives

$$\delta_{\ell-1} = c_\eta \frac{\eta_5 - \eta_\ell}{\eta_\ell - \eta_0} q_\ell, \quad (30)$$

where

$$q_\ell := \frac{\det([\eta_0, \eta_i] \mathbf{f}, [\eta_0, \eta_i, \eta_j] \mathbf{f}, [\eta_0, \eta_i, \eta_j, \eta_k] \mathbf{f})}{\det([\eta_i, \eta_5] \mathbf{f}, [\eta_i, \eta_j, \eta_5] \mathbf{f}, [\eta_i, \eta_j, \eta_k, \eta_5] \mathbf{f})}, \quad (31)$$

and the constant

$$c_\eta := \prod_{i=1}^4 \frac{\eta_i - \eta_0}{\eta_5 - \eta_i} \quad (32)$$

does not depend on  $\ell$ . The second factor in (30) clearly decreases with  $\ell$ , so the Lagrange interpolation problem will have an admissible solution if the discrete torsion does not change sign, and it varies with parameters rather slightly. This condition is obviously satisfied if the curvature and the torsion of the curve  $\mathbf{f}$  do not change the sign, and the parameters  $\eta_\ell$  are close enough.

Though in practice one would use the Bézier form of the interpolant (4), a closed Lagrange form of the interpolant could sometimes be useful too. Since  $t_\ell$  should be distinct, it is right at hand. Let us introduce cubic polynomials

$$\psi_i(t) := \prod_{\substack{j=1 \\ j \neq i}}^4 (t - t_j), \quad i = 1, 2, 3, 4.$$

If we rewrite  $\psi_i$  in the Bernstein basis, we obtain

$$\psi_i = \psi_i(0)B_0^3 + \left( \psi_i(0) + \frac{1}{3}\dot{\psi}_i(0) \right) B_1^3 + \left( \psi_i(1) - \frac{1}{3}\dot{\psi}_i(1) \right) B_2^3 + \psi_i(1)B_3^3.$$

But

$$\psi_i(t_j) = \delta_{i,j}\psi_i(t_i), \quad i, j = 1, 2, 3, 4.$$

So the rational functions

$$\begin{aligned} \mathcal{L}_i := \mathcal{L}_{i,t} := & \frac{\pi(t_i)}{\psi_i(t_i)} \left( \psi_i(0)r_0 + \frac{1}{w_1} \left( \psi_i(0) + \frac{1}{3}\dot{\psi}_i(0) \right) r_1 \right. \\ & \left. + \frac{1}{w_2} \left( \psi_i(1) - \frac{1}{3}\dot{\psi}_i(1) \right) r_2 + \psi_i(1)r_3 \right), \quad i = 1, 2, 3, 4, \end{aligned} \quad (33)$$

with  $r_k : [0, 1] \rightarrow \mathbb{R}$ ,  $k = 0, 1, 2, 3$ , defined in (4), satisfy  $\mathcal{L}_i(t_j) = \delta_{i,j}$ ,  $i, j = 1, 2, 3, 4$ . Also, from (6),

$$\mathbf{P}_0 = \sum_{i=1}^4 \left( -\frac{\dot{\omega}_0(t_0)}{\dot{\omega}_0(t_i)} \frac{\pi(t_i)}{\psi_i(t_i)} \right) \mathbf{P}_i = \sum_{i=1}^4 \mathbf{P}_i \mathcal{L}_i(t_0),$$

and similarly at  $t_5$ . Thus, if the interpolant  $\mathbf{r}$  exists, it admits a closed form representation

$$\mathbf{r} = \sum_{i=1}^4 \mathbf{P}_i \mathcal{L}_i.$$

Of course, the particular choice of parameters  $t_1, t_2, t_3, t_4$  could be replaced by any subset of four distinct values from (2). Also, the rational cubic geometric interpolation scheme reproduces cubic rational Bézier curves with nonvanishing both the curvature and the torsion uniquely. This implies the identity

$$\sum_{i=1}^4 \frac{p(t_i)}{\pi(t_i)} \mathcal{L}_i = \frac{p}{\pi},$$

where  $p$  denotes any polynomial of degree  $\leq 3$ .

Cubic rational Bézier curves  $\mathbf{r} : [0, 1] \rightarrow \mathbb{R}^3$  may be unbounded if the denominator  $\pi$  vanishes in  $(0, 1)$ . But for an interpolant that interpolates six data points one would usually try to avoid this situation in practical applications. Since the weights  $w_i$  could be derived in a closed form, the following corollary elaborates this possibility further.

**Corollary 3.** *Suppose that the assumptions  $D_{123} \neq 0$  and (23) of Corollary 2 are satisfied, and the weights  $w_1, w_2$  are determined by (22), (25), (26), and (27). The denominator  $\pi$  is positive on  $(0, 1)$  if  $w_1 \geq 0$  and  $w_2 \geq 0$ . If at least one of the weights  $w_i$  is negative then  $\pi$  remains positive iff*

$$4w_1^3 - 3w_1^2w_2^2 - 6w_1w_2 + 4w_2^3 + 1 > 0.$$

PROOF. Note that due to the convex hull property  $\pi(t) > 0$ ,  $t \in [0, 1]$ , for all weights  $(w_1, w_2)$  in the first quadrant

$$\mathbb{R}_+^2 := \{(w_1, w_2) \mid w_1 \geq 0, w_2 \geq 0\} \subset \mathbb{R}^2.$$

Let  $\mathcal{D} \supset \mathbb{R}_+^2$  be the largest connected open set that determines the weights for which  $\pi$  does not vanish in  $[0, 1]$ . Since  $\pi(0) = \pi(1) = 1$ ,  $\pi$  must have a double zero at each point  $(w_1, w_2)$  of the boundary of  $\overline{\mathcal{D}}$ . The first polynomial of the Gröbner basis of the polynomials  $\{\pi, \dot{\pi}\}$  reads

$$g(w_1, w_2) := 4w_1^3 - 3w_1^2w_2^2 - 6w_1w_2 + 4w_2^3 + 1.$$

Thus double zeroes of  $\pi$  must lie on the variety  $g(w_1, w_2) = 0$ . There is precisely one branch of this variety in  $\mathbb{R}^2 \setminus \mathbb{R}_+^2$  (see Figure 2).

In order to observe this note that  $g(-1/3, -1/3) = 0$ . For each  $w_2 \in \mathbb{R}$ ,  $w_2 \geq -1/3$ , there is one sign change in the coefficient sequence

$$(-4, -3w_2^2, +6w_2, 4w_2^3 + 1),$$

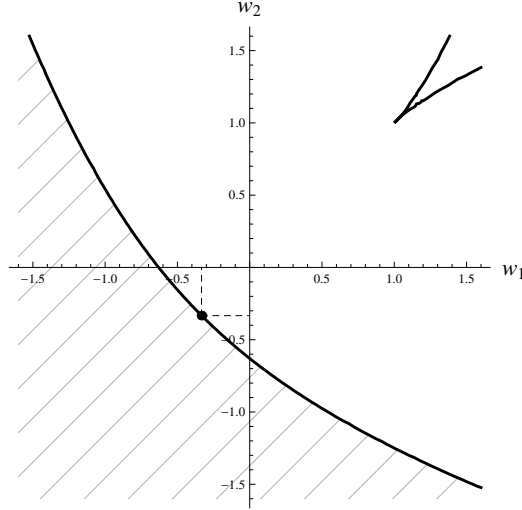


Figure 2: The variety  $g(w_1, w_2) = 0$  (black curves) and the region where  $\mathbf{r}$  has poles (filled region).

so by the Descartes' rule of signs there exists precisely one  $w_1 = w_1(w_2) < 0$  such that  $g(w_1(w_2), w_2) = 0$ . If we interchange the role of  $w_1$  and  $w_2$ , we obtain the other half of the branch concerned. Since  $g(0, 0) = 1 > 0$ , the polynomial  $\pi$  stays positive on  $(0, 1)$  if  $g(w_1, w_2) > 0, (w_1, w_2) \in \mathbb{R}^2 \setminus \mathbb{R}_+^2$ . On the other side, if  $g(w_1, w_2) \leq 0, (w_1, w_2) \in \mathbb{R}^2 \setminus \mathbb{R}_+^2$ , (21) implies that the denominator  $\pi$  has a unique minimum at  $t^* \in (0, 1)$ ,  $\dot{\pi}(t^*) = 0$ . The quadratic equation yields

$$t^* := t^*(w_1, w_2) := \begin{cases} \frac{1}{3} + \frac{1}{3} \frac{1-w_2}{1-w_1 + \sqrt{(1-w_2)^2 + (1-w_1)(w_2-w_1)}}, & w_1 < w_2, \\ \frac{1}{2}, & w_1 = w_2, \\ \frac{2}{3} - \frac{1}{3} \frac{1-w_1}{1-w_2 + \sqrt{(1-w_1)^2 + (1-w_2)(w_1-w_2)}}, & w_1 > w_2. \end{cases}$$

Since

$$\frac{\partial \pi(t^*)}{\partial w_1} = B_1^3(t^*) + \dot{\pi}(t^*) \frac{\partial t^*}{\partial w_1} = B_1^3(t^*) > 0,$$

for any  $(w_1, w_2) \in \mathbb{R}^2 \setminus \mathbb{R}_+^2$  such that  $g(w_1, w_2) < 0$  the denominator  $\pi$  strictly increases with  $w_1$  to the value 0 at  $g(w_1, w_2) = 0$ . The proof is complete.  $\square$

#### 4. Asymptotic analysis

In this section, we show that the assumptions of Corollary 2 are fulfilled for quite a significant class of data. Some expansions involved have been obtained by using a computer algebra system.

**Theorem 4.** *If the data  $(\mathbf{P}_\ell)_{\ell=0}^5$  are sampled from an analytic curve  $\mathbf{f} : [a, b] \rightarrow \mathbb{R}^3$  with nonvanishing both the curvature as well as the torsion, and the parameter interval  $[a, b]$  is small enough, then there exists a unique geometric Lagrange cubic rational Bézier interpolant.*

PROOF. Without loss of generality we may assume that  $\mathbf{f}$  is parameterized by the arc-length  $s \in [0, h]$ , with data points determined by

$$\mathbf{P}_\ell = \mathbf{f}(h\eta_\ell), \quad \ell = 0, 1, \dots, 5, \quad (34)$$

where

$$0 = \eta_0 < \eta_1 < \dots < \eta_4 < \eta_5 = 1, \quad \boldsymbol{\eta} := (\eta_\ell)_{\ell=0}^5.$$

We may also additionally assume

$$\mathbf{f}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{F}\mathbf{f}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (35)$$

where  $\mathcal{F}\mathbf{f}$  denotes the Frenet frame of the curve  $\mathbf{f}$ . Let the curvature  $\kappa$  and the torsion  $\tau$  of  $\mathbf{f}$  expand as

$$\kappa(s) = \kappa_0 + \kappa_1 \frac{s}{1!} + \kappa_2 \frac{s^2}{2!} + \dots, \quad \tau(s) = \tau_0 + \tau_1 \frac{s}{1!} + \tau_2 \frac{s^2}{2!} + \dots, \quad s \in [0, h],$$

with  $\kappa_0, \tau_0 \neq 0$  by the assumption. The Frenet-Serret formulas, with the choice (35), give the curve expansion that shortens to

$$\mathbf{f}(s) = \begin{pmatrix} s - \frac{1}{6}\kappa_0^2 s^3 - \frac{1}{8}\kappa_0 \kappa_1 s^4 + \mathcal{O}(s^5) \\ \frac{1}{2}\kappa_0 s^2 + \frac{1}{6}\kappa_1 s^3 + \frac{1}{24}(\kappa_2 - \kappa_0(\kappa_0^2 + \tau_0^2))s^4 + \mathcal{O}(s^5) \\ \frac{1}{6}\kappa_0 \tau_0 s^3 + \frac{1}{24}(2\kappa_1 \tau_0 + \kappa_0 \tau_1)s^4 + \mathcal{O}(s^5) \end{pmatrix}.$$

From here, it is straightforward to determine by (34) the data and the data difference expansions. The key determinant expands as

$$D_{123} = \frac{1}{12} \mathcal{V}(\eta_1, \eta_2, \eta_3, \eta_4) \kappa_0^2 \tau_0 h^6 + \mathcal{O}(h^7),$$

and it obviously does not vanish for  $h$  small enough since  $\mathcal{V}(\eta_1, \eta_2, \eta_3, \eta_4) > 0$  (see (29)). Further,  $q_\ell$  defined in (31) expands as

$$q_\ell = 1 - \frac{2\kappa_1\tau_0 + \kappa_0\tau_1}{4\kappa_0\tau_0}h + \mathcal{O}(h^2)$$

independently of  $\ell$ . So (30) gives the quotients

$$\delta_{i-1} = c_\eta \frac{1 - \eta_i}{\eta_i} \left( 1 - \frac{2\kappa_1\tau_0 + \kappa_0\tau_1}{4\kappa_0\tau_0}h \right) + \mathcal{O}(h^2)$$

that clearly satisfy (20) for  $h$  small enough. The proof is completed.  $\square$

The analysis of the asymptotic approximation order requires a rather precise expansion of the scalar unknowns. The parameters  $t_i$  expand as

$$t_i = \eta_i + (1 - \eta_i)\eta_i(c_{i,1}h + c_{i,2}h^2 + c_{i,3}h^3) + \mathcal{O}(h^4), \quad i = 1, 2, 3, 4. \quad (36)$$

Here,  $c_{i,k}$  are expressions that depend on the data constants only. In particular,  $c_{i,k}$  involves coefficients  $\kappa_j, \tau_j, j = 0, 1, \dots, k$ . The terms

$$c_{i,1} = -\frac{1}{6} \left( \frac{\kappa_1}{\kappa_0} + \frac{\tau_1}{2\tau_0} \right)$$

are independent of  $\eta_\ell$ , the terms  $c_{i,2}$  are linear functions of  $\eta_i$ ,

$$c_{i,2} = \frac{1}{360} \left( \frac{2\kappa_1\tau_1}{\kappa_0\tau_0} - \frac{18\kappa_2}{\kappa_0} + 6\kappa_0^2 + \frac{20\kappa_1^2}{\kappa_0^2} + \frac{6\tau_0^4 - 6\tau_2\tau_0 + 5\tau_1^2}{\tau_0^2} \right) - \frac{1}{180} \left( \frac{2\kappa_1\tau_1}{\kappa_0\tau_0} + \frac{12\kappa_2}{\kappa_0} + 6\kappa_0^2 - \frac{10\kappa_1^2}{\kappa_0^2} + \frac{6\tau_0^4 + 9\tau_2\tau_0 - 10\tau_1^2}{\tau_0^2} \right) \eta_i,$$

with coefficients that do not depend on  $i$ , and  $c_{i,3}$  are polynomials of total degree 2 in constants  $\eta_\ell$ . This illuminates the fact, verified for  $k = 3$  directly, that the following divided differences of the terms  $c_{i,k}$  vanish,

$$[\eta_j, \dots, \eta_{j+k}](c_{i,k})_{i=j}^{k+j} := \sum_{i=j}^{k+j} \frac{c_{i,k}}{\prod_{\substack{\ell=j \\ \ell \neq i}}^{k+j} (\eta_i - \eta_\ell)} = 0, \quad j = 1, 2, \dots, 4-k; \quad k = 1, 2, 3. \quad (37)$$

At last, we obtain expansions of the weights  $w_i$  from (22) as

$$w_i = 1 - d_2 h^2 + \mathcal{O}(h^3), \quad i = 1, 2, \quad (38)$$

with

$$d_2 := \frac{36\kappa_0^4\tau_0^2 + \kappa_0^2(36\tau_0^4 + 24\tau_2\tau_0 - 35\tau_1^2) + 4\kappa_0\tau_0(3\kappa_2\tau_0 - 2\kappa_1\tau_1) - 20\kappa_1^2\tau_0^2}{720\kappa_0^2\tau_0^2}.$$

The expansions obtained allow us to prove the following theorem. But first of all, let us recall the parametric distance (see e.g., [17]) as a measure of distance between parametric curves  $\mathbf{f} : [a, b] \rightarrow \mathbb{R}^d$  and  $\mathbf{g} : [c, d] \rightarrow \mathbb{R}^d$ , defined as

$$\text{dist}_P(\mathbf{f}, \mathbf{g}) := \inf_{\varphi} \max_{a \leq t \leq b} \|\mathbf{f}(t) - \mathbf{g}(\varphi(t))\|,$$

where the infimum is taken among all diffeomorphisms  $\varphi : [a, b] \rightarrow [c, d]$ , and  $\|\cdot\|$  is the usual Euclidean norm.

**Theorem 5.** *Suppose that the interpolation data*

$$\mathbf{P}_\ell = \mathbf{f}(h\eta_\ell), \quad \ell = 0, 1, \dots, 5, \quad 0 = \eta_0 < \eta_1 < \dots < \eta_4 < \eta_5 = 1,$$

*are sampled from an analytic curve  $\mathbf{f} : [0, h] \rightarrow \mathbb{R}^3$  with nonvanishing curvature and torsion. The asymptotic parametric approximation order is optimal, i.e., 6.*

PROOF. Suppose that  $h$  is small enough, so that the assertions of Theorem 4 hold, and let  $\mathbf{r}_h = \frac{1}{\pi_h}\mathbf{p}_h$  denote the rational cubic interpolant, obtained for a particular  $h$ . Let us recall the triangle inequality

$$\text{dist}_P(\mathbf{f}, \mathbf{r}_h) \leq \text{dist}_P(\mathbf{f}, \mathbf{q}) + \text{dist}_P(\mathbf{q}, \mathbf{r}_h). \quad (39)$$

A good guess of the inserted curve  $\mathbf{q}$  could make it easier to bound two right-hand terms than the distance  $\text{dist}_P(\mathbf{f}, \mathbf{r}_h)$  directly. We choose  $\mathbf{q} = \mathbf{q}_h$  as a polynomial curve of degree  $\leq 5$  determined by the interpolation conditions

$$\mathbf{q}_h(t_\ell) = \mathbf{P}_\ell, \quad \ell = 0, 1, \dots, 5.$$

This way we may use the Newton error remainder form to bound each of the terms involved. In order to bound the first one we reparameterize  $\mathbf{f}$  by an interpolating polynomial  $\varphi$  of degree  $\leq 5$ , determined by the conditions

$$\varphi(t_\ell) = h\eta_\ell, \quad \ell = 0, 1, \dots, 5.$$

The expansions (36) show that  $t_\ell$  are separated for  $h$  small enough, and  $\varphi$  is well defined. Let us recall  $t_0 = \eta_0 = 0$ ,  $t_5 = \eta_5 = 1$ , and (36). If we expand entries of the divided difference table for the data  $(t_i, h\eta_i)_{i=0}^5$ , the relations (37) imply

$$[t_i, t_{i+1}] \varphi = h + \mathcal{O}(h^2), \quad i = 0, 1, \dots, 4,$$

and

$$[t_i, t_{i+1}, \dots, t_{i+j}] \varphi = \mathcal{O}(h^j), \quad i = 0, 1, \dots, 5 - j; j = 2, 3, \dots, 5.$$

Thus

$$\varphi'(t) = h + \mathcal{O}(h^2), \quad \varphi^{(j)}(t) = \mathcal{O}(h^j), \quad j = 2, 3, \dots, 5, \quad (40)$$

and, since  $\varphi$  is a particular regular reparameterization, we obtain the bound

$$\text{dist}_P(\mathbf{f}, \mathbf{q}_h) \leq \max_{0 \leq t \leq 1} \|\mathbf{f}(\varphi(t)) - \mathbf{q}_h(t)\|.$$

But the polynomial curve  $\mathbf{q}_h$  agrees with  $\mathbf{f} \circ \varphi$  at six parameter values  $t_\ell, \ell = 0, 1, \dots, 5$ . So the interpolation error is

$$\mathbf{f}(\varphi(t)) - \mathbf{q}_h(t) = (t - t_0)(t - t_1) \cdots (t - t_5) [t_0, t_1, \dots, t_5, t] (\mathbf{f} \circ \varphi) = \mathcal{O}(h^6).$$

The last equality follows from the chain rule applied to  $\frac{d^6}{dt^6} \mathbf{f}(\varphi(t))$ , and (40), which proves more generally

$$\frac{d^r}{dt^r} \mathbf{f}(\varphi(t)) = \mathcal{O}(h^r), \quad r = 1, 2, \dots, 6. \quad (41)$$

The second term in (39) is bounded from above by

$$\text{dist}_P(\mathbf{q}_h, \mathbf{r}_h) \leq \max_{0 \leq t \leq 1} \|\mathbf{q}_h(t) - \mathbf{r}_h(t)\|,$$

and

$$\mathbf{q}_h(t) - \mathbf{r}_h(t) = (t - t_0)(t - t_1) \cdots (t - t_5) [t_0, t_1, \dots, t_5, t] \mathbf{r}_h,$$

since  $\mathbf{q}_h$  interpolates  $\mathbf{r}_h$  at  $t_\ell, \ell = 0, 1, \dots, 5$  too. But  $\pi_h \mathbf{r}_h$  is a polynomial curve of degree  $\leq 3$ , and the Leibniz rule reveals

$$\begin{aligned} \mathbf{0} &= [t_0, t_1, \dots, t_5, t] (\pi_h \mathbf{r}_h) = \\ &= \pi_h(t) [t_0, t_1, \dots, t_5, t] \mathbf{r}_h + \sum_{i=1}^3 [t_0, t_1, \dots, t_{i-1}, t] \pi_h [t_{i-1}, t_i, \dots, t_5] \mathbf{r}_h. \end{aligned}$$



From the expansion (38), the closed derivatives form and the convex hull property of Bézier curves it is straightforward to conclude

$$\pi_h(t) = 1 + \mathcal{O}(h^2), \quad \pi'_h(t) = \mathcal{O}(h^2), \quad \pi_h^{(i)}(t) = \mathcal{O}(h^i), \quad i = 2, 3. \quad (42)$$

Since  $\mathbf{r}_h$  interpolates  $\mathbf{f} \circ \varphi$  at  $t_\ell$ , (41) and (42) imply

$$\begin{aligned} & (1 + \mathcal{O}(h^2)) [t_0, t_1, \dots, t_5, t] \mathbf{r}_h = \\ & - \sum_{i=1}^3 \underbrace{[t_0, t_1, \dots, t_{i-1}, t] \pi_h}_{\mathcal{O}(h^i)} \underbrace{[t_{i-1}, t_i, \dots, t_5] (\mathbf{f} \circ \varphi)}_{\mathcal{O}(h^{6-i})} = \mathcal{O}(h^6). \end{aligned}$$

This concludes the proof.  $\square$

## 5. Numerical example: subdivision

In this section, we illustrate the cubic rational Lagrange geometric interpolation at six points by using it as a basic step of a geometric subdivision scheme. The example is entirely numerical. All the basic theoretical questions that accompany each subdivision scheme, such as the correctness and the convergence, the smoothness of the limit curve etc., are yet to be answered.

Suppose that an initial data points sequence  $(\mathbf{P}_\ell^0)$  is such that a rational cubic interpolant  $\mathbf{r}_\ell^0$  that interpolates the points  $\mathbf{P}_{\ell+i}^0$ ,  $i = -2, -1, \dots, 3$ , exists for all  $\ell$ . The subdivision is defined by the rules

$$\begin{aligned} \mathbf{P}_{2\ell}^{k+1} &= \mathbf{P}_\ell^k, \\ \mathbf{P}_{2\ell+1}^{k+1} &= \sum_{i=1}^4 \mathcal{L}_{i, \mathbf{t}_\ell^k}((t_{\ell,2}^k + t_{\ell,3}^k)/2) \mathbf{P}_{\ell+i-2}^k = \mathbf{r}_\ell^k((t_{\ell,2}^k + t_{\ell,3}^k)/2), \end{aligned}$$

where

$$\mathbf{t}_\ell^k := (t_{\ell,j}^k)_{j=0}^5, \quad t_{\ell,0}^k = 0 < t_{\ell,1}^k < \dots < t_{\ell,5}^k = 1,$$

are the parameter values at which the rational cubic curve  $\mathbf{r}_\ell^k$  interpolates the points

$$\mathbf{P}_{\ell-2}^k, \mathbf{P}_{\ell-1}^k, \dots, \mathbf{P}_{\ell+3}^k,$$

and  $\mathcal{L}_{i, \mathbf{t}_\ell^k}$  are the corresponding Lagrange basis functions, determined by (33).

Let us consider the curve  $\mathbf{f} : [0, 10] \rightarrow \mathbb{R}^3$ , defined as follows

$$\mathbf{f}(t) := \frac{1}{\sqrt{5}} \begin{pmatrix} \sqrt{5} \ln(t+1) \cos t \\ \sqrt{t^2+1} + 2 \ln(t+1) \sin t \\ 2\sqrt{t^2+1} - \ln(t+1) \sin t \end{pmatrix}. \quad (43)$$

Let the boundary data points be given by

$$\mathbf{P}_\ell^0 = \mathbf{f} \left( \frac{\ell}{100} \right), \quad \mathbf{P}_{m_0-1-\ell}^0 = \mathbf{f} \left( 10 - \frac{\ell}{100} \right), \quad \ell = 0, 1, 2, \quad (44)$$

with  $m_0 = 20$ , and let the rest of the initial data points  $(\mathbf{P}_\ell^0)_{\ell=3}^{m_0-4}$  be sampled from  $\mathbf{f}$  in two different ways, equidistantly and randomly, just to investigate the influence of the initial data distribution. In both cases, the subdivision carries through, and we obtain sequences of points

$$(\mathbf{P}_\ell^k)_{\ell=0}^{m_k-1}, \quad k = 0, 1, \dots, \quad m_k = 15 \cdot 2^k + 5. \quad (45)$$

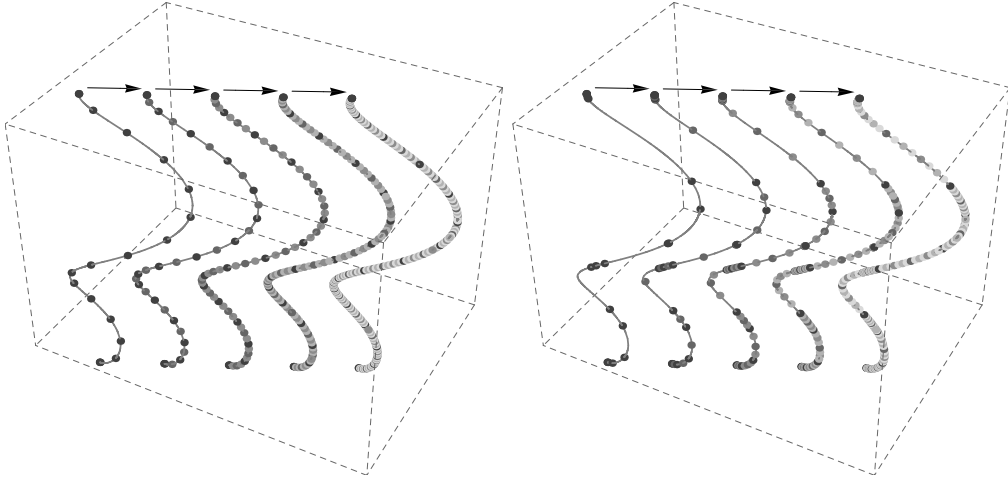


Figure 3: A superposition of the first five subdivision steps  $(\mathbf{P}_\ell^k)_{\ell=0}^{m_k-1}$ ,  $k = 0, 1, \dots, 4$ , starting with the equidistant data distribution (5 curves on the left), and a random one (5 curves on the right).

Superpositions of the resulting first five steps are shown in Fig 3. In order to exploit the numerical evidence of (45) further, we consider the sequence

$$\mathcal{E}^k, \quad k = 0, 1, \dots, 8,$$

of polygons, based upon the points  $(\mathbf{P}_\ell^k)_{\ell=0}^{m_k-1}$ . The limit curve would be continuous if  $(\mathcal{E}^k)_{k \geq 0}$  forms componentwise a Cauchy sequence in the uniform norm. To observe this numerically, we compute a particular parametric

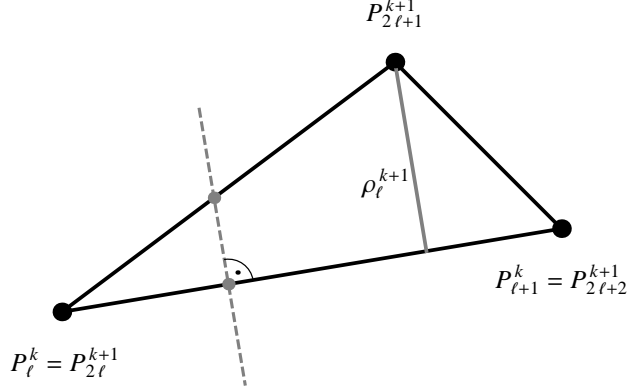


Figure 4: Regular parameterization of the segment of  $\mathcal{E}^{k+1}$  by the orthogonal projection to the line segment  $\mathbf{P}_\ell^k \mathbf{P}_{\ell+1}^k$ .

distance (see Figure 4). The segment of  $\mathcal{E}^{k+1}$ , determined by the points  $\mathbf{P}_{2\ell}^{k+1}, \mathbf{P}_{2\ell+1}^{k+1}, \mathbf{P}_{2\ell+2}^{k+1}$ , can be regularly parameterized by the orthogonal projection onto the line segment  $\mathbf{P}_\ell^k \mathbf{P}_{\ell+1}^k$  to the point

$$u\mathbf{P}_\ell^k + (1-u)\mathbf{P}_{\ell+1}^k, \quad u \in [0, 1],$$

provided

$$\lambda_\ell^{k+1} := \max \left\{ \frac{\|\mathbf{P}_{2\ell+1}^{k+1} - \mathbf{P}_{2\ell}^{k+1}\|}{\|\mathbf{P}_{\ell+1}^k - \mathbf{P}_\ell^k\|}, \frac{\|\mathbf{P}_{2\ell+2}^{k+1} - \mathbf{P}_{2\ell+1}^{k+1}\|}{\|\mathbf{P}_{\ell+1}^k - \mathbf{P}_\ell^k\|} \right\} < 1.$$

In the latter case, for this particular parametric distance, we obtain for the segment considered

$$\max_{0 \leq u \leq 1} \|\mathcal{E}^{k+1}(u) - \mathcal{E}^k(u)\|_\infty = \rho_\ell^{k+1},$$

where  $\rho_\ell^{k+1}$  denotes the Euclidean length of the altitude in the triangle  $\mathbf{P}_\ell^k \mathbf{P}_{\ell+1}^k \mathbf{P}_{2\ell+1}^{k+1}$  with the base  $\mathbf{P}_\ell^k \mathbf{P}_{\ell+1}^k$  (see Figure 4). Figure 5 clearly indicates that

$$\rho^k := \max_\ell \rho_\ell^k$$

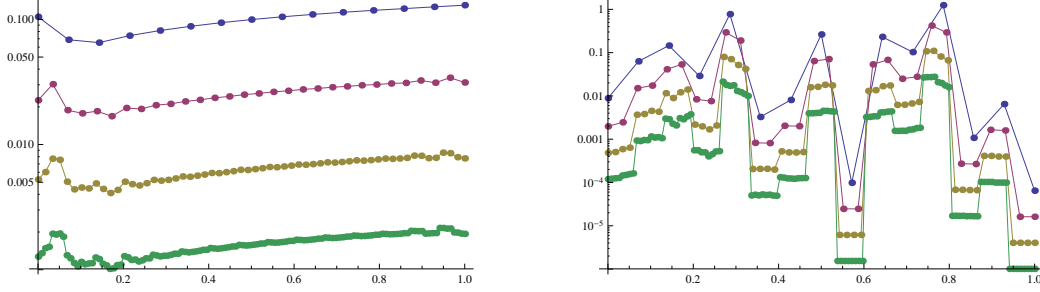


Figure 5: The distances  $(\rho_\ell^k)$ ,  $k = 1, 2, 3, 4$ , for the equidistant data start-up distribution (left), and a random one (right).

$k$	$\lambda^k$	$\rho^k$	$\rho^k / \rho^{k-1}$	$\psi^k$	$\psi^k / \psi^{k-1}$
1	0.600	$1.29810 \times 10^{-1}$		$7.41337 \times 10^{-1}$	
2	0.536	$3.41883 \times 10^{-2}$	0.263	$3.78804 \times 10^{-1}$	0.511
3	0.519	$8.61251 \times 10^{-3}$	0.252	$1.94666 \times 10^{-1}$	0.514
4	0.513	$2.15623 \times 10^{-3}$	0.250	$9.95511 \times 10^{-2}$	0.511
5	0.509	$5.39930 \times 10^{-4}$	0.250	$5.00913 \times 10^{-2}$	0.503
6	0.507	$1.35229 \times 10^{-4}$	0.250	$2.50837 \times 10^{-2}$	0.501
7	0.505	$3.38780 \times 10^{-5}$	0.251	$1.25537 \times 10^{-2}$	0.500
8	0.505	$8.48945 \times 10^{-6}$	0.251	$6.27875 \times 10^{-3}$	0.500

Table 1: Convergence of the subdivision to the limit  $G^1$  curve, with equidistantly sampled starting data.

decreases with growing  $k$ , independently of the starting data distribution.

More precisely, the numerical evidence, given in Tables 1 and 2 clearly supports the conclusion that

$$\lambda^k := \max_\ell \lambda_\ell^k$$

stays bounded well below 1, and the sequence  $(\rho^k)$  decreases by a factor larger than 2. This supports a conjecture that the sequence  $(\mathcal{E}^k)$  converges to a continuous curve. Further, let  $\psi_\ell^k$ ,

$$\psi_\ell^k := \angle(\mathbf{P}_{\ell-1}^k \mathbf{P}_\ell^k, \mathbf{P}_\ell^k \mathbf{P}_{\ell+1}^k), \quad \psi^k := \max_\ell \psi_\ell^k,$$

be an angle between the consecutive line segments of  $\mathcal{E}^k$ . If  $\psi^k \rightarrow 0$  uniformly with growing  $k$ , the limit curve would be  $G^1$  too (see e.g., [18]). Figure 6

$k$	$\lambda^k$	$\rho^k$	$\rho^k/\rho^{k-1}$	$\psi^k$	$\psi^k/\psi^{k-1}$
1	0.614	$1.24849 \times 10^0$		$9.91372 \times 10^{-1}$	
2	0.566	$4.21494 \times 10^{-1}$	0.338	$5.96720 \times 10^{-1}$	0.602
3	0.555	$1.09887 \times 10^{-1}$	0.261	$3.41928 \times 10^{-1}$	0.573
4	0.550	$2.76940 \times 10^{-2}$	0.252	$1.76209 \times 10^{-1}$	0.515
5	0.547	$6.95455 \times 10^{-3}$	0.251	$8.69146 \times 10^{-2}$	0.493
6	0.550	$1.76050 \times 10^{-3}$	0.253	$4.36649 \times 10^{-2}$	0.502
7	0.559	$4.47925 \times 10^{-4}$	0.254	$2.18677 \times 10^{-2}$	0.501
8	0.572	$1.44841 \times 10^{-4}$	0.323	$1.13530 \times 10^{-2}$	0.519

Table 2: Convergence of the subdivision to the limit  $G^1$  curve, with randomly sampled starting data.

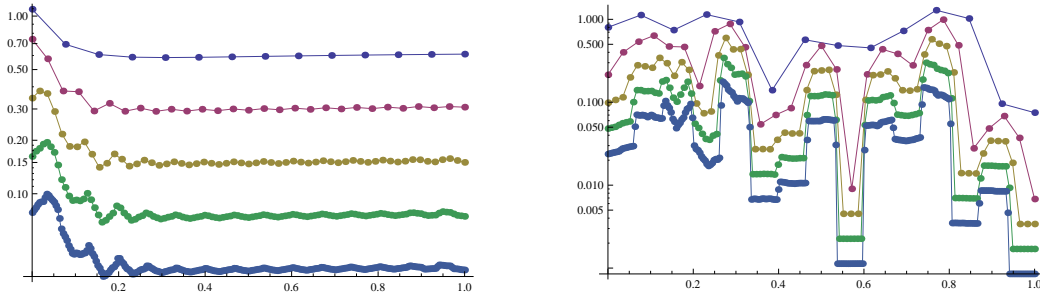


Figure 6: The angles  $(\psi_\ell^k)$ ,  $k = 0, 1, \dots, 4$ , for the equidistant data start-up distribution (left), and the random one (right).

and Tables 1 and 2 give support to this conjecture too. One is tempted to conjecture even  $G^2$  smoothness though the polynomial counterparts (see e.g., [19]) are discouraging. But a brief numerical evidence shows that this is not to be expected in general.

Let us conclude the paper with a numerical estimate of the asymptotic approximation order of the limit curve. We choose the boundary points by (44), and the other points are distributed equidistantly, starting with

$$m_0 = 20, 40, \dots, 120,$$

initial points. For each  $m_0$ , we estimate the Hausdorff distance between the curve  $\mathbf{f}$  and the polygon obtained after three subdivision steps. Table 3 clearly indicates that the approximation order is 6.

$m_0$	$\text{dist}_H(\mathbf{f}, \mathcal{E}^3)$	decay
20	$6.69477 \times 10^{-3}$	
40	$9.47456 \times 10^{-5}$	-6.143
60	$9.01141 \times 10^{-6}$	-5.802
80	$1.25521 \times 10^{-6}$	-6.852
100	$3.08374 \times 10^{-7}$	-6.291
120	$1.04793 \times 10^{-7}$	-5.920

Table 3: The approximation order after three subdivision steps, for the number of initial points 20, 40,  $\dots$ , 120.

## Appendix A.

In this section the proof of the geometric result on existence and uniqueness of a solution of the geometric interpolation problem for the unordered data, provided by the referee, is given. The result is well-known in algebraic and projective geometry, but it seems that it is hard to be found in the literature, so we include it for completeness.

**Theorem 6.** *Let  $\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_5$  be given spatial points. If no quadruple of points is coplanar, there exists a unique geometric rational cubic curve, which interpolates given data.*

PROOF. Take one of the given points in the projective space, say  $\mathbf{P}_5$ , and project all the others from  $\mathbf{P}_5$  onto a plane  $\epsilon$  (not passing through  $\mathbf{P}_5$ ). Denote the image points by  $\overline{\mathbf{P}}_0, \overline{\mathbf{P}}_2, \overline{\mathbf{P}}_3, \overline{\mathbf{P}}_4$ . The assumption implies that no triple of these points are collinear. So there is a unique non-degenerate quadratic conic  $\mathcal{C}_1$  passing through them (i.e., an interpolating conic). Obviously, there is a unique quadratic cone  $\mathcal{K}_1$  with the vertex  $\mathbf{P}_5$ , passing through that conic  $\mathcal{C}_1$ . By construction, all the six given points lie on it.

Now interchange the roles of  $\mathbf{P}_5$  and another of the given points, say  $\mathbf{P}_4$ , getting a second quadratic cone  $\mathcal{K}_2$  which also contains all the six given points. The intersection of these cones (being an algebraic variety of fourth order) splits into the common generator  $\mathbf{P}_4\mathbf{P}_5$  and a spatial rational cubic. (There is only one projective type of them, called “normal cubic” and denoted by  $\mathcal{C}_3$ .) Since every such cubic passes through the vertex of any quadratic cone on which it is lying, it passes also through  $\mathbf{P}_4$  and  $\mathbf{P}_5$ , hence through all the six given points.

Uniqueness is implied by construction since the variety of all chords (including tangents) of  $\mathcal{C}_3$  is a one-parameter family of quadratic cones (like  $\mathcal{K}_1$  and  $\mathcal{K}_2$ ) with vertices running along this  $\mathcal{C}_3$  and the common intersection of all these cones is just the  $\mathcal{C}_3$  itself. This completes the proof.  $\square$

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