High order parametric polynomial approximation of quadrics in $\mathbb{R}^d$

Gašper Jaklič[a,b,c], Jernej Kozak[a,b], Marjeta Krajnc[a,b], Vito Vitrih[c,d,*], Emil Žagar[a,b]

[a] *FMF, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia*
[b] *IMFM, Jadranska 19, Ljubljana, Slovenia*
[c] *PINT, University of Primorska, Muzejski trg 2, Koper, Slovenia*
[d] *FAMNIT, University of Primorska, Glagoljaška 8, Koper, Slovenia*

Abstract

In this paper an approximation of implicitly defined quadrics in $\mathbb{R}^d$ by parametric polynomial hypersurfaces is considered. The construction of the approximants provides the polynomial hypersurface in a closed form, and it is based on the minimization of the error term arising from the implicit equation of a quadric. It is shown that this approach also minimizes the normal distance between the quadric and the polynomial hypersurface. Furthermore, the asymptotic analysis confirms that the distance decreases at least exponentially as the polynomial degree grows. Numerical experiments for spatial quadrics illustrate the obtained theoretical results.

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1. Introduction

Implicitly defined hypersurfaces are important objects in mathematical analysis and in the areas such as computer aided geometric design (CAGD). Among them, the ones defined by algebraic implicit equations are the most widely used. From the computational point of view it is often convenient that

*Corresponding author.
Email address: vito.vitrih@upr.si (Vito Vitrih)

the degree of the implicit equation is small, but the corresponding objects should still provide enough shape flexibility. This makes quadratic implicit equations in $\mathbb{R}^d$, which define hypersurfaces known as $(d-1)$-dimensional quadrics, first to be considered. For their use in fitting, blending, offsetting, and intersection problems see e.g. [1–9].

However, implicit representation is not suitable to deal with all problems encountered. As it turns out, the parametric representations of objects are often more appropriate. For quadrics, it is well known that they can be globally parameterized by trigonometric or hyperbolic functions, and even by rational quadratics (e.g. [10], [11]). Unfortunately, they do not admit a polynomial parametric representation in general. Since the polynomial representation is often required in practical applications, it is reasonable to replace the exact parameterization by an approximate one, but to keep the polynomial form. Several authors considered this problem (e.g. [12–15]). However, most of these results are obtained for some special types of quadrics of a particular dimension.

This paper provides a high order approximation scheme for all types of quadrics in any dimension. The closed form parametric polynomial approximants are derived in such a way that the implicit equation of a quadric is satisfied approximately. Furthermore, it is proven that the distance between the obtained polynomial hypersurface and the quadric decreases at least exponentially as the polynomial degree grows.

As a motivation, let us look at the following example. Consider the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$, which can be parameterized as

$$\begin{aligned} x_1 &= \cos\varphi_1 \, \cos\varphi_2, \\ x_2 &= \sin\varphi_1 \, \cos\varphi_2, \qquad \varphi_1 \in [-\pi, \pi], \quad \varphi_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \\ x_3 &= \sin\varphi_2, \end{aligned} \tag{1}$$

A straightforward approach to a polynomial approximation would take a Taylor expansion of sine and cosine up to the degree n. For $n = 5$ we obtain a parametric polynomial approximant

$$\begin{aligned} r_1(v_1, v_2) &= c_5(v_1)\, c_5(v_2), \\ r_2(v_1, v_2) &= s_5(v_1)\, c_5(v_2), \qquad (v_1, v_2) \in [-3, 3] \times [-1.59, 1.59]\,, \\ r_3(v_1, v_2) &= s_5(v_2), \end{aligned} \tag{2}$$

where

$$c_5(v) = 1 - \frac{v^2}{2} + \frac{v^4}{24}, \quad s_5(v) = v - \frac{v^3}{6} + \frac{v^5}{120}.$$

The error term in the implicit sphere representation equals

$$r_1^2(v_1,v_2)+r_2^2(v_1,v_2)+r_3^2(v_1,v_2)-1=\varepsilon(v_1,v_2),\ \varepsilon(v_1,v_2)=\frac{1}{360}\left(v_1^6+v_2^6\right)+\dots,$$

with the maximum value 0.71. But Fig. 1 shows that this approximation is clearly not satisfying. Furthermore, this Taylor approximant does not even

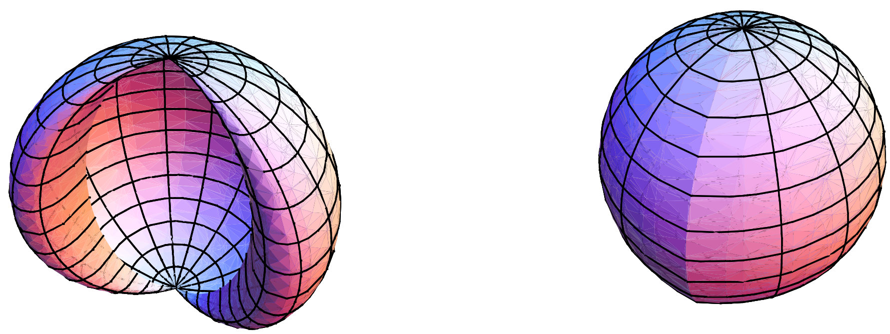

Figure 1: Approximation of the unit sphere by the parametric polynomial surface (2), left, and by the surface (3), right.

yield a closed surface. Naturally, we expect a better approximation, if the error term ε could be made smaller. Let us choose

$$\begin{aligned} r_1(u_1,u_2) &= p_5(u_1)\,p_5(u_2), \\ r_2(u_1,u_2) &= q_5(u_1)\,p_5(u_2), \quad (u_1,u_2)\in[-0.846,0.846]\times[-0.47,0.47]\,, \\ r_3(u_1,u_2) &= q_5(u_2), \end{aligned} \tag{3}$$

where

$$\begin{aligned} p_5(u) &= 1-(3+\sqrt{5})u^2+(1+\sqrt{5})u^4, \\ q_5(u) &= (1+\sqrt{5})u-(3+\sqrt{5})u^3+u^5. \end{aligned}$$

The approximating hypersurface is closed and the corresponding error term

$$r_1^2(u_1,u_2)+r_2^2(u_1,u_2)+r_3^2(u_1,u_2)-1=u_1^{10}\,p_5^2(u_2)+u_2^{10}$$

is bounded by 0.19. Fig. 1 confirms that the second approximating polynomial surface does it much better than the Taylor expansion.

The goal of this paper is to show that minimizing the error term in the implicit equation minimizes the normal distance between the surfaces, and to construct parametric polynomials with sufficiently small error terms. It is not surprising that the asymptotic approximation order is $2\,n$, since the approximation can be viewed as a special case of geometric interpolation of surfaces by polynomials in $\mathbb{R}^2$ (see, e.g., [16] and [17]). The approximation order $2\,n$ somehow follows from the very well known conjecture on the approximation order for parametric polynomial approximation [18].

The paper is organized as follows. In Section 2 the normal form of quadrics is presented. The following section provides a general approach to a parametric approximation of implicitly defined hypersurfaces. Section 4 recalls the results obtained in [15] for the approximation of conic sections. The construction of polynomial approximants together with the error analysis is given in Section 5 and Section 6. The paper is concluded by applying the results to spatial quadrics and presenting some numerical examples.

2. Quadrics in a normal form

A $(d-1)$-dimensional quadric is a hypersurface in $\mathbb{R}^d$, defined as the variety of a quadratic polynomial. In particular coordinates $\boldsymbol{x} = (x_i)_{i=1}^d$, a general quadric is defined by an algebraic implicit equation

$$\boldsymbol{x}^T A\,\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{x} + c = \sum_{i,j=1}^{d} a_{i,j}\,x_i\,x_j + \sum_{i=1}^{d} b_i\,x_i + c = 0, \tag{4}$$

where $A = (a_{i,j})_{i,j=1}^d \in \mathbb{R}^{d\times d}$ is a symmetric matrix, $\boldsymbol{b} := (b_i)_{i=1}^d \in \mathbb{R}^d$, and $c \in \mathbb{R}$.

By a suitable change of variables, any quadric can be written in a normal form by choosing coordinate directions as the principal axes of the quadric. More precisely, since the matrix A is symmetric, it can be diagonalized as $A = U\,\Lambda\,U^T$, where U is an orthogonal matrix and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, d$. By introducing new coordinates

$$\boldsymbol{y} := (y_i)_{i=1}^d := U^T\boldsymbol{x} \quad \text{and} \quad \boldsymbol{\beta} := (\beta_i)_{i=1}^d := U^T\boldsymbol{b},$$

the equation (4) simplifies to a normal form

$$\sum_{i=1}^{d} \lambda_i \, y_i^2 + \sum_{i=1}^{d} \beta_i \, y_i + c = 0. \tag{5}$$

It is easy to see that quadrics with at least one zero eigenvalue have an exact polynomial parameterization (elliptic paraboloid, hyperbolic paraboloid, etc.) or the problem of polynomial approximation reduces to lower dimensional quadrics (cylinder, etc.). Therefore we will from now on assume that all the eigenvalues λ_i are nonzero. Equation (5) can then be simplified to

$$\sum_{i=1}^{d} \lambda_i \left(y_i + \frac{\beta_i}{2\lambda_i} \right)^2 = \sum_{i=1}^{d} \frac{\beta_i^2}{4\lambda_i^2} - c. \tag{6}$$

After a translation, rotation, scaling and permutation of variables, (6) further simplifies to

$$\sum_{i=1}^{K} x_i^2 - \sum_{i=K+1}^{d} x_i^2 = \sigma, \quad K \in \{1, 2, \ldots, d\}, \quad \sigma \in \{0, 1\}. \tag{7}$$

Quite clearly, the coordinates $\boldsymbol{x}$ in (7) differ from those introduced in (4), but for the sake of simplicity we keep the same notation.

3. Parametric approximation of implicit hypersurfaces

3.1. General hypersurfaces

Let $\boldsymbol{x} = (x_i)_{i=1}^d \in \mathbb{R}^d$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function. Suppose that the implicit equation

$$f(\boldsymbol{x}) = 0 \tag{8}$$

defines a smooth regular hypersurface S,

$$S = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \ f(\boldsymbol{x}) = 0 \right\}.$$

Further, let

$$\boldsymbol{r} = (r_i)_{i=1}^d : \Delta \subset \mathbb{R}^{d-1} \to \mathbb{R}^d, \quad \boldsymbol{u} := (u_i)_{i=1}^{d-1} \mapsto (r_i(\boldsymbol{u}))_{i=1}^d \,, \tag{9}$$

be a parametric approximation of the hypersurface S that satisfies the implicit equation (8) approximately, i.e.,

$$f(\boldsymbol{r}(\boldsymbol{u})) = \varepsilon(\boldsymbol{u}), \quad \boldsymbol{u} \in \Delta \subset \mathbb{R}^{d-1}. \tag{10}$$

For ε small enough we expect that the hypersurface and the polynomial approximation are "close together". To be more precise, let

$$\mathcal{T} := \{\boldsymbol{r}(\boldsymbol{u}) : \ \boldsymbol{u} \in \Delta\}$$

denote the approximating hypersurface defined by (9), and let $\mathcal{S} \subset S$ be a part of the hypersurface S that is approximated by $\mathcal{T}$. The distance between $\mathcal{S}$ and $\mathcal{T}$ can be measured by a well known Hausdorff distance. Since it is computationally too expensive, one can use the *normal distance* as its upper bound.

For each point $\boldsymbol{x} \in \mathcal{S}$ the normal distance is defined as

$$\rho(\boldsymbol{x}) := \|\boldsymbol{r}(\boldsymbol{u}) - \boldsymbol{x}\|_2,$$

where the parameter $\boldsymbol{u} \in \Delta$ is determined in such a way that $\boldsymbol{r}(\boldsymbol{u})$ is the intersection point of $\mathcal{T}$ and the normal of $\mathcal{S}$ at a particular point $\boldsymbol{x}$ (see Fig. 2).

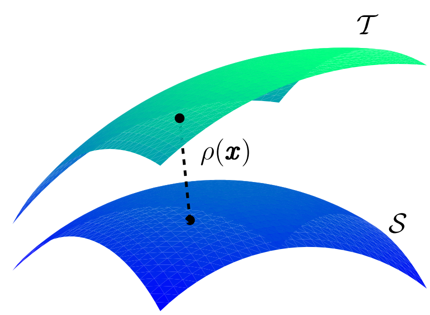

Figure 2: The normal distance $\rho(\boldsymbol{x})$.

The equations that determine $\boldsymbol{u}$ are given as

$$\nabla f(\boldsymbol{x}) \wedge (\boldsymbol{r}(\boldsymbol{u}) - \boldsymbol{x}) = \boldsymbol{0}, \tag{11}$$

where $\nabla f := (f_{x_i})_{i=1}^d$ is the gradient, $f_{x_i} = \frac{\partial f}{\partial x_i}$, and $\wedge$ denotes the wedge product. Note that for $d = 3$, the wedge product is the well-known cross product. The equations (11) can be rewritten as

$$f_{x_i}(\boldsymbol{x})\,(r_k(\boldsymbol{u}) - x_k) = f_{x_k}(\boldsymbol{x})\,(r_i(\boldsymbol{u}) - x_i)\,,\ \ i \neq k,\ \ i, k \in \{1, 2, \ldots, d\}. \quad (12)$$

Moreover, the first order expansion of the equation (10) together with (8) reveal

$$\nabla f(\boldsymbol{x}) \cdot (\boldsymbol{r}(\boldsymbol{u}) - \boldsymbol{x}) = \varepsilon(\boldsymbol{u}) + \delta(\boldsymbol{u}), \quad (13)$$

where $\delta(\boldsymbol{u})$ denotes higher order terms in differences $r_i(\boldsymbol{u}) - x_i$. From (11) and (13) it now follows

$$r_i(\boldsymbol{u}) - x_i = \frac{f_{x_i}(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|_2^2}(\varepsilon(\boldsymbol{u}) + \delta(\boldsymbol{u})), \quad i = 1, 2, \ldots, d,$$

and the normal distance at a point $\boldsymbol{x} \in \mathcal{S}$ simplifies to

$$\rho(\boldsymbol{x}) = \frac{|\varepsilon(\boldsymbol{u}) + \delta(\boldsymbol{u})|}{\|\nabla f(\boldsymbol{x})\|_2}.$$

Quite clearly, the equation (11) might not have a solution $\boldsymbol{u} \in \Delta$, or the solution might not be unique. But if ε is small enough and Δ is such that the map

$$\tau : \mathcal{S} \to \mathcal{T}, \quad \boldsymbol{x} \mapsto \boldsymbol{u},$$

where $\boldsymbol{u}$ is determined by (11), is bijective, then the normal distance is

$$d_N(\mathcal{S}, \mathcal{T}) := \max_{\boldsymbol{x} \in \mathcal{S}} \rho(\boldsymbol{x}).$$

3.2. Quadrics

In this subsection the normal distance between quadrics in a normal form and their parametric approximants is outlined. Since quadrics in a normal form are defined by the particular algebraic equation of order two, the Taylor expansion of (10) is

$$\sum_{i=1}^{d} f_{x_i}(\boldsymbol{x})\ (r_i(\boldsymbol{u}) - x_i) + \frac{1}{2}\sum_{i=1}^{d} f_{x_i\,x_i}(\boldsymbol{x})\,(r_i(\boldsymbol{u}) - x_i)^2 = \varepsilon(\boldsymbol{u}), \quad (14)$$

where $f_{x_i x_i}(\boldsymbol{x}) = \frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{x})$. Suppose that $\nabla f(\boldsymbol{x}) \neq \mathbf{0}$. Then $f_{x_\ell}(\boldsymbol{x}) \neq 0$ for at least one $\ell \in \{1, 2, \ldots, d\}$. From (12) it then follows

$$r_i(\boldsymbol{u}) - x_i = \frac{f_{x_i}(\boldsymbol{x})}{f_{x_\ell}(\boldsymbol{x})}\,(r_\ell(\boldsymbol{u}) - x_\ell)\,, \quad i \neq \ell, \quad i \in \{1, 2, \ldots, d\}, \quad (15)$$

and (14) simplifies to a quadratic equation

$$\|\nabla f(\boldsymbol{x})\|_2^2 \,\frac{r_\ell(\boldsymbol{u}) - x_\ell}{f_{x_\ell}(\boldsymbol{x})} + \frac{1}{2}\sum_{i=1}^{d} f_{x_i\,x_i}(\boldsymbol{x})\, f_{x_i}^2(\boldsymbol{x}) \left(\frac{r_\ell(\boldsymbol{u}) - x_\ell}{f_{x_\ell}(\boldsymbol{x})}\right)^2 - \varepsilon(\boldsymbol{u}) = 0$$

for the difference $r_\ell(\boldsymbol{u}) - x_\ell$ with solutions

$$r_\ell(\boldsymbol{u}) - x_\ell = \frac{2\,\varepsilon(\boldsymbol{u})\, f_{x_\ell}(\boldsymbol{x})}{\|\nabla f\|_2^2 \pm \sqrt{\|\nabla f\|_2^4 + 2\left(\sum_{i=1}^{d} f_{x_i\,x_i}(\boldsymbol{x}) f_{x_i}^2(\boldsymbol{x})\right)\varepsilon(\boldsymbol{u})}}. \tag{16}$$

Since only one solution is needed, it is obvious to choose the one that satisfies $r_\ell(\boldsymbol{u}) - x_\ell \to 0$ when $|\varepsilon(\boldsymbol{u})| \to 0$, as the basis for the reparameterization, i.e., the plus sign. Furthermore, by using (12) the normal distance simplifies to

$$\begin{aligned} \rho(\boldsymbol{x}) &= \left|\frac{r_\ell(\boldsymbol{u}) - x_\ell}{f_{x_\ell}(\boldsymbol{x})}\right| \|\nabla f\|_2 \\ &= \frac{2\,|\varepsilon(\boldsymbol{u})|\, \|\nabla f\|_2}{\|\nabla f\|_2^2 + \sqrt{\|\nabla f\|_2^4 + 2\left(\sum_{i=1}^{d} f_{x_i\,x_i}(\boldsymbol{x}) f_{x_i}^2(\boldsymbol{x})\right)\varepsilon(\boldsymbol{u})}}. \end{aligned} \tag{17}$$

Note that $\nabla f(\boldsymbol{x}) = \boldsymbol{0}$ only for $\boldsymbol{x} = \boldsymbol{0}$ in the equation (7). This singular point will be treated separately.

4. Polynomial approximation of conic sections

In [15], particular high order parametric polynomial approximants for an ellipse and for a hyperbola are derived. In this section some of those results are summarized to be later on applied for an approximation of quadrics.

Let us define

$$\Xi(u) := (-1)^n \prod_{k=0}^{n-1} \left(u\, e^{-\mathrm{i}\,\frac{2k+1}{2n}\pi} - 1\right),$$

and

$$p_{n,+}(u) := \mathrm{Re}(\Xi(u)), \quad q_{n,+}(u) := \mathrm{Im}(\Xi(u)).$$

Then $p_{n,+}$ and $q_{n,+}$ are polynomials of degree $\leq n$ that satisfy

$$p_{n,+}^2(u) + q_{n,+}^2(u) = 1 + u^{2n}, \quad p_{n,+}(0) = 1, \quad q_{n,+}(0) = 0. \tag{18}$$

The explicit formulas for the coefficients of $p_{n,+}$ and $q_{n,+}$ can be found in [15, Thm. 3]. Recall also that the polynomial $p_{n,+}$ is an even and $q_{n,+}$ is an odd function.

Let the unit circle be parameterized as

$$x_1 = \cos\varphi, \quad x_2 = \sin\varphi, \quad \varphi \in \mathbb{R}.$$

From (12) it follows that the normal reparameterization $\varphi \mapsto u = u(\varphi)$ is defined through the solution of

$$\frac{q_{n,+}(u)}{p_{n,+}(u)} = \tan\varphi. \tag{19}$$

In [15, Sec. 6.1], it is shown that the equation (19) is equivalent to $\psi_{n,+}(u) = \varphi$, where

$$\psi_{n,+}(u) := \sum_{k=0}^{n-1} \arctan\left(\frac{u\sin\left(\frac{2k+1}{2n}\pi\right)}{1 - u\cos\left(\frac{2k+1}{2n}\pi\right)}\right).$$

Moreover it is proven that for any $\varphi \in \left[-\frac{n\pi}{4}, \frac{n\pi}{4}\right]$, there exists a unique solution

$$u \in [-1, 1], \quad u = \psi_{n,+}^{-1}(\varphi) =: \phi_{n,+}(\varphi),$$

and the series expansion of the reparameterization, obtained by computer algebra system, is

$$\phi_{n,+}(\varphi) = \omega_n\,\varphi - \frac{\omega_n^3\varphi^3}{9 - 12\,\omega_n^2} + \mathcal{O}\left((\omega_n\varphi)^5\right), \quad \omega_n := \sin\left(\frac{\pi}{2n}\right). \tag{20}$$

Furthermore, the normal distance in the approximation of the whole circle equals

$$\begin{aligned}\frac{\phi_{n,+}^{2n}(\pi)}{1 + \sqrt{1 + \phi_{n,+}^{2n}(\pi)}} &\le \frac{1}{2}\,(\omega_n\pi)^{2n} + \mathcal{O}\left((\omega_n\pi)^{4n}\right)\\ &\sim \frac{1}{2}\left(\frac{\pi^2}{2n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi^2}{2n}\right)^{2n+1}\right).\end{aligned}$$

For the approximation of the unit hyperbola, the polynomials

$$p_{n,-}(u) := p_{n,+}(\mathrm{i}\,u), \quad q_{n,-}(u) := -\,\mathrm{i}\,q_{n,+}(\mathrm{i}\,u),$$

are applied. They are real polynomials of degree $\leq n$ that satisfy

$$p_{n,-}^2(u) - q_{n,-}^2(u) = 1 + (-1)^n u^{2n}, \quad p_{n,-}(0) = 1, \quad q_{n,-}(0) = 0. \tag{21}$$

The coefficients of $p_{n,-}$ and $q_{n,-}$ are nonnegative, the polynomial $p_{n,-}$ is an even and $q_{n,-}$ is an odd function.

In [15, Sec. 6.2], it is shown that for the unit hyperbola, parameterized as

$$x_1 = \cosh\varphi, \quad x_2 = \sinh\varphi, \quad \varphi \in \mathbb{R}, \tag{22}$$

the normal reparameterization $\varphi \mapsto u = u(\varphi)$ is defined through the solution of the equation $\psi_{n,-}(u;\varphi) = 0$, where

$$\psi_{n,-}(u;\varphi) := \sinh\varphi\, p_{n,-}(u) + \cosh\varphi\, q_{n,-}(u) - \sinh(2\varphi), \tag{23}$$

such that $\mathrm{sign}(u) = \mathrm{sign}(\varphi)$. Since $p_{n,-}$ and $\cosh\varphi$ are even, and $q_{n,-}$ and $\sinh\varphi$ are odd functions, it is enough to consider only positive u and φ. From the nonnegativeness of the coefficients of $p_{n,-}$ and $q_{n,-}$ it follows that $\psi_{n,-}(u;\varphi)$ is a strictly increasing function in u. Furthermore,

$$\psi_{n,-}(0;\varphi) = \sinh\varphi - \sinh(2\varphi) < 0, \quad \lim_{u\to\infty} \psi_{n,-}(u;\varphi) = \infty.$$

Thus for any $\varphi \in \mathbb{R}$ there exists a unique solution

$$u = \mathrm{sign}(\varphi)\ \left(\psi_{n,-}^{-1}(0;|\varphi|)\right) =: \phi_{n,-}(\varphi)$$

of (23), which (again by the help of computer algebra system) expands as

$$\phi_{n,-}(\varphi) = \omega_n\,\varphi + \frac{\omega_n^3\varphi^3}{9 - 12\,\omega_n^2} + \mathcal{O}\left((\omega_n\varphi)^5\right) \tag{24}$$

(see [15, Cor. 8]). Note further that $\phi_{n,-}(\varphi) \in (-1,1)$ for any $\varphi \in (-C_n^*, C_n^*)$ where $C_n^* \sim 0.9\,\omega_n^{-1} > \dfrac{n}{2}$ is the solution of $\psi_{n,-}(1;\varphi) = 0$. The normal distance in the approximation of the hyperbola (22) with $|\varphi| < M < C_n^*$ is bounded by

$$\phi_{n,-}^{2n}(M) \sim \left(\frac{\pi\, M}{2n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi\, M}{2n}\right)^{2n+1}\right).$$

5. Polynomial approximation of quadrics in $\mathbb{R}^d$

In this section, a parametric polynomial approximation of quadrics in $\mathbb{R}^d$, defined by the implicit equation (4), and with a small error term ε, is outlined. As explained in Section 2, it is enough to consider the approximants for quadrics in a normal form (7) only.

Polynomials will be constructed by using the conic section's approximants $p_{n,\pm}$ and $q_{n,\pm}$. The general procedure will follow the idea explained in the next simple example. Take a unit sphere in $\mathbb{R}^3$. One of its possible parameterizations is given by (1). Now, by replacing cosines by $p_{n,+}$ and sines by $q_{n,+}$, we obtain a parametric polynomial approximant

$$\begin{aligned} r_1(u_1, u_2) &= p_{n,+}(u_1)\, p_{n,+}(u_2), \\ r_2(u_1, u_2) &= q_{n,+}(u_1)\, p_{n,+}(u_2), \\ r_3(u_1, u_2) &= q_{n,+}(u_2), \end{aligned}$$

where u_1 and u_2 belong to some new domain of interest. For a general quadric in a normal form, a parameterization might involve not only cosines and sines, but also hyperbolic cosines and hyperbolic sines. These functions are then replaced by $p_{n,-}$ and $q_{n,-}$. The aim of this section is to provide a construction of an approximant for a quadric, together with the error term.

As the above example suggests, each component of the approximating polynomial $\boldsymbol{r} = (r_i)_{i=1}^d$ for a general dimension d will be a tensor product of univariate polynomials. Throughout this section the maximal degree of the univariate polynomials involved is fixed to n. To shorten the notation, the parameters are written in the following way

$$\boldsymbol{u}_{j,\ell} := (u_i)_{i=j}^{\ell}, \quad \boldsymbol{u}_\ell := \boldsymbol{u}_{1,\ell}.$$

Consider first an approximation of the hypersphere

$$x_1^2 + x_2^2 + \cdots + x_k^2 = 1, \quad k \geq 2. \tag{25}$$

Our goal is to derive the polynomials $\boldsymbol{w}_k = (w_{k,i})_{i=1}^k : \mathbb{R}^{k-1} \to \mathbb{R}^k$, $\deg(w_{k,i}) \leq n$, that satisfy

$$\sum_{i=1}^k w_{k,i}^2 = 1 + \varepsilon_k \tag{26}$$

for some small error term ε_k. One of the possible solutions is

$$\begin{aligned}
w_{k,1}(\boldsymbol{u}_{\ell,k+\ell-2}) &:= \prod_{j=\ell}^{k+\ell-2} p_{n,+}(u_j), \\
w_{k,i}(\boldsymbol{u}_{\ell,k+\ell-2}) &:= q_{n,+}(u_{i+\ell-2}) \prod_{j=i+\ell-1}^{k+\ell-2} p_{n,+}(u_j), \quad i = 2, 3, \ldots, k-1, \\
w_{k,k}(\boldsymbol{u}_{\ell,k+\ell-2}) &:= q_{n,+}(u_{k+\ell-2}), \qquad w_{1,1} := 1,
\end{aligned} \tag{27}$$

with the error term given in the following lemma.

Lemma 1. *Let $k \geq 2$. If the functions $w_{k,i} : \mathbb{R}^{k-1} \to \mathbb{R}$ are defined by* (27) *and*

$$\boldsymbol{u}_{\ell,k+\ell-2} \in \Delta_k := [-\phi_{n,+}(\pi), \phi_{n,+}(\pi)] \times \left[-\phi_{n,+}\left(\frac{\pi}{2}\right), \phi_{n,+}\left(\frac{\pi}{2}\right)\right]^{k-2}, \tag{28}$$

then $\boldsymbol{w}_k$ satisfies (26) *with the error term*

$$\varepsilon_k(\boldsymbol{u}_{\ell,k+\ell-2}) := \sum_{i=\ell}^{k+\ell-2} u_i^{2n} \prod_{j=i+1}^{k+\ell-2} p_{n,+}^2(u_j) \leq \|\boldsymbol{u}_{\ell,k+\ell-2}\|_{2n}^{2n}. \tag{29}$$

Proof. Recall (18). From (27) it is easy to see that

$$\begin{aligned}
&\varepsilon_k(\boldsymbol{u}_{\ell,k+\ell-2}) = \sum_{i=\ell}^{k+\ell-1} w_{k,i-\ell+1}^2(\boldsymbol{u}_{\ell,k+\ell-2}) - 1 \\
&= \prod_{j=\ell}^{k+\ell-2} p_{n,+}^2(u_j) + \sum_{i=2}^{k} q_{n,+}^2(u_{i+\ell-2}) \prod_{j=i+\ell-1}^{k+\ell-2} p_{n,+}^2(u_j) - 1 \\
&= p_{n,+}^2(u_{k+\ell-2})\left(\varepsilon_{k-1}(\boldsymbol{u}_{\ell,k+\ell-3}) + 1\right) + q_{n,+}^2(u_{k+\ell-2}) - 1 \\
&= \varepsilon_{k-1}(\boldsymbol{u}_{\ell,k+\ell-3})\, p_{n,+}^2(u_{k+\ell-2}) + u_{k+\ell-2}^{2n} \\
&= \varepsilon_{k-2}(\boldsymbol{u}_{\ell,k+\ell-4})\, p_{n,+}^2(u_{k+\ell-3})\, p_{n,+}^2(u_{k+\ell-2}) + u_{k+\ell-3}^{2n}\, p_{n,+}^2(u_{k+\ell-2}) + u_{k+\ell-2}^{2n} \\
&= \cdots = \sum_{i=\ell}^{k+\ell-2} u_i^{2n} \prod_{j=i+1}^{k+\ell-2} p_{n,+}^2(u_j).
\end{aligned}$$

Note that $p_{n,+}(u_i)$ decreases from 1 to 0 as φ_i runs from 0 to $\phi_{n,+}\left(\frac{\pi}{2}\right)$. Therefore $p_{n,+}^2(u_i) \leq 1$ for $u_i \in \left[-\phi_{n,+}\left(\frac{\pi}{2}\right), \phi_{n,+}\left(\frac{\pi}{2}\right)\right]$ and

$$0 \leq \varepsilon_k(\boldsymbol{u}_{\ell,k+\ell-2}) \leq \sum_{i=\ell}^{k+\ell-2} u_i^{2n} = \|\boldsymbol{u}_{\ell,k+\ell-2}\|_{2n}^{2n}. \tag{30}$$

The proof is completed. $\square$

Remark 1. The choice of parameter domain (28) implies that the obtained polynomial approximation defines a closed hypersurface. Furthermore, at the parameter value $\boldsymbol{u} = \boldsymbol{0}$, the point $(1, 0, \ldots, 0)$ is interpolated.

Let us now consider quadrics in a normal form (7). Their polynomial approximation together with the error term is given in the next theorem.

Theorem 1. *Suppose that a quadric has a normal form* (7) *with* $K < d$ *and the polynomial approximant* $\boldsymbol{r} := \boldsymbol{r}_{K,d,\sigma} := (r_i)_{i=1}^d : \mathbb{R}^{d-1} \to \mathbb{R}^d$ *is defined as*

$$r_i(\boldsymbol{u}) := \begin{cases} w_{K,i}\left(\boldsymbol{u}_{K-1}\right) \, v_{1,\sigma}(u_{d-1}), & i = 1, 2, \ldots, K, \\ w_{d-K,i-K}\left(\boldsymbol{u}_{K,d-2}\right) \, v_{2,\sigma}(u_{d-1}), & i = K+1, K+2, \ldots, d, \end{cases} \tag{31}$$

where $\boldsymbol{u} = \boldsymbol{u}_{d-1}$ *and* $v_{\ell,\sigma} : \mathbb{R} \to \mathbb{R}$, $\ell = 1, 2$, *are defined as*

$$\begin{aligned} &v_{1,0}(u) := v_{2,0}(u) := u, \\ &v_{1,1}(u) := p_{n,-}(u), \quad v_{2,1}(u) := q_{n,-}(u). \end{aligned}$$

Let the parameter domain be chosen as

$$\Delta_{K,d} := \Delta_K \times \Delta_{d-K} \times \mathbb{R}, \quad \Delta_1 := \emptyset. \tag{32}$$

Then the polynomial $\boldsymbol{r}$ *satisfies the implicit equation* (7) *approximately with the error term*

$$\begin{aligned} \varepsilon_{K,d}(\boldsymbol{u}) &:= \sigma\,(-1)^n u_{d-1}^{2n} + v_{1,\sigma}^2(u_{d-1})\,\varepsilon_K(\boldsymbol{u}_{K-1}) - v_{2,\sigma}^2(u_{d-1})\,\varepsilon_{d-K}(\boldsymbol{u}_{K,d-2}) \\ &\leq \sigma\,(-1)^n u_{d-1}^{2n} + v_{1,\sigma}^2(u_{d-1})\,\left\|\boldsymbol{u}_{K-1}\right\|_{2n}^{2n}, \end{aligned} \tag{33}$$

where $\varepsilon_1 := 0$.

Proof. From (27) and (7) it follows

$$\begin{aligned} \varepsilon_{K,d}(\boldsymbol{u}) &= v_{1,\sigma}^2(u_{d-1}) \sum_{i=1}^{K} w_{K,i}^2(\boldsymbol{u}_{K-1}) - v_{2,\sigma}^2(u_{d-1}) \sum_{i=K+1}^{d} w_{d-K,i-K}^2(\boldsymbol{u}_{K,d-2}) - \sigma \\ &= v_{1,\sigma}^2(u_{d-1})\,\varepsilon_K(\boldsymbol{u}_{K-1}) - v_{2,\sigma}^2(u_{d-1})\,\varepsilon_{d-K}(\boldsymbol{u}_{K,d-2}) \\ &+ v_{1,\sigma}^2(u_{d-1}) - v_{2,\sigma}^2(u_{d-1}) - \sigma. \end{aligned}$$

The equation (21) implies

$$v_{1,\sigma}^2(u_{d-1}) - v_{2,\sigma}^2(u_{d-1}) - \sigma = \sigma\,(-1)^n u_{d-1}^{2n},$$

and by Lemma 1 the proof is completed. $\square$

Remark 2. For a quadric in a normal form (7) with $K = d$ and $\sigma = 1$, the polynomial approximant is defined as $\boldsymbol{r}_{d,d,1} := \boldsymbol{w}_d$, while for $\sigma = 0$ the quadric reduces to the point $\mathbf{0}$.

6. Normal reparameterization and error analysis

In this section it is shown that the normal distance between a quadric in a normal form and its polynomial approximant is well defined. Furthermore, the upper bound for the normal distance is established and the asymptotic behaviour of the error is outlined.

6.1. Hypersphere

The hypersphere $\mathcal{S}_k$, $k \geq 2$, defined by (25), can be represented in the parametric form as

$$(x_1, x_2, \ldots, x_k) = (h_{k,1}(\boldsymbol{\varphi}), h_{k,2}(\boldsymbol{\varphi}), \ldots, h_{k,k}(\boldsymbol{\varphi})),$$

where

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_{k-1} \in \Omega_k := [-\pi, \pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^{k-2},$$

and

$$\begin{aligned} h_{k,1}(\boldsymbol{\varphi}_{\ell,k+\ell-2}) &:= \prod_{j=\ell}^{k+\ell-2} \cos\varphi_j, \\ h_{k,i}(\boldsymbol{\varphi}_{\ell,k+\ell-2}) &:= \sin\varphi_{i+\ell-2} \prod_{j=i+\ell-1}^{k+\ell-2} \cos\varphi_j, \quad i = 2, 3, \ldots, k-1, \\ h_{k,k}(\boldsymbol{\varphi}_{\ell,k+\ell-2}) &:= \sin\varphi_{k+\ell-2}, \qquad h_{1,1} := 1. \end{aligned} \tag{34}$$

The following theorem gives the upper bound for the normal distance between the hypersphere and its polynomial approximant. Throughout this section we will assume that $d \ll 4^n$.

Theorem 2. *Let the polynomial* $\boldsymbol{w}_k$ *be given by* (27) *and let the degree* $n > 4$*. The polynomial hypersurface*

$$\mathcal{P}_k := \{\boldsymbol{w}_k(\boldsymbol{u}), \quad \boldsymbol{u} \in \Delta_k\}$$

approximates the hypersphere (34) *with the normal distance bounded by*

$$d_N(\mathcal{P}_k, \mathcal{S}_k) = \max_{\boldsymbol{\varphi}\in\Omega_k} \rho(\boldsymbol{\varphi}) \le \frac{1}{2}\left(\frac{\pi^2}{2n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi^2}{2n}\right)^{2n+1}\right).$$

PROOF. Let us first prove that the normal reparameterization of $\boldsymbol{w}_k$, introduced in Section 3, is well defined on Δ_k. Note that for $f(\boldsymbol{x}_{k-1}) = \sum_{i=1}^{k} x_i^2 - 1$, the gradient $\nabla f(\boldsymbol{x}_{k-1}) \neq 0$ for all $\boldsymbol{x}_{k-1} \in \mathcal{S}_k$. The equations

$$x_{i+1}\, \boldsymbol{w}_{k,i}(\boldsymbol{u}_{k-1}) = x_i\, \boldsymbol{w}_{k,i+1}(\boldsymbol{u}_{k-1}), \quad i = 1, 2, \ldots, k-1,$$

that by (12) define the normal reparameterization $\boldsymbol{\varphi}_{k-1} \to \boldsymbol{u}_{k-1}$, are

$$\frac{q_{n,+}(u_1)}{p_{n,+}(u_1)} = \tan\varphi_1, \tag{35}$$

$$\frac{q_{n,+}(u_i)}{p_{n,+}(u_i)} = \tan\varphi_i\, \frac{q_{n,+}(u_{i-1})}{\sin\varphi_{i-1}}, \quad i = 2, 3, \ldots, k-1. \tag{36}$$

Conditions (18) imply that $\mathbf{0} \mapsto \mathbf{0}$. From the analysis of the equation (19) in Section 4 we conclude that for $\varphi_1 \in [-\pi, \pi]$ there exists a unique solution $u_1 = \phi_{n,+}(\varphi_1)$ of the equation (35). Furthermore, $|u_1| < 1$. Let

$$\rho_i := \frac{\varepsilon_i}{1 + \sqrt{1 + \varepsilon_i}}.$$

From (16) and (26) we obtain

$$\frac{q_{n,+}(u_{i-1})}{\sin\varphi_{i-1}} = 1 + \frac{\varepsilon_i(\boldsymbol{u}_{i-1})}{1 + \sqrt{1 + \varepsilon_i(\boldsymbol{u}_{i-1})}} = 1 + \rho_i(\boldsymbol{u}_{i-1}), \tag{37}$$

and from (17), (18) and (33), it follows

$$\rho_1 = 0, \qquad \rho_2(\boldsymbol{u}_1) = \frac{u_1^{2n}}{1 + \sqrt{1 + u_1^{2n}}}.$$

Since $\varphi_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for $i \ge 2$, $\tan\varphi_i$ is a smooth function and

$$\frac{q_{n,+}(u_{i-1})}{\sin\varphi_{i-1}} \tan\varphi_i = \tan\Phi_i,$$

where

$$\Phi_i = \Phi_i(\boldsymbol{\varphi}_i) := \arctan\left(\frac{q_{n,+}(u_{i-1})}{\sin\varphi_{i-1}}\tan\varphi_i\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Again from the analysis of the equation (19) it follows that there exists a unique

$$u_i = u_i(\boldsymbol{\varphi}_i) = \phi_{n,+}\left(\Phi_i\right), \quad i = 2, 3, \ldots, k-1,$$

that solves (36). Expansion (20) and $\omega_n = \dfrac{\pi}{2n} + \mathcal{O}\left(\left(\dfrac{\pi}{2n}\right)^3\right)$ imply

$$\begin{aligned} u_1(\varphi_1) &= \frac{\pi\,\varphi_1}{2n} + \mathcal{O}\left(\left(\frac{\pi\varphi_1}{2n}\right)^3\right), \\ u_i(\boldsymbol{\varphi}_i) &= \frac{\pi\,\Phi_i}{2n} + \mathcal{O}\left(\left(\frac{\pi\,\Phi_i}{2n}\right)^3\right), \quad i = 2, 3, \ldots, k-1. \end{aligned}$$

Let us now derive the upper bound for the normal distance. By (30), the equation (17) simplifies for the hypersphere $\mathcal{S}_k$ to

$$\rho(\boldsymbol{\varphi}_{k-1}) = \frac{\varepsilon_k(\boldsymbol{u}_{k-1})}{1 + \sqrt{1 + \varepsilon_k(\boldsymbol{u}_{k-1})}} \le \frac{1}{2}\,\varepsilon_k(\boldsymbol{u}_{k-1}).$$

Since $\varphi_1 \in [-\pi, \pi]$ and $\Phi_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\begin{aligned} \max_{\boldsymbol{\varphi}_{k-1}\in\Omega_k} \rho(\boldsymbol{\varphi}_{k-1}) &\le \max_{\boldsymbol{\varphi}_{k-1}\in\Omega_k} \frac{1}{2}\sum_{i=1}^{k-1} u_i^{2n} \\ &= \frac{1}{2}\left(\frac{\pi^2}{2n}\right)^{2n} + \frac{k-2}{2}\left(\frac{\pi^2}{4n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi^2}{2n}\right)^{2n+1}\right). \end{aligned}$$

By using the assumption $k \ll 4^n$, the proof is completed. $\square$

6.2. Quadrics with nonzero eigenvalues

The quadric $\mathcal{Q}_{K,d,\sigma}$, defined by (7) with $K < d$, can be represented in the parametric form as

$$x_i(\boldsymbol{\varphi}) := \begin{cases} h_{K,i}\left(\boldsymbol{\varphi}_{K-1}\right)\, g_{1,\sigma}(\varphi_{d-1}), & i = 1, 2, \ldots, K, \\ h_{d-K,i-K}\left(\boldsymbol{\varphi}_{K,d-2}\right)\, g_{2,\sigma}(\varphi_{d-1}), & i = K+1, K+2, \ldots, d, \end{cases} \tag{38}$$

where

$$
\begin{aligned}
&g_{1,0}(\varphi) := g_{2,0}(\varphi) := \varphi, \\
&g_{1,1}(\varphi) := \cosh(\varphi), \quad g_{2,1}(\varphi) := \sinh(\varphi),
\end{aligned}
$$

and

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_{d-1} \in \Omega_{K,d} := \Omega_K \times \Omega_{d-K} \times \mathbb{R}, \qquad \Omega_1 := \emptyset.$$

The upper bound for the normal distance between a quadric (7) and its polynomial approximant (31) is given in the following theorem.

Theorem 3. *Let the polynomial $\boldsymbol{r} = \boldsymbol{r}_{K,d,\sigma}$ be defined by* (31) *and let $n > 4$. The polynomial hypersurface*

$$\mathcal{P} := \mathcal{P}_{K,d,\sigma} := \{\boldsymbol{r}\left(\boldsymbol{u}\right), \quad \boldsymbol{u} \in \Delta_{K,d}\}$$

approximates the quadric $\mathcal{Q} = \mathcal{Q}_{K,d,\sigma}$, given by (38), *and*

$$\boldsymbol{\varphi} \in \Omega_{K,d} \cap \{|\varphi_{d-1}| \le M\}, \quad M \le \frac{n}{2},$$

with the normal distance bounded by

$$d_N(\mathcal{P}, \mathcal{Q}) \le \frac{1}{2\sqrt{2}} M \left(\frac{\pi^2}{2n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi^2}{2n}\right)^{2n+1}\right)$$

for $\sigma = 0$, and by

$$d_N(\mathcal{P}, \mathcal{Q}) \le \left(\frac{\pi}{2n} M\right)^{2n} + \cosh^2(M) \left(\frac{\pi^2}{2n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi(\pi + M)}{2n}\right)^{2n+1}\right)$$

for $\sigma = 1$.

Proof. Let us first prove that the normal reparameterization $\Omega_{K,d} \to \Delta_{K,d}$ is well defined, where $\Delta_{K,d}$ is given by (32). If $f(\boldsymbol{x}) = 0$ is the implicit equation of the quadric $\mathcal{Q}$, then $\nabla f(\boldsymbol{x}) \neq \boldsymbol{0}$ for all $\boldsymbol{x} \in \mathcal{Q}$ except for $\boldsymbol{x} = \boldsymbol{0}$ in the case $\sigma = 0$. But since the point $(\sigma, 0, \ldots, 0)$ is interpolated, the theory of Section 3 can be applied for all points of the quadric.

From the analysis of the hypersphere it follows that for any $\boldsymbol{\varphi}_{d-2} \in \Omega_K \times \Omega_{d-K}$, if $d > 2$, there exists a unique $\boldsymbol{u}_{d-2}$, satisfying equations (15). Further, the equation in (15), that determines u_{d-1}, simplifies to

$$x_d\, r_K + x_K\, r_d - 2\, x_K\, x_d = 0. \tag{39}$$

Suppose first that $\sigma = 0$. In this case it is straightforward to verify that

$$u_{d-1} = \frac{2\,\varphi_{d-1}}{\frac{q_{n,+}(u_{K-1})}{\sin(\varphi_{K-1})} + \frac{q_{n,+}(u_{d-2})}{\sin(\varphi_{d-2})}} \tag{40}$$

is the unique solution of (39). Note that

$$\frac{q_{n,+}(u_{K-1})}{\sin(\varphi_{K-1})} = 1 + \rho_K(\boldsymbol{u}_{1,K-1}) \leq 1 + \frac{1}{2}\|\boldsymbol{u}_{K-1}\|_{2n}^{2n}, \tag{41}$$

and

$$\frac{q_{n,+}(u_{d-2})}{\sin(\varphi_{d-2})} = 1 + \rho_{d-K}(\boldsymbol{u}_{K,d-2}) \leq 1 + \frac{1}{2}\|\boldsymbol{u}_{K,d-2}\|_{2n}^{2n}, \tag{42}$$

which follows from (29) and (37). This implies $u_{d-1} = \varphi_{d-1} + \mathcal{O}\left(\|\boldsymbol{u}_{d-2}\|_{2n}^{2n}\right)$. The case $\sigma = 1$ is more complicated. The equation (39) is equivalent to

$$\begin{aligned}\sinh(\varphi_{d-1})\, p_{n,-}(u_{d-1}) \left(\frac{q_{n,-}(u_{K-1})}{\sin(\varphi_{K-1})}\right) + \cosh(\varphi_{d-1})\, q_{n,-}(u_{d-1}) \left(\frac{q_{n,-}(u_{d-2})}{\sin(\varphi_{d-2})}\right) \\ = \sinh(2\varphi_{d-1}).\end{aligned} \tag{43}$$

By (41) and (42), it simplifies to

$$\psi_{n,-}(u_{d-1};\varphi_{d-1}) = \widetilde{\psi}_n(u_{d-1};\varphi_{d-1}), \tag{44}$$

where

$$\begin{aligned}&\widetilde{\psi}_n(u_{d-1};\varphi_{d-1}) := \\ &-\left(\sinh(\varphi_{d-1})\, p_{n,-}(u_{d-1})\rho_K(\boldsymbol{u}_{1,K-1}) + \cosh(\varphi_{d-1})\, q_{n,-}(u_{d-1})\rho_{d-K}(\boldsymbol{u}_{K,d-2})\right).\end{aligned}$$

Without loss of generality we can assume that $\varphi_{d-1}, u_{d-1} \geq 0$. Since ρ_K, ρ_{d-K} are nonnegative and $p_{n,-}$, $q_{n,-}$ are monotonically increasing, $\psi_{n,-}(u_{d-1};\varphi_{d-1})$ is an increasing and $\widetilde{\psi}_n(u_{d-1};\varphi_{d-1})$ is a decreasing function. Furthermore, inequality $\rho_K(\boldsymbol{u}_{1,K-1}) \leq 1$ implies

$$\begin{aligned}&\widetilde{\psi}_n(0;\varphi_{d-1}) - \psi_{n,-}(0;\varphi_{d-1}) \\ &= \sinh(\varphi_{d-1})\left(2\cosh(\varphi_{d-1}) - 1 - \rho_K(\boldsymbol{u}_{1,K-1})\right) \geq 0,\end{aligned}$$

which proves that for any φ_{d-1} there exists a unique solution u_{d-1} of the equation (44). Thus the normal reparameterization is well defined, i.e., for every $\boldsymbol{\varphi} \in \Omega_{K,d}$ there exists a unique $\boldsymbol{u} \in \Delta_{K,d}$.

The normal distance (17) for a quadric (7) simplifies to

$$\rho(\boldsymbol{\varphi}) = \frac{|\varepsilon_{K,d}(\boldsymbol{u})| \|\boldsymbol{x}\|}{\|\boldsymbol{x}\|^2 + \sqrt{\|\boldsymbol{x}\|^4 + \sigma \varepsilon_{K,d}(\boldsymbol{u})}} \leq \frac{|\varepsilon_{K,d}(\boldsymbol{u})|}{(2\|\boldsymbol{x}\|)^{1-\sigma}}.$$

The last inequality holds since $\|\boldsymbol{x}\| \geq 1$ for $\sigma = 1$. Suppose first that $\sigma = 0$. From (38) it follows that $\|\boldsymbol{x}\| = \sqrt{2}\varphi_{d-1}$, and further by (40) we obtain

$$\begin{aligned} \frac{|\varepsilon_{K,d}(\boldsymbol{u})|}{2\|\boldsymbol{x}\|} &= \frac{|\varepsilon_{K,d}(\boldsymbol{u})|}{2\sqrt{2}\varphi_{d-1}} = \frac{1}{2\sqrt{2}\,\varphi_{d-1}} \left| u_{d-1}^2 \, \varepsilon_K(\boldsymbol{u}_{K-1}) - u_{d-1}^2 \, \varepsilon_{d-K}(\boldsymbol{u}_{K,d-2}) \right| \\ &\leq \frac{u_{d-1}^2}{2\sqrt{2}\varphi_{d-1}} \sum_{i=1}^{d-2} u_i^{2n} \leq \frac{1}{2\sqrt{2}} \, M \left(\frac{\pi^2}{2n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi^2}{2n}\right)^{2n+1}\right). \end{aligned}$$

For $\sigma = 1$, (33) implies

$$\varepsilon_{K,d}(\boldsymbol{u}) \leq (-1)^n u_{d-1}^{2n} + p_{n,-}^2(u_{d-1}) \sum_{i=1}^{K} u_i^{2n}.$$

Note that (43) is a perturbed equation $\psi_{n,-}(u_{d-1}; \varphi_{d-1}) = 0$. From the perturbation theory for polynomial equations it follows

$$|u_{d-1} - \phi_{n,-}(\varphi_{d-1})| \leq \frac{2 \sinh(2\,\varphi_{d-1})}{\frac{d\psi_{n,-}(u; \varphi_{d-1})}{du}\Big|_{u=\phi_{n,-}(\varphi_{d-1})}} \delta + \mathcal{O}\left(\delta^2\right), \tag{45}$$

where $\delta \leq \frac{1}{2}\|\boldsymbol{u}_{d-2}\|_{2n}^{2n}$. From the nonnegativeness of the coefficients of $p_{n,-}$, $q_{n,-}$ and $q_{n,-}(0) = 0$ we obtain

$$\frac{d\psi_{n,-}(u; \varphi)}{du} \geq \cosh\varphi \, q_{n,-}'(u) > \cosh\varphi \, \frac{q_{n,-}(u)}{u} = \frac{1}{2\,u} \sinh(2\,\varphi_{d-1}) \, \frac{q_{n,-}(u)}{\sinh\varphi_{d-1}}.$$

Moreover, from (16), (21), (24) and the assumption $M \leq \frac{n}{2}$, it follows

$$\begin{aligned} &\frac{d\psi_{n,-}(u; \varphi_{d-1})}{du}\Big|_{u=\phi_{n,-}(\varphi_{d-1})} > \\ &\quad \frac{\sinh(2\,\varphi_{d-1})}{2} \left(\frac{1}{\phi_{n,-}(\varphi_{d-1})} - \frac{(-1)^n \phi_{n,-}^{2n-1}(\varphi_{d-1})}{C}\right) > \frac{\sinh(2\,\varphi_{d-1})}{4}, \end{aligned}$$

where $C \geq 1$. By (45) this implies

$$u_{d-1} \leq \phi_{n,-}(\varphi_{d-1}) + 8\,\delta + \mathcal{O}\left(\delta^2\right) = \phi_{n,-}(\varphi_{d-1}) + \mathcal{O}\left(\|\boldsymbol{u}_{d-2}\|_{2n}^{2n}\right).$$

From (16) it then follows

$$p_{n,-}(u_{d-1}) = \cosh(\varphi_{d-1})\left(1 + \mathcal{O}\left(\|\boldsymbol{u}_{d-1}\|_{2n}^{2n}\right)\right).$$

From the expansions (20) and (24) we obtain

$$\|\boldsymbol{u}_{d-2}\|_{2n}^{2n} \leq \left(\frac{\pi^2}{2\,n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi^2}{2\,n}\right)^{2n+1}\right)$$

and

$$\phi_{n,-}^{2n}(\varphi_{d-1}) \leq \left(\frac{\pi\,M}{2\,n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi\,M}{2\,n}\right)^{2n+1}\right),$$

and finally

$$|\varepsilon_{K,d}(\boldsymbol{u})| \leq \left(\frac{\pi}{2n}M\right)^{2n} + \cosh^2(M)\left(\frac{\pi^2}{2n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi(\pi+M)}{2n}\right)^{2n+1}\right),$$

which completes the proof. □

7. Quadrics in $\mathbb{R}^3$

Results from previous sections will now be applied to quadrics in $\mathbb{R}^3$, known also as quadric surfaces. The normal form (7) yields only four different cases shown in Table 1. The remaining ones with at least one nonzero eigenvalue have either an exact polynomial representation (elliptic paraboloid, hyperbolic paraboloid, parabolic cylinder) or their parameterization follows directly from the parameterization of conic sections (elliptic cylinder, hyperbolic cylinder).

For the ellipsoid, hyperboloid of one or two sheets, and a cone the polynomial approximants obtained from (27) and (31) are shown in Table 2. Moreover, Table 3 numerically illustrates how the normal distance decreases to zero with the growing degree n. The polynomial surfaces for $n = 4, 5, 6$ are shown in Fig. 4.

Table 1: Quadric surfaces with no polynomial parameterization.

Ellipsoid	$x_1^2 + x_2^2 + x_3^2 = 1$
Hyperboloid of one sheet	$x_1^2 + x_2^2 - x_3^2 = 1$
Hyperboloid of two sheets	$x_1^2 - x_2^2 - x_3^2 = 1$
Cone	$x_1^2 + x_2^2 - x_3^2 = 0$

Table 2: Polynomial approximants for quadric surfaces given in Table 1.

quadric	polynomial approximant	domain
$x_1^2 + x_2^2 + x_3^2 = 1$	$(p_{n,+}(u_1)p_{n,+}(u_2),\ q_{n,+}(u_1)p_{n,+}(u_2),\ q_{n,+}(u_2))$	Δ_3
$x_1^2 + x_2^2 - x_3^2 = 1$	$(p_{n,+}(u_1)p_{n,-}(u_2),\ q_{n,+}(u_1)p_{n,-}(u_2),\ q_{n,-}(u_2))$	$\Delta_2 \times \mathbb{R}$
$x_1^2 - x_2^2 - x_3^2 = 1$	$(p_{n,-}(u_2),\ p_{n,+}(u_1)q_{n,-}(u_2),\ q_{n,+}(u_1)q_{n,-}(u_2))$	$\Delta_2 \times [0, \infty)$
$x_1^2 + x_2^2 - x_3^2 = 0$	$(p_{n,+}(u_1)u_2,\ q_{n,+}(u_1)u_2,\ u_2)$	$\Delta_2 \times \mathbb{R}$

Furthermore, an interesting phenomena of a sequential approximation, already observed in [15], is presented in Fig. 3. Namely, the polynomial approximant cycles the sphere several times (the number of cycles increases with growing degree n) and all sequential approximations are surprisingly good.

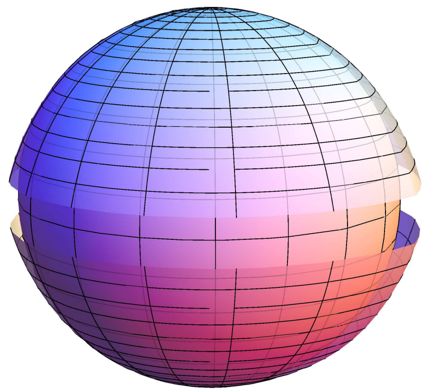

Figure 3: An example of a sequential approximation for degree $n = 5$.

Table 3: The upper bound for the normal distance for the polynomial approximation of the ellipsoid, hyperboloid of one sheet and the cone with $M = \frac{1}{2}$.

n	ellipsoid	hyperboloid	cone
5	0.09430	1.11514	0.15503
6	0.01399	0.12183	0.01694
7	0.00138	0.00952	0.00132
8	0.00009	0.00056	0.00008
9	$4.8 \cdot 10^{-6}$	0.00003	$3.5 \cdot 10^{-6}$
10	$1.9 \cdot 10^{-7}$	$9.3 \cdot 10^{-7}$	$1.3 \cdot 10^{-7}$
11	$6.1 \cdot 10^{-9}$	$2.8 \cdot 10^{-8}$	$3.9 \cdot 10^{-9}$
12	$1.6 \cdot 10^{-10}$	$7.0 \cdot 10^{-10}$	$9.7 \cdot 10^{-11}$
13	$3.5 \cdot 10^{-12}$	$1.5 \cdot 10^{-11}$	$2.0 \cdot 10^{-12}$
14	$6.6 \cdot 10^{-14}$	$2.7 \cdot 10^{-13}$	$3.7 \cdot 10^{-14}$
15	$1.1 \cdot 10^{-15}$	$4.2 \cdot 10^{-15}$	$5.8 \cdot 10^{-16}$

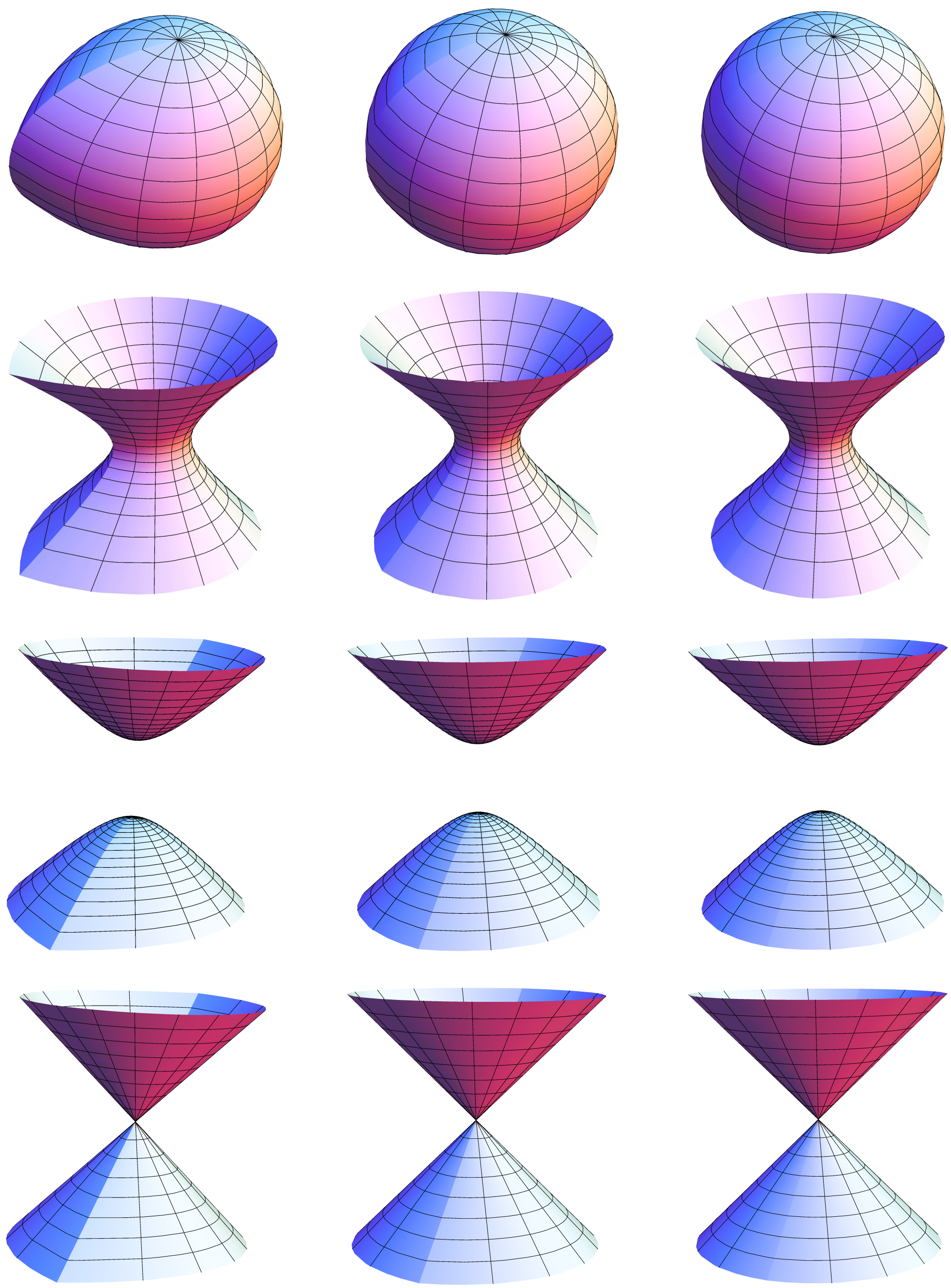

Figure 4: Approximants for quadric surfaces from Table 1 of degrees 4, 5 and 6.

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