Advances in Computational Mathematics manuscript No.
(will be inserted by the editor)

An approach to geometric interpolation by Pythagorean-hodograph curves

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Received: date / Accepted: date

Abstract The problem of geometric interpolation by Pythagorean-hodograph (PH) curves of general degree n is studied independently of the dimension $d \geq 2$. In contrast to classical approaches, where special structures that depend on the dimension are considered (complex numbers, quaternions, etc.), the basic algebraic definition of a PH property together with geometric interpolation conditions is used. The analysis of the resulting system of nonlinear equations exploits techniques such as the cylindrical algebraic decomposition and relies heavily on a computer algebra system. The nonlinear equations are written entirely in terms of geometric data parameters and are independent of the dimension. The analysis of the boundary regions, construction of solutions for particular data and homotopy theory are used to establish the existence and (in some cases) the number of admissible solutions. The general approach is applied to the cubic Hermite and Lagrange type of interpolation. Some known results are extended and numerical examples provided.

Mathematics Subject Classification (2000) 65D05 · 65D17

1 Introduction

Polynomial Pythagorean-hodograph (PH) curves have been widely studied in the last two decades. Since the Euclidean norm of their hodograph is the absolute value of a polynomial, they possess several practically important properties like a rational offset, polynomial arc length, etc. This makes them a useful tool in Computer Aided Geometric Design applications.

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Several approaches to PH curve characterization have been proposed (see, e.g., (Farouki, 2008)). Most of them depend on the dimension of the underlying Euclidean space $\mathbb{R}^d$. Planar PH curves can be characterized through some identities between complex numbers (Farouki, 1994) and for spatial curves perhaps the best known approach to define a hodograph is to use quaternions. An alternative way to define them is through a Hopf map representation (see, e.g., (Farouki, 2008)). A generalization to higher dimensions can be done with the help of Clifford algebra (see, e.g., (Choi et al, 2002)). These particular approaches usually simplify computations by PH curves, but on the other hand they have to be developed for each space dimension separately.

Many approximation schemes that involve PH curves can be found in the literature, most of them are of the Hermite type (Farouki and Neff, 1995; Albrecht and Farouki, 1996; Meek and Walton, 1997; Jüttler, 2001; Farouki et al, 2003; Jaklič et al, 2010a; Kwon, 2010). Much less work has been carried out on Lagrange interpolation problems. In the planar case the only analyzed scheme is given in (Jaklič et al, 2008), while for spatial curves no results for this type of interpolation are available in the literature. This is due to the fact that Lagrange interpolation problems are much more difficult to handle. It is clear that PH curves are important objects in $\mathbb{R}^2$ and in $\mathbb{R}^3$. But several practical applications lead to study curves in higher dimensions too. In motion design, e.g., curves in $\mathbb{R}^4$ are frequently used to construct rational motions.

In this paper, we examine the most basic characterization of a PH curve that follows directly from its definition. A parametric polynomial curve $\boldsymbol{r}_n : [0,1] \to \mathbb{R}^d$ of degree $\leq\ n$ is a PH curve if the Euclidean norm of its hodograph $\|\boldsymbol{r}_n'\| = \sqrt{\boldsymbol{r}_n'^T \boldsymbol{r}_n'}$ is the absolute value of a polynomial of degree $\leq\ n-1$. For regular curves it is actually a polynomial, since the norm of the hodograph is strictly positive. Although the above characterization motivated the study of PH curves in detail, it has usually not been used to solve interpolation or approximation problems involving such curves. Here we use it directly to two most common geometric interpolation problems, the Hermite and the Lagrange case and could be extended to the interpolation of higher order geometric data such as curvature, e.g. Although we use the expression Hermite interpolation, it is formally correct only for the cubic case. For interpolation schemes using higher degree polynomial curves it is actually Lagrange interpolation with additional geometric continuity conditions at the boundary points.

A general idea of the approach outlined in the paper is quite simple. A parametric polynomial curve $\boldsymbol{r}_n : [0,1] \to \mathbb{R}^d$ that interpolates prescribed data can be expressed in the polynomial basis that is dual to the interpolation functionals involved. The basis $\boldsymbol{w}$ depends on particular parameters, yet to be determined. This gives the norm of the hodograph as

$$\|\boldsymbol{r}_n'\| = \sqrt{\boldsymbol{w}'^T G \boldsymbol{w}'},$$

where $G \in \mathbb{R}^{d\times d}$ denotes the corresponding Gram matrix that depends only on the prescribed data. All the geometric properties of the data interpolated are thus encapsulated in the symmetric positive semidefinite matrix G. This also enables us to study the interpolation problem in the original d-dimensional space without any projections on its suitable subspace. Further, the remaining free parameters should be chosen in such a way that the PH condition is fulfilled. However, the equations derived this way do not depend on the dimension of the space that the data belong to. This unifies the analysis of the nonlinear systems obtained, which could be carried out entirely in terms of elements of G. The semidefiniteness of G provides the relations that should be satisfied by the elements of G if the matrix originates from an interpolation problem. It is obvious that such an approach is convenient in practical computations too. Even though, it is clear from (Jaklič et al, 2008) that the existence analysis is quite a challenge even for a low degree n. Our approach is an algebraic one, and relies heavily on the topological Brouwer's degree argument, a method that turned out fruitful in similar problems (Kozak and Žagar, 2004; Kozak and Krajnc, 2007a,b; Jaklič et al, 2008).

The paper is organized as follows. The introductory section is followed by three sections covering the general case. In Section 2, a characterization of PH curves in $\mathbb{R}^d$ is derived as a system of polynomial equations for the coefficients of $\boldsymbol{r}_n'$. The next section defines the Hermite and Lagrange geometric interpolation problems. Section 4 introduces the homotopy method approach to be used in the existence analysis, and proves two rather general assertions that are used in the existence considerations.

In Section 5, the results derived for general degree curves are applied to study the cubic case thoroughly. The first part of the section briefly revisits the Hermite interpolation problem, but from a new point of view. As a result, geometrically intuitive necessary and sufficient conditions for the existence of solutions are obtained. Although this interpolation problem has already been considered in (Wagner and Ravani, 1997; Jüttler and Mäurer, 1999; Pelosi et al, 2005), the results obtained there do not cover all the data configurations where the solutions exist. However, recently in (Kwon, 2010), a complete characterization has been provided and conditions for the existence of the solution were independently derived for all possible data.

The second part of Section 5 is devoted to cubic PH Lagrange interpolation. This problem turned out to be much more difficult than the Hermite one. Though the results obtained are generally comparable with those of the Hermite case, there are also some significant differences: the number of solutions may reach six compared to two Hermite solutions, etc. The paper is concluded with some remarks and numerical examples. We are convinced that the approach suggested could be carried out to some extent for higher degree problems too given a suitable computer power at will.

2 Characterization of Pythagorean hodograph curves in $\mathbb{R}^d$

Let $\boldsymbol{r}_n : [0,1] \to \mathbb{R}^d : t \mapsto \boldsymbol{r}_n(t) = \left(r_{n,i}(t)\right)_{i=1}^d$ be a parametric polynomial curve, where $r_{n,i} \in \mathbb{P}_n$, $i = 1, 2, \ldots, d$. Here $\mathbb{P}_n$ denotes the space of real polynomials of degree $\leq n$. Then $\boldsymbol{r}_n$, $n \geq 1$, is a Pythagorean-hodograph (PH) curve if and only if its hodograph, i.e., $\boldsymbol{r}_n' = \left(r_{n,i}'\right)_{i=1}^d$, satisfies

$$\left\|\boldsymbol{r}_n'(t)\right\| = \sqrt{\boldsymbol{r}_n'(t)^T \boldsymbol{r}_n'(t)} = \sqrt{\sum_{i=1}^d \left(r_{n,i}'(t)\right)^2} = \left|\sigma_{n-1}(t)\right|, \quad t \in [0,1], \tag{1}$$

for some polynomial $\sigma_{n-1} \in \mathbb{P}_{n-1}$. If in addition $\boldsymbol{r}_n$ is a regular curve then we may assume $\sigma_{n-1} > 0$. Throughout the paper, we shall assume the obvious requirement $n \geq 1$. The norm $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$, induced by the inner product $\boldsymbol{u}^T \boldsymbol{v}$ and $\angle(\boldsymbol{u}, \boldsymbol{v})$ stands for the angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

The following theorem reveals the relations between the coefficients of the hodograph that imply the PH property.

Theorem 1 *Suppose that a polynomial curve* $\boldsymbol{r}_n : [0,1] \to \mathbb{R}^d$ *of degree* $\leq n$ *is regular at* $t = 0$*, and let*

$$\left\|\boldsymbol{r}_n'(t)\right\|^2 = \sum_{i=0}^{2n-2} \alpha_i t^i. \tag{2}$$

Then the curve $\boldsymbol{r}_n$ *has a Pythagorean hodograph if and only if the coefficients* α_i *satisfy*

$$\alpha_i = \sum_{j=i+1-n}^{n-1} \beta_j \, \beta_{i-j}, \quad i = n, n+1, \ldots, 2n-2, \tag{3}$$

where all β_i *depend on* $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ *only, and are defined recursively as*

$$\beta_0 := \sqrt{\alpha_0}, \quad \beta_i := \frac{1}{2\beta_0}\left(\alpha_i - \sum_{j=1}^{i-1} \beta_j \, \beta_{i-j}\right), \quad i = 1, 2, \ldots, n-1. \tag{4}$$

Furthermore, $\sigma_{n-1}(t) = \sum_{i=0}^{n-1} \beta_i t^i > 0,\ t \in [0,b] \subseteq [0,1]$*, for some* $0 < b \leq 1$.

Proof The proof follows straightforwardly by comparing the coefficients of (2) and σ_{n-1}^2. □

Remark 1 Suppose that $\boldsymbol{r}_n$ is regular on $[0,b]$. Since the kernel of the k-th divided difference $[\tau_i, \tau_{i+1}, \ldots, \tau_{i+k}]$ is precisely the space of polynomials of degree $\leq k-1$, the equations (3) are equivalent to

$$[\tau_0, \tau_1, \ldots, \tau_j] \left\|\boldsymbol{r}_n'(\cdot)\right\| = 0, \quad j = n, n+1, \ldots, 2n-2, \tag{5}$$

where $0 \leq \tau_0 < \tau_1 < \cdots < \tau_{2n-2} \leq b$ are chosen arbitrarily.

As an example, for $n = 3$ the system (3) (for $\alpha_0 \neq 0$) can be rewritten in an equivalent form as

$$8\alpha_0^2\alpha_3 - 4\alpha_0\alpha_2\alpha_1 + \alpha_1^3 = 0, \quad 64\alpha_0^3\alpha_4 - 5\alpha_1^4 + 24\alpha_0\alpha_2\alpha_1^2 - 32\alpha_0^2\alpha_3\alpha_1 - 16\alpha_0^2\alpha_2^2 = 0, \tag{6}$$

and

$$\sigma_2(t) = \frac{1}{\sqrt{\alpha_0}}\left(\left(\frac{\alpha_2}{2} - \frac{\alpha_1^2}{8\alpha_0}\right)t^2 + \frac{\alpha_1}{2}t + \alpha_0\right). \tag{7}$$

3 Geometric interpolation

Let us now turn our attention to the geometric interpolation, and let us consider the Lagrange case first. Suppose that a sequence of data points

$$\boldsymbol{T}_j \in \mathbb{R}^d, \quad j = 0, 1, \ldots, n, \quad \boldsymbol{T}_j \neq \boldsymbol{T}_{j+1},$$

in the Euclidean space $\mathbb{R}^d$, $d \geq 2$, is given. A parametric PH polynomial curve $\boldsymbol{r}_n : [0,1] \to \mathbb{R}^d$ of degree $\leq n$ has to be found, for which

$$\boldsymbol{r}_n(t_j) = \boldsymbol{T}_j, \quad j = 0, 1, \ldots, n, \tag{8}$$

where

$$t_0 := 0 < t_1 < \cdots < t_{n-1} < t_n := 1. \tag{9}$$

Note that the interval of the parameterization can clearly be chosen as $[0, 1]$, since a linear reparameterization does not affect the interpolation problem considered. However, the parameters $\boldsymbol{t} := \left(t_j\right)_{j=1}^{n-1}$ are left unknown. Regardless of this fact, the closed Lagrange form of the interpolating curve is right at hand, namely

$$\boldsymbol{r}_n = \sum_{j=0}^{n} \boldsymbol{T}_j \, \ell_j,$$

where ℓ_j is the j-th Lagrange basis polynomial with respect to the interpolation parameters (9).

The interpolating polynomial curve has $n-1$ free parameters $\boldsymbol{t}$, which is precisely the number of equations in (3) that $\boldsymbol{r}_n$ has to satisfy to become a PH curve. Let us examine $\boldsymbol{r}_n'^T \boldsymbol{r}_n'$ more thoroughly. Since $\sum_{j=0}^{n} \ell_j \equiv 1$,

$$\boldsymbol{r}_n'(t) = \sum_{j=0}^{n} \boldsymbol{T}_j \ell_j'(t) = \sum_{j=1}^{n} \left(\boldsymbol{T}_j - \boldsymbol{T}_0\right) \ell_j'(t) = \sum_{j=1}^{n} \Delta\boldsymbol{T}_{j-1} w_j'(t), \tag{10}$$

where

$$\Delta\boldsymbol{T}_{j-1} := \boldsymbol{T}_j - \boldsymbol{T}_{j-1}, \quad w_j(t) := \sum_{k=j}^{n} \ell_k(t) = 1 - \sum_{k=0}^{j-1} \ell_k(t), \quad \boldsymbol{w} := \left(w_j\right)_{j=1}^{n}. \tag{11}$$

This gives

$$\boldsymbol{r}_n'(t)^T \boldsymbol{r}_n'(t) = \boldsymbol{w}'(t)^T G \boldsymbol{w}'(t), \tag{12}$$

where G is the Gram data difference matrix, $G := \left(\Delta\boldsymbol{T}_i^T \Delta\boldsymbol{T}_j\right)_{i,j=0}^{n-1}$. It is well known (Boyd and Vandenberghe, 2004; Dattorro, 2009) that G carries all the data information. Let

$$\delta_i := \left\|\Delta\boldsymbol{T}_i\right\|, \quad c_{ij} := \cos\theta_{ij}, \quad \theta_{ij} := \angle\left(\Delta\boldsymbol{T}_i, \Delta\boldsymbol{T}_j\right). \tag{13}$$

The matrix G can be written as

$$G = DCD, \tag{14}$$

where $D = \operatorname{diag}(\delta_0, \delta_1, \ldots, \delta_{n-1})$ and $C = (c_{ij})_{i,j=0}^{n-1}$ is the interpoint angle matrix (Dattorro, 2009).

Positive semidefiniteness of the interpoint angle matrix C implies positive semidefiniteness of the Gram matrix G, which yields the following observation.

Lemma 1 *Let the vectors $\Delta\boldsymbol{T}_i$, $i = 0, 1, \ldots, n-1$, be linearly independent. If the interpolating curve $\boldsymbol{r}_n$ exists, it is regular on any bounded parameter interval.*

Proof The assumption implies that G is actually positive definite. So the norm $\left\|\boldsymbol{r}_n'(t)\right\|$ may by (12) vanish only if $\boldsymbol{w}'(t^*) = \boldsymbol{0}$ at some t^*. However, the relation (11) implies $\ell_i'(t^*) = 0$, $i = 0, 1, \ldots, n$, which is clearly a contradiction. Namely, the identity $t = \sum_{i=0}^{n} t_i \ell_i(t)$, derived at t^*, would yield $1 = 0$. $\square$

A slight modification of the Lagrange interpolation problem gives the G^1 interpolation, i.e., geometric Hermite interpolation. In this case, data points to be interpolated are given as $\boldsymbol{T}_j \in \mathbb{R}^d,\ j = 1, 2, \ldots, n-1$, and additionally, the tangent directions $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$, $\|\boldsymbol{d}_1\| = \|\boldsymbol{d}_2\| = 1$, at $\boldsymbol{T}_1$ and $\boldsymbol{T}_{n-1}$, respectively, should match too. So the parametric PH polynomial curve $\boldsymbol{r}_n : [0,1] \to \mathbb{R}^d$ of degree $\leq n$ should satisfy

$$\begin{aligned} &\boldsymbol{r}_n'(t_1) = \lambda_1 \boldsymbol{d}_1, \quad \boldsymbol{r}_n(t_j) = \boldsymbol{T}_j,\ j = 1, 2, \ldots, n-1, \quad \boldsymbol{r}_n'(t_{n-1}) = \lambda_2 \boldsymbol{d}_2, \\ &t_1 := 0 < t_2 < \cdots < t_{n-2} < t_{n-1} := 1, \quad \lambda_1 > 0, \quad \lambda_2 > 0. \end{aligned} \tag{15}$$

The unknowns here are the parameter values $t_2, t_3, \ldots, t_{n-2}$ as well as the tangent lengths λ_1 and λ_2. Though one, equipped with the knowledge on the Hermite interpolation in the functional case, is tempted to assume that the G^1 interpolation could be understood completely just by following the limit case $\Delta t_0 \to 0$, $\Delta t_{n-1} \to 0$ of the Lagrange case, not all of the properties could be established this way (see, e.g., (Kozak and Krajnc, 2007a)). Nevertheless, a PH characterization of the interpolant can be derived in a similar way. The closed form of the interpolant may be written as

$$\boldsymbol{r}_n = \lambda_1 \boldsymbol{d}_1 h_0 + \sum_{j=1}^{n-1} \boldsymbol{T}_j\, h_j + \lambda_2 \boldsymbol{d}_2,$$

where h_j are Hermite basis polynomials. Therefore, one can express the derivative $\boldsymbol{r}_n'$ similarly to (10) as

$$\boldsymbol{r}_n'(t) = \lambda_1 \boldsymbol{d}_1 h_0'(t) + \sum_{i=2}^{n-1} \Delta\boldsymbol{T}_{i-1} \widetilde{w}_i'(t) + \lambda_2 \boldsymbol{d}_2 h_n'(t), \quad \widetilde{w}_i(t) := \sum_{k=i}^{n-1} h_k(t),\ i = 2, \ldots, n-1.$$

If we define $\widetilde{\boldsymbol{w}} := (\lambda_1 h_0, \widetilde{w}_2, \widetilde{w}_3, \ldots, \widetilde{w}_{n-1}, \lambda_2 h_n)$, and use the notation (13) and (14) with the assumption

$$\Delta\boldsymbol{T}_0 \to \boldsymbol{d}_1,\ \Delta\boldsymbol{T}_{n-1} \to \boldsymbol{d}_2,\ \ \delta_0 \to 1,\ \delta_{n-1} \to 1, \tag{16}$$

the derivative length reveals as $\|\boldsymbol{r}_n'\| = \sqrt{\widetilde{\boldsymbol{w}}'^T G \widetilde{\boldsymbol{w}}'}$, in the form, familiar from (12). Again, all the data are hidden in the Gram matrix G.

4 Homotopy approach

Determining the existence and the uniqueness of a solution of a system of polynomial equations that depends on some data parameters, with unknowns restricted to a given admissible open set $\mathcal{D} \subset \mathbb{R}^d$, is a difficult task. Nevertheless, there are several known general approaches one might choose, and one of them is a homotopy analysis, which turned out as a very efficient tool in many geometric interpolation problems (Kozak and Žagar, 2004; Kozak and Krajnc, 2007a,b). With this approach, the first, but crucial step is to understand precisely which data configurations force the unknowns to approach the boundary of the admissible set $\mathcal{D}$. This way, a part of geometric conditions that may imply the change in the solution variety is revealed. The other, complementary part consists of a set where the kernel of the Jacobian of the system is nontrivial since the real branches of the variety may turn complex and vice versa. The boundary analysis could be quite tedious since the number of different possibilities to be examined increases exponentially with the number of unknowns. In this section some general results for the

Lagrange and Hermite PH interpolation problem are stated that significantly reduce the number of possibilities to be examined.

In the Lagrange case, the admissible parameter set is given as a simplex

$$\mathcal{D} = \left\{ (t_i)_{i=1}^{n-1} \in \mathbb{R}^{n-1} \,|\, t_0 := 0 < t_1 < t_2 < \cdots < t_{n-1} < 1 =: t_n \right\}.$$

A point $\boldsymbol{t} \in \mathcal{D}$ approaches the boundary of $\mathcal{D}$ if the parameter spacing $\Delta t_i := t_{i+1} - t_i$ tends to zero for at least one i. This obviously implies that the interpolatory curve grows unboundedly at least between two points being interpolated.

Lemma 2 *Suppose that the Gram matrix G is nonsingular, and the parameter sequence $\boldsymbol{t}$ approaches the boundary of $\mathcal{D}$ in such a way that for some constant $\varepsilon_0 > 0$, and for each $i \in \{1, 2, \ldots, n\}$ either*

$$\Delta t_{i-1} = \mathcal{O}(\varepsilon) \quad \text{or} \quad \varepsilon = o(\Delta t_{i-1})$$

holds for all ε, $0 < \varepsilon \le \varepsilon_0$. Let

$$\nu_i := \begin{cases} 0, & \varepsilon = o(\Delta t_{i-1}) \\ \max\limits_{0 \le j \le i-1} \{ i - j \,|\, \Delta t_\ell = \mathcal{O}(\varepsilon), j \le \ell \le i-1 \}, & \text{otherwise} \end{cases}, \; i = 1, 2, \ldots, n.$$

If the sequence $(\nu_i)_{i=1}^n$ has a unique maximum, the corresponding choice of parameters $\boldsymbol{t}$ does not determine a regular interpolating curve $\boldsymbol{r}_n$.

Proof Let $\omega(t) := \prod\limits_{k=0}^{n} (t - t_k)$. Then

$$\ell_j(t) = \prod_{\substack{k=0 \\ k \ne j}}^{n} \frac{t - t_k}{t_j - t_k} = \frac{1}{\omega'(t_j)} \left(\frac{\omega(t)}{t - t_j} \right).$$

Let now i be such that $\nu_i > \nu_j$, $j = 1, 2, \ldots, i-1, i+1, \ldots, n$. Then

$$\omega'(t_j) = (t_j - t_0)(t_j - t_1) \cdots (t_j - t_{j-1})(t_j - t_{j+1}) \cdots (t_j - t_n) = \mathcal{O}\left(\varepsilon^{\nu_i}\right) \tag{17}$$

for $i - \nu_i \le j \le i$, and

$$\varepsilon^{\nu_i} = o\left(\omega'(t_j)\right), \quad j < i - \nu_i, \; j > i. \tag{18}$$

Also, for k such that $i - \nu_i \le k \le i$, we observe $t - t_k = t - t_{i-\nu_i} + \mathcal{O}(\varepsilon)$, therefore

$$\begin{aligned} \frac{\omega(t)}{t - t_j} &= \prod_{k=0}^{i-\nu_i-1} (t - t_k) \cdot \prod_{\substack{k=i-\nu_i \\ k \ne j}}^{i} (t - t_k) \cdot \prod_{k=i+1}^{n} (t - t_k) = \\ &= \prod_{k=0}^{i-\nu_i-1} (t - t_k) \cdot (t - t_{i-\nu_i})^{\nu_i} \cdot \prod_{k=i+1}^{n} (t - t_k) + \mathcal{O}(\varepsilon) =: q(t) + \mathcal{O}(\varepsilon). \end{aligned}$$

Let us use these expansions in (11). From (17) and (18) it follows that the Lagrange basis polynomials with indices $i - \nu_i \le j \le i$, are large compared with the others. Thus

$$w_r(t) = \sum_{k=r}^{n} \ell_k(t) = \sum_{\substack{k=r \\ i-\nu_i \le k \le i}}^{n} \ell_k(t) + \mathcal{O}(\varepsilon) = \text{const}\, q(t) + \mathcal{O}(\varepsilon).$$

Since G is nonsingular, from (12) it follows

$$\left\|\boldsymbol{r}_n'(t)\right\|^2 = \boldsymbol{w}'(t)^T G \boldsymbol{w}'(t) = \text{const}\ \left(q'(t)\right)^2 + \mathcal{O}\left(\varepsilon\right).$$

This concludes the proof since q' obviously has $n-1$ zeros in $[0,1]$. □

For the cubic case, there are in general six possibilities that should be considered: $(t_1 \to 0, t_1 < t_2 < 1)$, $(t_1 \to 0, t_2 \to 0)$, $(t_1 \to 0, t_2 \to 1)$, $(0 < t_1 < t_2, t_2 \to 1)$, $(t_1 \to 1, t_2 \to 1)$, $(0 < t_1 \to t_2 < 1)$. However, by the lemma only one is left to be analysed, namely $t_1 \to 0, t_2 \to 1$, where $\nu_1 = 1$, $\nu_2 = 0$, and $\nu_3 = 1$. For the quintic case the number of critical possibilities to be examined increases to eight. The number of all possibilities that have to be considered in general is given in (Jaklič et al, 2010b).

Lemma 3 *The parameter sequence $\boldsymbol{t}$ cannot approach the boundary of $\mathcal{D}$ such that*

$$\Delta t_{i-1} = \mathcal{O}\left(\varepsilon\right),\ \varepsilon \to 0, \quad \Delta t_{j-1} \geq const > 0, \quad j \neq i.$$

Proof Observe that w_i is large compared with the rest of $w_r, r \neq i$. The proof follows then in a way similar to the proof of Lemma 2. □

The admissible parameter domain for the G^1 case changes to

$$\mathcal{D} = \left\{(\lambda_1, \lambda_2) \times (t_i)_{i=2}^{n-2} \,|\, \lambda_1, \lambda_2 > 0,\ t_1 = 0 < t_2 < t_3 < \cdots < t_{n-2} < 1 = t_{n-1}\right\}.$$

The next lemma explains the behaviour of λ_1 and λ_2 at the boundary.

Lemma 4 *Suppose that the parameter sequence satisfies $t_1 = 0 < t_2 < t_3 < \cdots < t_{n-2} < 1 = t_{n-1}$. Any single λ_i, defined in (15), may not grow unboundedly. If they both grow unboundedly, then they can grow only as*

$$0 < const_1 \leq \frac{\lambda_1}{\lambda_2} \leq const_2 < \infty, \tag{19}$$

and then the conditions

$$[\tau_0, \tau_1, \ldots, \tau_i]\sqrt{\lambda_1^2 h_0'^2\left(\cdot\right) + \lambda_2^2 h_n'^2\left(\cdot\right)} = 0, \quad i = n, n+1, \ldots, 2n-3, \tag{20}$$

should be satisfied, where τ_j, $j = 0, 1, \ldots, n-2$, and τ_j, $j = n-1, n, \ldots, 2n-3$, are zeros of h_0' and h_n', respectively. Further, $c_{0,n-1}$ should satisfy the equation

$$[\tau_0, \tau_1, \ldots, \tau_{2n-2}]\sqrt{\lambda_1^2 h_0'^2\left(\cdot\right) + \lambda_2^2 h_n'^2\left(\cdot\right) + 2\lambda_1\lambda_2 h_0'\left(\cdot\right) h_n'\left(\cdot\right) c_{0,n-1}} = 0, \tag{21}$$

where $\tau_{2n-2} \in [0,1]$ is any value, distinct from all the other τ_i.

Proof The first assertion is obvious. Suppose that (19) is satisfied. Then, as $\lambda_i \to \infty$, the main part of $\left\|\boldsymbol{r}_n'\right\|$ simplifies to

$$\left\|\boldsymbol{r}_n'\left(t\right)\right\| = \sqrt{\lambda_1^2 h_0'^2\left(t\right) + \lambda_2^2 h_n'^2\left(t\right) + 2\lambda_1\lambda_2 h_0'\left(t\right) h_n'\left(t\right) c_{0,n-1}} + \mathcal{O}\left(\frac{1}{\lambda_i}\right).$$

From the basis $(h_i)_{i=0}^n$ it is straightforward to conclude that each of the polynomials h_0' and h_n' has $n-1$ distinct zeros in $[0,1]$, but no common one. So we can apply the defining PH equations in the form (5) based upon these zeros and one additional point τ_{2n-2}. The assertion follows. □

5 Cubic case

The general results from previous sections will now be used to analyze the cubic Hermite and Lagrange interpolation problem. As usual, it turns out that the Lagrange problem is much more difficult than the Hermite one. In the Hermite case, not only the conditions for the existence of admissible solutions, but also the explicit formulae can be given. In the Lagrange case the equations are much more complicated, and using homotopy seems the most convenient approach.

By (14), the Gram matrix is of the form

$$G = \begin{pmatrix} \delta_0 & & \\ & \delta_1 & \\ & & \delta_2 \end{pmatrix} \begin{pmatrix} 1 & c_{01} & c_{02} \\ c_{01} & 1 & c_{12} \\ c_{02} & c_{12} & 1 \end{pmatrix} \begin{pmatrix} \delta_0 & & \\ & \delta_1 & \\ & & \delta_2 \end{pmatrix},$$

where in the Hermite case an additional assumption (16) has to be applied.

By the theory on Euclidean distance matrices (Dattorro, 2009), the constants c_{ij} are not independent, namely

$$(c_{02} - c_{01}c_{12})^2 \leq \left(1 - c_{01}^2\right)\left(1 - c_{12}^2\right). \tag{22}$$

The equality is reached if and only if the vectors $\boldsymbol{\Delta T}_0$, $\boldsymbol{\Delta T}_1$, and $\boldsymbol{\Delta T}_2$ are coplanar. Furthermore, the rank of the matrix G drops to 1 only if $|c_{01}| = |c_{12}| = |c_{02}| = 1$, in the case $\delta_i > 0$, for all i. This corresponds to the Lagrange case with data lying on the same line. The rank of G is equal to 1 also if $\delta_1 = 0$ and $|c_{02}| = 1$ (in the Hermite case $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ coincide). Thus the nonplanar data must satisfy $\delta_i > 0$, $i = 0, 1, 2$, and $(c_{02} - c_{01}c_{12})^2 < \left(1 - c_{01}^2\right)\left(1 - c_{12}^2\right)$, which implies $|c_{01}| < 1$, and $|c_{12}| < 1$.

Note that the condition (22) has a clear geometric meaning. It is equivalent to

$$\frac{(c_{01} + c_{12})^2}{2(1 + c_{02})} + \frac{(c_{01} - c_{12})^2}{2(1 - c_{02})} \leq 1, \quad \text{if } c_{02} \in (-1, 1), \tag{23}$$

or $c_{02} = \pm 1,\ c_{01} = \pm c_{12}$. For every fixed $|c_{02}| < 1$, the boundary of the region defined by (22) is an ellipse that reduces to a line segment at $|c_{02}| = 1$.

Remark 2 From the theoretical as well as from the practical point of view it is important to know in advance when the interpolatory PH curve transforms continuously to the planar case solution as the data do. And vice versa. Quite clearly, a true space solution may turn unbounded or irregular if the data continuously reduce to the planar case. In the opposite direction, a planar solution could always be extended to a nearby true space case if the Implicit function theorem could be applied on equations that determine curve parameters.

5.1 Hermite case

The analysis of the cubic Hermite PH interpolation is straightforward, and can be done without a homotopy approach. The interpolating curve is given as

$$\boldsymbol{r}_3(t) = \lambda_1 t(1-t)^2 \boldsymbol{d}_1 + (1-t)^2(2t+1)\boldsymbol{T}_1 + (3-2t)t^2\boldsymbol{T}_2 - \lambda_2 t^2(1-t)\boldsymbol{d}_2,$$

and

$$h_0'(t) = (1-t)(1-3t), \quad h_3'(t) = t(3t-2).$$

The coefficients α_i, defined in Theorem 1, are

$$\alpha_0 = \lambda_1^2, \qquad \alpha_1 = -4\lambda_1 \left(-3\delta_1 c_{01} + c_{02}\lambda_2 + 2\lambda_1\right),$$
$$\alpha_2 = -12\delta_1 \left(5c_{01}\lambda_1 + 2c_{12}\lambda_2\right) + 22\lambda_1 \left(c_{02}\lambda_2 + \lambda_1\right) + 36\delta_1^2 + 4\lambda_2^2,$$
$$\alpha_3 = -12 \left(-\delta_1 \left(7c_{01}\lambda_1 + 5c_{12}\lambda_2\right) + 3c_{02}\lambda_1\lambda_2 + 6\delta_1^2 + 2\lambda_1^2 + \lambda_2^2\right),$$
$$\alpha_4 = 9 \left(-4\delta_1 \left(c_{01}\lambda_1 + c_{12}\lambda_2\right) + 2c_{02}\lambda_1\lambda_2 + 4\delta_1^2 + \lambda_1^2 + \lambda_2^2\right).$$

With α_i expressed this way, the equations (6) turn out as a system of two polynomial equations of total degree 12 for the unknowns λ_1 and λ_2. The first equation of the corresponding Gröbner basis, computed by using the lexicographic monomial ordering with respect to variables (λ_2, λ_1), factors as

$$\delta_1^3\lambda_1^2(1-c_{01}^2)^2 \left(-\left(1-c_{02}\right)\left(1+2c_{02}\right)\lambda_1^2 + 3\delta_1\left(3c_{01} + \left(1-4c_{02}\right)c_{12}\right)\lambda_1 - 18\delta_1^2(1- \right.$$
$$\left. - c_{12}^2)\right)\Big(- \left(1-c_{02}\right)\left(1+2c_{02}\right)\lambda_1^2 + 3\delta_1\left(3c_{01} - \left(4c_{02}+1\right)c_{12}\right)\lambda_1 - 18\delta_1^2(1-c_{12}^2)\Big).$$

The solution $\lambda_1 = 0$ is not admissible, and four possible solutions are left. If we combine these four zeros λ_1 with the rest of the Gröbner basis, we obtain four possible solution pairs. With the help of the functions

$$g_1\left(x,y,w\right) := (1-w)x + \left(\frac{1}{2}+w\right)y,$$
$$g_2\left(x,y,w\right) := g_1(x,y,w)^2 - (1-w)\left(\frac{1}{2}+w\right)\left(1-(x-y)^2\right), \tag{24}$$
$$\zeta^{\pm}\left(x,y,w\right) := \frac{3\delta_1\left(1-(x-y)^2\right)}{g_1(x,y,w) \pm \sqrt{g_2(x,y,w)}},$$

the four solution pairs $\left(\lambda_{1,i}, \lambda_{2,i}\right), \ i = 1,2,3,4$, are simplified to

$$\begin{aligned}
\lambda_{1,1} &= \zeta^{+}\left(\frac{c_{01}+c_{12}}{2}, \frac{c_{01}-c_{12}}{2}, c_{02}\right), \ \lambda_{2,1} = \zeta^{+}\left(\frac{c_{01}+c_{12}}{2}, \frac{c_{12}-c_{01}}{2}, c_{02}\right),\\
\lambda_{1,2} &= \zeta^{-}\left(\frac{c_{01}+c_{12}}{2}, \frac{c_{01}-c_{12}}{2}, c_{02}\right), \ \lambda_{2,2} = \zeta^{-}\left(\frac{c_{01}+c_{12}}{2}, \frac{c_{12}-c_{01}}{2}, c_{02}\right),\\
\lambda_{1,3} &= \zeta^{-}\left(\frac{c_{01}-c_{12}}{2}, \frac{c_{01}+c_{12}}{2}, -c_{02}\right), \ \lambda_{2,3} = \zeta^{+}\left(\frac{c_{12}-c_{01}}{2}, \frac{c_{01}+c_{12}}{2}, -c_{02}\right),\\
\lambda_{1,4} &= \zeta^{+}\left(\frac{c_{01}-c_{12}}{2}, \frac{c_{01}+c_{12}}{2}, -c_{02}\right), \ \lambda_{2,4} = \zeta^{-}\left(\frac{c_{12}-c_{01}}{2}, \frac{c_{01}+c_{12}}{2}, -c_{02}\right).
\end{aligned} \tag{25}$$

Quite clearly, $g_2\left(x,y,c_{02}\right) > g_1^2\left(x,y,c_{02}\right), \ -1 < c_{02} < -\frac{1}{2}, \ |x-y| < 1$. So the first pair $\left(\lambda_{1,1}, \lambda_{2,1}\right)$ is admissible for this c_{02} range, but the second one $\left(\lambda_{1,2}, \lambda_{2,2}\right)$ is not since

$$\zeta^{+}\left(x,y,w\right)\zeta^{-}\left(x,y,w\right) = 18\delta_1^2\frac{1-(x-y)^2}{(1-w)(1+2w)} \tag{26}$$

implies $\lambda_{1,2}, \lambda_{2,2} < 0$. As $c_{02} \uparrow -\frac{1}{2}$, we observe that $\lambda_{1,1}, \lambda_{2,1} \to \infty$ if

$$g_1\left(\frac{c_{01}+c_{12}}{2}, \frac{c_{01}-c_{12}}{2}, -\frac{1}{2}\right) = \frac{3}{4}\left(c_{01}+c_{12}\right) \le 0.$$

However, if $c_{01} + c_{12} > 0$, this pair continuously crosses the boundary $c_{02} = -\frac{1}{2}$ (Fig. 1, left). A cross-cut at $c_{02} = -\frac{1}{4}$ is shown in Fig. 1 (right).

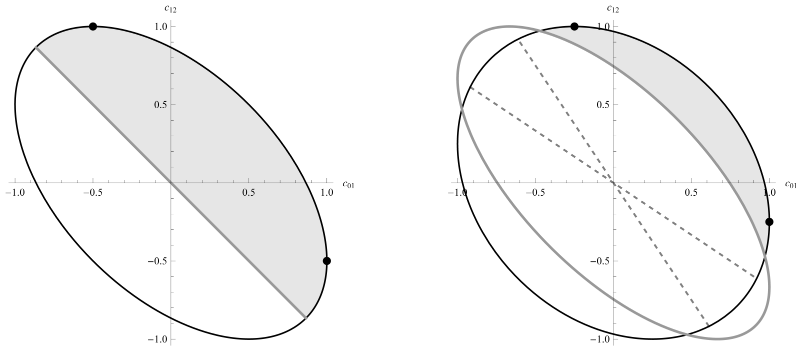

Fig. 1 The admissible parameter regions at $c_{02} = -\frac{1}{2}$ (left) and at $c_{02} = -\frac{1}{4}$ (right). The black boundary is determined by (22), the dark gray one by $g_2 = 0$, the dashed line segments correspond to $g_1\left(\frac{c_{01}+c_{12}}{2}, \frac{c_{01}-c_{12}}{2}, c_{02}\right) = 0$, $g_1\left(\frac{c_{01}+c_{12}}{2}, \frac{c_{12}-c_{01}}{2}, c_{02}\right) = 0$, and the admissible area is coloured bright gray.

If $c_{02} > -\frac{1}{2}$, the equation (26) shows that the first and the second pair are both admissible or not at the same time. From the symmetry $g_2(x, y, w) = g_2(-x, y, w) = g_2(x, -y, w)$ we observe that it is enough to require

$$g_2\left(\frac{c_{01}+c_{12}}{2}, \frac{c_{01}-c_{12}}{2}, c_{02}\right) \geq 0$$

or, simplified

$$\frac{3\left(c_{01}+c_{12}\right)^2}{4\left(1+2c_{02}\right)} + \frac{3\left(c_{01}-c_{12}\right)^2}{8\left(1-c_{02}\right)} \geq 1, \tag{27}$$

to make the pairs real. Note that for every $c_{02} \in \left(-\frac{1}{2}, 1\right)$ we have $g_2\left(x, y, c_{02}\right) < g_1^2\left(x, y, c_{02}\right),\ |x-y| < 1$, so it seems necessary to verify

$$g_1\left(\frac{c_{01}+c_{12}}{2}, \frac{c_{01}-c_{12}}{2}, c_{02}\right) > 0, \quad g_1\left(\frac{c_{01}+c_{12}}{2}, \frac{c_{12}-c_{01}}{2}, c_{02}\right) > 0$$

in order to guarantee $\lambda_{1,2} > 0, \lambda_{2,2} > 0$. However, this is not needed. Since (27) should hold, this can be simplified to $c_{01}+c_{12} > 0$. By a similar approach, it can be shown that $\left(\lambda_{1,3}, \lambda_{2,3}\right)$ and $\left(\lambda_{1,4}, \lambda_{2,4}\right)$ cannot be admissible. The admissible region as a whole is shown in Fig. 2.

Let us summarize the discussion.

Theorem 2 *Suppose that nonplanar data* $\boldsymbol{d}_1$, $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, *and* $\boldsymbol{d}_2$ *are prescribed, i.e.,*

$$\left(c_{02} - c_{01}c_{12}\right)^2 < \left(1 - c_{01}^2\right)\left(1 - c_{12}^2\right).$$

Then there is precisely one interpolant (determined by $\left(\lambda_{1,1}, \lambda_{2,1}\right)$ *in* (25)*), iff*

$$-1 < c_{02} < -\frac{1}{2} \quad or \quad c_{02} = -\frac{1}{2},\ c_{01} + c_{12} > 0.$$

If $c_{02} > -\frac{1}{2}$ then the interpolation problem has two solutions (given by the pairs $\left(\lambda_{1,1}, \lambda_{2,1}\right)$, and $\left(\lambda_{1,2}, \lambda_{2,2}\right)$), iff

$$c_{01} + c_{12} > 0, \quad g_2\left(\frac{c_{01} + c_{12}}{2}, \frac{c_{01} - c_{12}}{2}, c_{02}\right) \geq 0. \tag{28}$$

Otherwise, there are no solutions. The two solution pairs coincide iff in the last relation of (28) the equality is reached.

If the data reduce to a planar problem, the solutions determined by Theorem 2 may be followed by continuity to the boundary. By Remark 2, the only possible exceptions are planar data where a space solution may turn unbounded or irregular.

Theorem 3 *Suppose that planar data that satisfy $\delta_1 > 0$, $|c_{01}|\,|c_{12}| < 1$, are reached as a limit of space data problems that satisfy requirements of Theorem 2. The solution pairs $\left(\lambda_{1,i}, \lambda_{2,i}\right)$ determined by Theorem 2 could be almost always continuously traced to planar data solution. The exceptions are*

a) $c_{02} = -\dfrac{1}{2}$, $c_{12} + c_{02} \leq 0$, *the solution $\left(\lambda_{1,1}, \lambda_{2,1}\right)$ becomes unbounded,*
b) $c_{02} = -\dfrac{1}{2}$, $c_{12} + c_{02} \geq 0$, *the solution $\left(\lambda_{1,2}, \lambda_{2,2}\right)$ becomes unbounded,*
c) $c_{01} = 1$, $(c_{12} = c_{02})$, *or* $c_{12} = 1$, $(c_{01} = c_{02})$, *the solution $\left(\lambda_{1,1}, \lambda_{2,1}\right)$ turns irregular.*

Proof The first two exceptions follow from (25), just by taking the limits $c_{02} \uparrow -\frac{1}{2}$ and $c_{02} \downarrow -\frac{1}{2}$ respectively. By the assumptions of the theorem, the solution may turn irregular only if one of lambdas becomes equal to 0. This follows from the fact that the only irregular planar cubic PH curves are line segments (see, e.g., (Farouki and Sakkalis, 1990)). From (24) and (25) we observe that $\lambda_{1,i}$ may vanish only if

$$\left(1 - c_{12}^2\right)(1 - c_{02})(1 + 2c_{02}) = 0.$$

The factor $1 - c_{02}$ cannot vanish since the only admissible choice is then $c_{02} = 1$, $c_{01} = 1$, $c_{12} = 1$, excluded by the assumption. The possibility $1 + 2c_{02} = 0$ must be

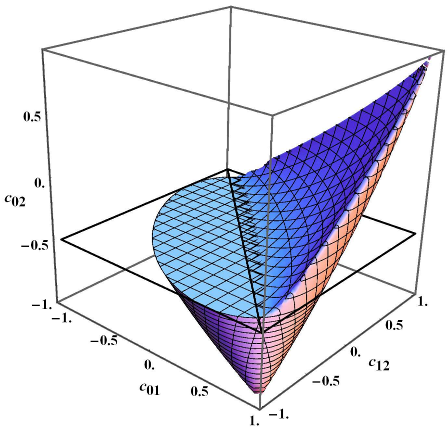

Fig. 2 The admissible choice of parameters c_{01}, c_{12}, and c_{02}.

studied for the solution $\left(\lambda_{1,1}, \lambda_{2,1}\right)$ only. If $c_{02} = -\frac{1}{2}$, $c_{01} + c_{12} > 0$, then

$$\left(\lambda_{1,1}, \lambda_{2,1}\right) = 2\delta_1 \left(\frac{1 - c_{12}^2}{c_{01} + c_{12}}, \frac{1 - c_{01}^2}{c_{01} + c_{12}} \right)$$

determines a regular solution if $|c_{01}| < 1$, $|c_{12}| < 1$. So only $|c_{12}| = 1$ may imply $\lambda_{1,i} = 0$. A brief inspection of the four possibilities in the closed form solutions (25) gives only one possible choice $c_{12} = 1, \lambda_{1,1} = 0$. The possibility $\lambda_{2,i} = 0$ follows similarly. □

It is straightforward to determine the irregular cubic curves observed in Theorem 3. They are line segments

$$\boldsymbol{T}_1 + \Delta\boldsymbol{T}_1\, t^3, \quad \boldsymbol{T}_2 - \Delta\boldsymbol{T}_1\, (1-t)^3. \tag{29}$$

So far we have assumed $\delta_1 > 0$. However, also the case $\boldsymbol{T}_1 = \boldsymbol{T}_2$ implies planar data. As $\boldsymbol{T}_2 \to \boldsymbol{T}_1$, the solutions obtained in Theorem 2 turn out irregular. However, the Jacobian of the system (6) becomes singular, and there are infinitely many planar only solutions, which confirms the well-known behaviour of Tschirnhausen cubics at the double point.

Theorem 4 *The planar problem $\boldsymbol{T}_1 = \boldsymbol{T}_2$, $\boldsymbol{d}_2 \neq \boldsymbol{d}_1$, has a solution only if $c_{02} = -\frac{1}{2}$. In this case, any pair $\lambda_1 = \lambda_2 > 0$ determines a regular interpolant.*

Proof Suppose that $\delta_1 = \|\Delta\boldsymbol{T}_1\| \to 0$. The closed form (25) gives a trivial solution only. For this reason we reexamine the equations (6) that simplify in this case to

$$\left(1 - c_{02}^2\right)(\lambda_1 + 2c_{02}\lambda_2) = 0, \quad -\left(1 - c_{02}^2\right)\left(7\lambda_1^2 + 22c_{02}\lambda_2\lambda_1 + 4\lambda_2^2\right) = 0. \tag{30}$$

Since $\boldsymbol{d}_1 \neq \boldsymbol{d}_2$, then $c_{02} \neq 1$, and since $c_{02} = -1$ does not imply a regular interpolant, it follows $1 - c_{02}^2 \neq 0$. The first equation in (30) then implies $\lambda_1 = -2c_{02}\lambda_2$ and consequently $c_{02} < 0$. Considering this in the second equation we obtain $\left(1 - 4c_{02}^2\right)\lambda_2^2 = 0$, and the assertion of the theorem follows. □

Let us conclude the Hermite case. It is straightforward to verify that we obtain the trivial 1-dimensional (line) case if $c_{01} = c_{12} \to 1$, $c_{02} \to 1$. Any pair $\lambda_1 > 0, \lambda_2 > 0$ that satisfies $6\delta_1(\lambda_1 + \lambda_2) > 9\delta_1^2 + \lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2$, or $\lambda_1 + \lambda_2 \leq 3\delta_1$ is admissible. Also, there is no regular solution if $c_{01} = c_{12} \to -1$, $c_{02} \to 1$, or $c_{01}c_{12} \to -1$, $c_{02} \to -1$.

5.2 Lagrange case

The analysis of the Lagrange PH interpolation is more complex. The curve $\boldsymbol{r}_3$ interpolates four data points. The interpolation conditions are given by the equations (8) where parameter values t_1 and t_2 are to be determined. The corresponding Gram matrix G depends on six data constants

$$\delta_0 > 0,\ \delta_1 > 0,\ \delta_2 > 0,\ -1 \leq c_{01} \leq 1,\ -1 \leq c_{12} \leq 1,\ -1 \leq c_{02} \leq 1. \tag{31}$$

The norm of the tangent vector is by (2), (12) and (7) given as

$$\left\|\boldsymbol{r}_3'(t)\right\| = \sqrt{\sum_{i=0}^{4} \alpha_i t^i} = \sqrt{\boldsymbol{w}'(t)^T G \boldsymbol{w}'(t)} = \sigma_2(t), \quad t \in [0,1]. \tag{32}$$

and the nonlinear equations that determine t_1 and t_2 are given by (6) with α_i, $i = 0, 1, \ldots, 4$, expressed from (32). It is straightforward to rewrite these equations in the polynomial form

$$e_1(t_1, t_2) = 0, \quad e_2(t_1, t_2) = 0. \tag{33}$$

The polynomials e_1 and e_2 are of total degree ≤ 15 and ≤ 14 respectively, and are too large to be written explicitly. Note that any admissible solution of (33) should satisfy $\alpha_0 \neq 0$ too.

Let us examine the boundary of the admissible t_i first. There are six possible ways how the unknown parameters t_1 and t_2 could approach the boundary of $\mathcal{D}$. However, Lemma 2 reduces the possibilities to be examined in the nonplanar case to a single one,

$$t_1 = \tau_1 \varepsilon + \mathcal{O}\left(\varepsilon^2\right), \quad t_2 = 1 - \tau_2 \varepsilon + \mathcal{O}\left(\varepsilon^2\right), \quad \tau_1, \tau_2 > 0. \tag{34}$$

Lemma 5 *Suppose that the solution of* (33) *has an expansion of the form* (34) *for all* ε *small enough. Then this expansion should read*

$$t_1 = \delta_0 \tau \varepsilon + \mathcal{O}\left(\varepsilon^2\right), \; t_2 = 1 - \delta_2 \tau \varepsilon + \mathcal{O}\left(\varepsilon^2\right), \; \tau > 0, \quad c_{02} = \cos\angle\left(\boldsymbol{\Delta T}_0, \boldsymbol{\Delta T}_2\right) = -\frac{1}{2}.$$

Proof Let us insert (34) in the equations (33). The expansions obtained are

$$\begin{aligned}
e_1(t_1, t_2) &= \delta_0^3 \delta_2^2 \tau_2 \left(1 - c_{02}^2\right) \left(2\delta_2 \tau_1 c_{02} + \delta_0 \tau_2\right) \varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) = 0, \\
e_2(t_1, t_2) &= \delta_0^2 \delta_2^2 \left(1 - c_{02}^2\right) \Big(4\left(\delta_0 \tau_2 + \delta_2 \tau_1\right)\left(\delta_0 \tau_2 - \delta_2 \tau_1\right) \\
&\quad - 11 \delta_0 \tau_2 \left(2\delta_2 \tau_1 c_{02} + \delta_0 \tau_2\right)\Big) \varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) = 0.
\end{aligned}$$

The first two possible asymptotic solutions $c_{02} = \pm 1 + \mathcal{O}(\varepsilon)$ are ruled out, since then

$$\sigma_2(t) = \frac{1}{\varepsilon}\left(3\left(\frac{\delta_0}{\tau_1} \pm \frac{\delta_2}{\tau_2}\right) t^2 - 2\left(2\frac{\delta_0}{\tau_1} \pm \frac{\delta_2}{\tau_2}\right) t + \frac{\delta_0}{\tau_1}\right) + \mathcal{O}(1),$$

and σ_2 necessarily changes sign in $[0, 1]$. On the other hand, if $2\delta_2 \tau_1 c_{02} + \delta_0 \tau_2 = 0$, the second asymptotic equation implies $\delta_2 \tau_1 = \delta_0 \tau_2$, and $c_{02} = -\frac{1}{2}$ follows. $\square$

In the planar case, additional approaches to the boundary of $\mathcal{D}$ may appear. Three possibilities are ruled out by Lemma 3, and there are the remaining two to be considered only.

Lemma 6 *The parameters* t_1 *and* t_2 *could approach the boundary of* $\mathcal{D}$ *if* $c_{01}^2 \to 1$ *or* $c_{12}^2 \to 1$ *too. In this, necessarily planar, case they expand as*

$$t_1 = \varepsilon + \mathcal{O}\left(\varepsilon^2\right), \quad t_2 = (1 + \tau_1)\varepsilon + \mathcal{O}\left(\varepsilon^2\right), \quad \tau_1 > 0, \tag{35}$$

and

$$t_1 = 1 - (1 + \tau_2)\varepsilon + \mathcal{O}\left(\varepsilon^2\right), \quad t_2 = 1 - \varepsilon + \mathcal{O}\left(\varepsilon^2\right), \quad \tau_2 > 0,$$

respectively.

Proof Let us assume (35). The second polynomial then expands as

$$e_2(t_1, t_2) = -4\delta_0^2 \delta_1^2 \left(1 - c_{01}^2\right)\left(\delta_0^2 \tau_1^2 - 2\delta_1 \delta_0 \tau_1 c_{01} + \delta_1^2\right) \varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right).$$

Since the term $\delta_0^2 \tau_1^2 - 2\delta_1 \delta_0 \tau_1 c_{01} + \delta_1^2$ cannot vanish, the first assertion is proved. The other one follows similarly. $\square$

If the data points are restricted to a line, obviously solutions exist only if $c_{01} = c_{12} = c_{02} = 1$, and in this case there are infinitely many of them. So we may from now on to the end of the section assume that the points are not taken from a line. Theorem 3 proves that a spatial PH curve may turn irregular when continuously changed to a planar one only if it is of the form (29). However, for the Lagrange case, this implies that limit planar data are obtained from a line segment. Since we have already excluded this possibility, the only possible planar exceptional data are determined by Lemma 5 and Lemma 6. In order to proceed we need the following lemma.

Lemma 7 *Suppose that $\varrho \in [0, 1)$. Let the data points be chosen as*

$$\boldsymbol{T}_0 = \begin{pmatrix} -1-\varrho \\ \sqrt{1-\varrho^2} \\ 0 \end{pmatrix}, \boldsymbol{T}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{T}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{T}_3 = \begin{pmatrix} 1+\varrho \\ \sqrt{1-\varrho^2} \\ 0 \end{pmatrix}. \tag{36}$$

The PH interpolation problem has a solution for any $\varrho \in [0, 1)$. The corresponding interpolation parameters $0 < t_1 < t_2 < 1$ are determined by

$$\begin{gathered} 4t_1(\varrho+1)(t_1(6t_1(2t_1(\varrho+1)-4\varrho-3)+19\varrho+8)-7\varrho+1)+4\varrho^2-1=0, \\ t_2 = 1 - t_1. \end{gathered} \tag{37}$$

For the range $\varrho \in [0, \frac{1}{2}]$, the admissible solution, denoted by $\left(t_1^+, t_2^+\right)$, is unique. For any $\varrho \in (\frac{1}{2}, 1)$ there are two distinct solution pairs $\left(t_1^\pm, t_2^\pm\right)$ that satisfy (37), and they are also the only solutions in the range $(\frac{1}{2}, \bar{\varrho})$, $\bar{\varrho} \approx 0.992989$ (Fig. 3). Both solutions are simple (i.e., Jacobian is nonsingular).

Proof From (36), it is straightforward to compute the corresponding Gram matrix parameters (31) as $\delta_0 = \delta_2 = 1$, $\delta_1 = 2$, $c_{01} = c_{12} = \varrho$, $c_{02} = 2\varrho^2 - 1$. Thus the symbolic equations (33) for the data (36) depend on three parameters t_1, t_2, and ϱ only. However, a direct approach that examines the structure of the ideal $\mathcal{I}(e_1, e_2)$ with the help of the Gröbner basis or resultants failed. This is due to the fact that the polynomials involved in equations (33) are of total degrees 21 and 32 in these three variables, and when expanded they consist of 584 and 1860 terms respectively. So the proof follows a rather long way around. At several places in the proof, we will use the notation $\pi_\ell(\dots)$ to denote a polynomial of the total degree $\leq \ell$ in variables $(\dots)$.

Let us divide the equations (6) by α_0^2 and α_0^3 respectively in order to exclude the possibility $\alpha_0 = 0$. Further, let us compute the Gröbner basis of the rationals involved with respect to variables $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_0)$. This reveals one of polynomial basis of the ideal of the equations (6) (with $\alpha_0 \neq 0$ assured) as

$$\begin{aligned} \boldsymbol{g} := (g_i)_{i=1}^9 := \Big(&\alpha_3^4 - 8\alpha_2\alpha_4\alpha_3^2 - 64\alpha_0\alpha_4^3 + 16\alpha_2^2\alpha_4^2,\ \alpha_3^3 - 4\alpha_2\alpha_4\alpha_3 + 8\alpha_1\alpha_4^2,\ -4\alpha_4\alpha_2^2 \\ &+ \alpha_3^2\alpha_2 + 16\alpha_0\alpha_4^2 + 2\alpha_1\alpha_3\alpha_4,\ -8\alpha_4\alpha_2^3 + 2\alpha_3^2\alpha_2^2 + 32\alpha_0\alpha_4^2\alpha_2 + \alpha_1\alpha_3^3 + 8\alpha_0\alpha_3^2\alpha_4, \\ &-\alpha_1\alpha_3^2 - 8\alpha_0\alpha_4\alpha_3 + 4\alpha_1\alpha_2\alpha_4,\ \alpha_1^2\alpha_4 - \alpha_0\alpha_3^2,\ \alpha_3\alpha_1^2 + 8\alpha_0\alpha_4\alpha_1 - 4\alpha_0\alpha_2\alpha_3, \\ &16\alpha_4\alpha_0^2 - 4\alpha_2^2\alpha_0 + 2\alpha_1\alpha_3\alpha_0 + \alpha_1^2\alpha_2,\ \alpha_1^3 - 4\alpha_0\alpha_2\alpha_1 + 8\alpha_0^2\alpha_3\Big). \end{aligned} \tag{38}$$

Let us now insert $\alpha_i = \alpha_i(t_1, t_2, \varrho)$ obtained from (32) for the particular data (36) in the basis (38). The polynomials g_i become rational functions of t_1, t_2, and ϱ, with the common denominator of the terms in g_i being either

$$(1-t_1)^4 t_1^4 (t_2-t_1)^4 (1-t_2)^4 t_2^4, \quad i \in \{2,3,5,6,7,8,9\},$$

or

$$(1-t_1)^6\, t_1^6\, (t_2-t_1)^6\, (1-t_2)^6\, t_2^6, \quad i \in \{1,4\}.$$

Thus we can bring these functions again to the polynomial form, and simplify them by leaving out constant factors or factors $1-\varrho^2$, since they cannot vanish. The resulting polynomials will be denoted $\boldsymbol{g}\,(t_1,t_2,\varrho) = (g_i\,(t_1,t_2,\varrho))_{i=1}^9$. By the construction, the system of polynomial equations $\boldsymbol{g}\,(t_1,t_2,\varrho) = \mathbf{0}$ is equivalent to the equations (33), and the fact that we have split two equations into several ones will help us to proceed with the proof. The second polynomial factors as

$$g_2\,(t_1,t_2,\varrho) = (1-(t_1+t_2))\,\left(1-(1+\varrho)(t_2-t_1)\right)\,\pi_{10}(t_1,t_2,\varrho), \tag{39}$$

with $\pi_{10}(t_1,t_2,\varrho)$ being a particular polynomial of total degree 10. The equation $g_2(t_1,t_2,\varrho) = 0$ implies that at least one of factors in (39) should vanish. Suppose first that $t_2 = 1-t_1$. Then the Gröbner basis of $\boldsymbol{g}\,(t_1, 1-t_1, \varrho)$, computed with respect to (t_1,ϱ), reads

$$\begin{aligned}(1-t_1)^2\, t_1^2\, \pi_6(t_1,\varrho)\,\Big(\varrho^3(\varrho+2)^3, -\varrho(\varrho+2)\,(\varrho(5\varrho(\varrho+3)+4)-12t_1)\,,\\ 12t_1\varrho+12t_1^2-(2\varrho(\varrho+4)+5)\varrho^2\Big),\end{aligned} \tag{40}$$

$$\pi_6(t_1,\varrho) := 4t_1(\varrho+1)\,(t_1\,(6t_1\,(2t_1(\varrho+1)-4\varrho-3)+19\varrho+8)-7\varrho+1)+4\varrho^2-1.$$

Note that (40) can vanish only if π_6 satisfies the equation $\pi_6(t_1,\varrho) = 0$, $\varrho \in [0,1)$, even at $\varrho = 0$. The solutions $t_1 = t_1(\varrho)$ of this equation can be found in a closed form

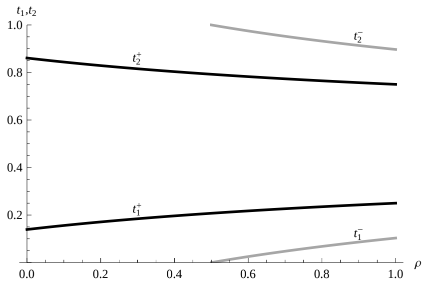

Fig. 3 The solution pairs $\left(t_1^{\pm}(\varrho), t_2^{\pm}(\varrho)\right)$, $\varrho \in [0,1)$ for the particular data (36), determined by the equations $\pi_6(t_1,\varrho) = 0, t_2 = 1-t_1$.

(Fig. 3), and are both simple. Thus the solutions could be continued to the solutions of a near true space problem. The existence assertion of the lemma is confirmed.

In order to complete the proof we have to show that there is no other admissible solution distinct from $t_2 = 1-t_1$, at least for $0 \le \varrho < \bar{\varrho}$, if the second or the third factor in (39) vanishes. Suppose that $1-(1+\varrho)(t_2-t_1) = 0$. Then

$$\varrho = -1 + \frac{1}{t_2-t_1}, \tag{41}$$

and the first equation simplifies to

$$g_1\left(t_1,t_2,-1+\frac{1}{t_2-t_1}\right) = 9\,(1-(t_1+t_2))^6\,\left(t_1^2+2\,(t_2-2)\,t_1+t_2^2\right)^2.$$

Since only the last factor could vanish, the only admissible possibility to be examined is $t_2 = -t_1 + 2\sqrt{t_1}$. However, such a t_2, inserted in (41), yields

$$\varrho = \frac{-2t_1 + 2\sqrt{t_1} - 1}{2\left(\sqrt{t_1} - 1\right)\sqrt{t_1}} \geq 1, \quad t_1 \in (0,1),$$

which rules out the possibility that the second factor in (39) is equal to zero.

Let us consider now the last possibility, $\pi_{10}(t_1, t_2, \varrho) = 0$. If the polynomials π_{10} and g_3 have a common root ϱ, their resultant $r_{2,3}$, computed with respect to ϱ, should vanish. It turns out as

$$r_{2,3}(t_1, t_2) = 9216\,(1-t_1)^2\, t_1^2\,(t_2 - t_1)^{12}\,(1-t_2)^2\, t_2^2\,(1 - t_1 - t_2)^4\, \pi_2^4\,(t_1, t_2)\, \pi_{14}\,(t_1, t_2)\,,$$

with $\pi_2\,(t_1, t_2) := t_1^2 + 2t_2t_1 - 4t_1 + t_2^2$, and $\pi_{14}\,(t_1, t_2)$ being a particular polynomial of total degree 14. Since the other factors in $r_{2,3}$ are extraneous, only π_{14} and π_2 have to be examined. Consider now Fig. 4. The figure shows that a variety, determined by the product $\pi_{14}\,\pi_2$ and the curves $\pi_{10}(t_1, t_2, \varrho) = 0$, $\varrho \in [0, 1)$, have no common solution point (t_1, t_2), except for a small part below the line $t_2 = t_1 + \frac{1}{2}$. First of all, it is

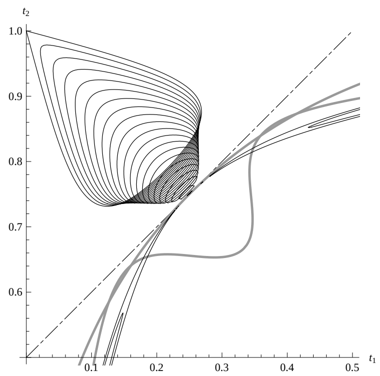

Fig. 4 Solutions of $\pi_{10}(t_1, t_2, \varrho) = 0$, $\varrho \in [0, 1)$ (black), and the curves determined by $\pi_2(t_1, t_2) = 0$ and $\pi_{14}(t_1, t_2) = 0$, respectively (gray). The dotted line $t_2 = t_1 + \frac{1}{2}$ separates the major part of both varieties.

straightforward to verify that $\pi_{14}(t_1, t_2) > 0$, $\pi_2(t_1, t_2) > 0$, $\frac{1}{2} \leq t_1 + \frac{1}{2} < t_2 < 1$, and only the points below the line $t_2 = t_1 + \frac{1}{2}$,

$$\Xi := \left\{ (t_1, t_2) \middle| \, 0 < t_1 < t_2 < 1,\; t_2 \leq t_1 + \frac{1}{2} \right\},$$

are to be considered. The polynomial π_{10} simplifies at the boundary of $\overline{\Xi}$, and with the help of the cylindrical algebraic decomposition it is easy to verify that any solution branch of the equation

$$\pi_{10}(t_1, t_2, \varrho) = 0, \quad \varrho \in [0, 1), \tag{42}$$

could cross the boundary of $\overline{\Xi}$ only at $(t_1, t_2) = (0, 0)$ or $(t_1, t_2) = (1, 1)$. This implies that any solution branch of (42) in $\overline{\Xi}$, distinct from an acnode, which is free of cusps, should include a point with tangent direction equal to $\pm(1,1)$ too. This gives us an impetus to study these points closely. They are determined by equations

$$\pi_{10}(t_1,t_2,\varrho)=0,\quad \frac{\partial\pi_{10}}{\partial t_1}(t_1,t_2,\varrho)+\frac{\partial\pi_{10}}{\partial t_2}(t_1,t_2,\varrho)=0. \tag{43}$$

The equations are easier to be considered in new variables $\nu_1 := t_2+t_1,\ \nu_2 := t_2-t_1$. The left-hand side of the second equation in (43) simplifies to

$$2\,(1-\nu_1)\,\pi_9(\nu_1,\nu_2,\varrho), \tag{44}$$

$$\begin{aligned}\pi_9(\nu_1,\nu_2,\varrho) &:= (\nu_2(\varrho+1)\,(\nu_2(\varrho+1)\,(\nu_2\varrho+\nu_2+5)+5)+1)\ \nu_1^2\\ &\quad -2\,(\nu_2(\varrho+1)\,(\nu_2(\varrho+1)\,(\nu_2\varrho+\nu_2+5)+5)+1)\ \nu_1\ +\nu_2\left(\nu_2^4\left(-(\varrho+1)^3\right)\right.\\ &\quad \left.-3\nu_2^3(\varrho+1)^2+2\nu_2^2(\varrho+1)(3\varrho+4)+2\nu_2(\varrho+2)(2\varrho+3)+2(\varrho+1)\right).\end{aligned}$$

Suppose that $\nu_1=1$. Then the polynomial π_{10} reduces to

$$\begin{aligned}\pi_{10}\left(\frac{1-\nu_2}{2},\ \frac{1+\nu_2}{2},\varrho\right) &= \frac{1}{4}\left(1-\nu_2^2\right)(\nu_2(\varrho+1)-1)\\ &\quad\cdot\left(\nu_2^4(\varrho+1)^2+2\nu_2^3(\varrho+1)-\nu_2^2(\varrho+2)(\varrho+4)+2\nu_2(\varrho+1)+1\right).\end{aligned}$$

Clearly, only the last factor can vanish, thus the equation $\pi_{10}\left(\frac{1-\nu_2}{2},\frac{1+\nu_2}{2},\varrho\right)=0$ has two solutions ν_2 for any $\varrho\in[0,1)$. These solutions both satisfy $\frac{1}{2}<\nu_2<1$ as suggested by Fig. 4. If $\nu_1\neq 1$, the polynomial π_9 in (44) must vanish. The first polynomial of the Gröbner basis of π_{10} and π_9, computed with respect to (ν_1,ν_2,ϱ), is equal to $\nu_2^2\pi_{12}(\nu_2,\varrho)$,

$$\begin{aligned}\pi_{12}(\nu_2,\varrho) &:= -\,\nu_2^4(\varrho+1)^2\left(4\varrho\left(4\varrho^2+2\varrho-5\right)-13\right)+4\nu_2^6(\varrho-1)(\varrho+1)^5\\ &\quad +16\nu_2^5(\varrho-1)(\varrho+1)^4-2\nu_2^3(\varrho+1)(2\varrho(5\varrho(2\varrho+3)-8)-25)\\ &\quad +\nu_2^2(4\varrho(2\varrho(2\varrho(\varrho+1)-3)-3)+13)+16\nu_2(\varrho-1)(\varrho+1)^2+4\varrho^2-4.\end{aligned}$$

This polynomial should vanish if the equations (43) hold. Additionally, $\pi_9(\nu_1,\nu_2,\varrho)=0$ should hold too. However, π_9 is a quadratic function in ν_1, with the discriminant equal to

$$4\,(\nu_2\varrho+\nu_2+1)\,(\nu_2(\varrho+1)\,(\nu_2\varrho+\nu_2+4)+1)\,\pi_8(\nu_2,\varrho),$$

$$\begin{aligned}\pi_8(\nu_2,\varrho) :=&\,\nu_2\Big(\nu_2^4(\varrho+1)^3+3\nu_2^3(\varrho+1)^2+\nu_2^2(\varrho+1)\\ &\cdot((\varrho-4)\varrho-7)+\nu_2(\varrho-4)\varrho-7\nu_2+3\varrho+3\Big)+1.\end{aligned}$$

The polynomial π_8 clearly decides the sign of the discriminant. The cylindrical algebraic decomposition, applied to

$$0\le\varrho<1,\ 0<\nu_2<1,\ \pi_{12}(\nu_2,\varrho)=0,\ \pi_8(\nu_2,\varrho)\ge 0, \tag{45}$$

reveals that the conditions (45) could only be met if $\varrho\in[\overline{\varrho},1)$, where $\overline{\varrho}\approx 0.992989$ is the only root of the polynomial

$$\begin{aligned}&32768\varrho^{12}+190464\varrho^{11}+77088\varrho^{10}-1775104\varrho^9-4372800\varrho^8-538464\varrho^7+10093184\varrho^6\\ &\quad +9967536\varrho^5-5709618\varrho^4-12201948\varrho^3-1981854\varrho^2+4357152\varrho+1860867\end{aligned}$$

in $[0,1)$. This shows that the intersection of the variety (42) and Ξ is empty for $\varrho < \overline{\varrho}$, and the lemma is confirmed. $\square$

Remark 3 Lemma 7 opens the door for possible additional admissible solutions in the range $\overline{\varrho} \leq \varrho < 1$. There are actually six admissible solutions in a subinterval of $[\overline{\varrho}, 1)$. Without going into details we provide the numerical evidence only, given in Fig. 5. Note also that at $\varrho = 1$, i.e., the data prescribed on a line, the number of admissible solutions is infinite.

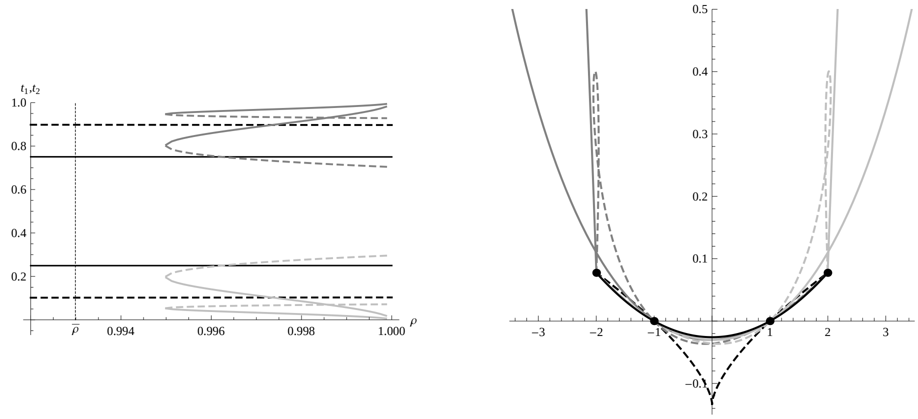

Fig. 5 Six solution parameter pairs $(t_1(\varrho), t_2(\varrho))$ for the data (36) (left) and six interpolants at $\varrho = \frac{997}{1000}$ (right).

Based upon Lemma 5 and Lemma 7, the following theorem gives sufficient conditions for the existence of a regular cubic interpolating PH curve for the particular c_{02} range.

Theorem 5 *Suppose that the data points*

$$\boldsymbol{T}_i \in \mathbb{R}^d, \quad i = 0, 1, 2, 3, \quad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1},$$

satisfy

$$-1 \leq c_{02} < -\frac{1}{2}, \quad |c_{01}| \neq 1, |c_{12}| \neq 1. \tag{46}$$

Then there exists a regular cubic PH curve that interpolates the given points.

Proof Let us choose the parameter ϱ introduced in Lemma 7 as $\varrho = \sqrt{\frac{1+c_{02}}{2}}$, and let the particular data points (36) be denoted by $\boldsymbol{T}_i^*$. The associated parameters are

$$c_{02}^* = 2\varrho^2 - 1 = c_{02} < -\frac{1}{2}, \; c_{01}^* = c_{12}^* = \varrho \geq 0, \quad \delta_0^* = \delta_2^* = 1, \; \delta_1^* = 2,$$

and the corresponding nonlinear system (33): $e_1^*(t_1, t_2) = 0$, $e_2^*(t_1, t_2) = 0$, has a unique solution $(t_1^*, t_2^*) \in \mathcal{D}$ that determines the Lagrange problem solution. Thus the Brouwer's degree of the particular map $(t_1, t_2) \to \left(e_1^*(t_1, t_2), e_2^*(t_1, t_2)\right), (t_1, t_2) \in \mathcal{D}$, is odd. It is straightforward to connect by a homotopy the particular map with a general one that corresponds to the data $\boldsymbol{T}_i$. At first step of the homotopy path, we change only lengths between data points by $\delta_i(\xi) := (1-\xi)\delta_i^* + \xi\delta_i$, $\xi \in [0,1]$. Then, since $c_{02}^* = c_{02} = \text{const} < -\frac{1}{2}$, we connect the particular pair (c_{01}^*, c_{12}^*) with the pair of general parameters (c_{01}, c_{12}) that satisfy (46), by a line segment which lies entirely inside the closed ellipse

$$\frac{(c_{01}+c_{12})^2}{2(1+c_{02})} + \frac{(c_{01}-c_{12})^2}{2(1-c_{02})} \leq 1,$$

and avoids points $|c_{01}| = 1$, $|c_{12}| = 1$. By Lemma 5 and Lemma 6 the homotopy constructed never vanishes at the boundary of $\mathcal{D}$. Thus the Brouwer's degree stays odd also for the general data, which implies the existence of the solution. $\square$

Theorem 5 leaves out the question of the uniqueness of the solution. Quite clearly, one should study the Jacobian along the solution of the equations (33). However, this turned out far too demanding for the computer power at will.

As already observed in the Hermite case, the range $c_{02} \geq -\frac{1}{2}$ is more difficult to analyse. This becomes even more evident in the Lagrange case. Let us conclude the section by considering some numerical observations for this c_{02} range (Fig. 6). The gray region indicates the region where at least two solutions should exist. Note that for some data there should exist several solutions, at least near the planar case discussed in Remark 3. The region is terminated by a black curve, that marks the cosines where the Jacobian of the system (33) at the solution is singular. The numerical results indicate that there are no solutions elsewhere.

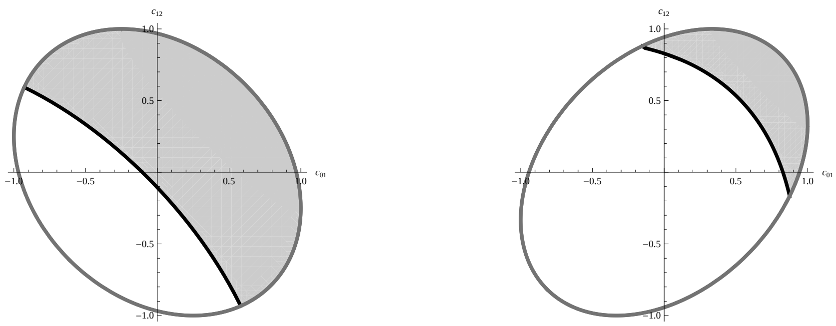

Fig. 6 The data where at least two solutions are expected (gray region) and the data where the solution is double (black curve) for $\boldsymbol{\delta} = (2,1,3)$ with $c_{02} = -\frac{1}{4}$ (left) and $c_{02} = \frac{1}{3}$ (right).

6 Numerical examples

It is straightforward to compute a numerical solution of the Hermite PH interpolation problem, but for the Lagrange case a remark should be added. The nonlinear system of equations (33) is a polynomial one. However, a straightforward application of a general solver that computes all the solutions of a polynomial system of equations would be a waste of computer time, and may lead to insufficient numerical accuracy of the result.

Instead, a continuation method (see (Allgower and Georg, 1990)) based upon theoretical existence considerations should be applied. With constants $\delta_0, \delta_1, \delta_2, c_{01}, c_{02}$, and c_{12} prescribed, Lemma 7 provides all solution pairs (t_1, t_2) for the problem

$$\varrho = \sqrt{\frac{1 + c_{02}}{2}},\ c_{02}^* = c_{02},\ c_{01}^* = c_{12}^* = \varrho,\ \delta_0^* = \delta_2^* = 1,\ \delta_1^* = 2.$$

If there is more than one solution pair, we select a particular one, which should be followed. Further, we construct a homotopy that connects the particular and the general data, and apply the continuation method. For most of possible data situations, a homotopy based upon lines

$$\begin{aligned} c_{01}(s) &= (1-s)\varrho + s\,c_{01} & \qquad c_{12}(s) &= (1-s)\varrho + s\,c_{12} \\ \delta_i(s) &= (1-s) + s\delta_i,\ i = 0, 2 & \qquad \delta_1(s) &= 2(1-s) + s\delta_1, \end{aligned}$$

where $s \in [0,1]$, is appropriate. This approach turned out to be very efficient, and only a few predictor steps along the homotopy path were usually required.

Let us conclude the paper with illustrative numerical examples, which demonstrate a cubic Lagrange interpolation in $\mathbb{R}^4$ and a quintic Lagrange interpolation in $\mathbb{R}^3$. Suppose that the given data points are

$$\boldsymbol{T}_0 = \begin{pmatrix} 0 \\ 10 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{T}_1 = \begin{pmatrix} 0 \\ 13 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{T}_2 = \begin{pmatrix} 2 \\ 16 \\ 4 \\ 3 \end{pmatrix}, \quad \boldsymbol{T}_3 = \begin{pmatrix} 10 \\ 20 \\ 10 \\ 5 \end{pmatrix}. \tag{47}$$

The corresponding parameters are

$$\delta_0 = \sqrt{11},\ \delta_1 = \sqrt{26},\ \delta_2 = 2\sqrt{30},\ c_{01} = 7\sqrt{\frac{2}{143}},\ c_{02} = \sqrt{\frac{10}{33}} > -\frac{1}{2},\ c_{12} = \frac{5}{2}\sqrt{\frac{5}{39}},$$

and two solutions exist,

$$(t_1^+, t_2^+) = (0.273230, 0.608113), \quad (t_1^-, t_2^-) = (0.087184, 0.791055),$$

which corresponds to numerical indications for $c_{02} > -1/2$.

For a geometric interpretation of the obtained cubic interpolants we borrow the results from the motion design. Every point in $\mathbb{R}^4$ (space of quaternions) uniquely determines a rotation in $\mathbb{R}^3$, and a curve in a quaternion space determines a spherical part of a motion of a rigid body (Farin et al, 2002, Chap. 29). The corresponding spherical motions of this example are shown in Fig. 7. The plus solution (left) seems nicer than the minus solution (right).

The second example is just a numerical step outside the rigorous analysis of the cubic case. Suppose that points $\boldsymbol{T}_i$,

$$\boldsymbol{T}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{T}_1 = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}, \boldsymbol{T}_2 = \begin{pmatrix} -4 \\ 7 \\ 4 \end{pmatrix}, \boldsymbol{T}_3 = \begin{pmatrix} -4 \\ 10 \\ 5 \end{pmatrix}, \boldsymbol{T}_4 = \begin{pmatrix} -4 \\ 12 \\ 6 \end{pmatrix}, \boldsymbol{T}_5 = \begin{pmatrix} -4 \\ 13 \\ 8 \end{pmatrix}$$

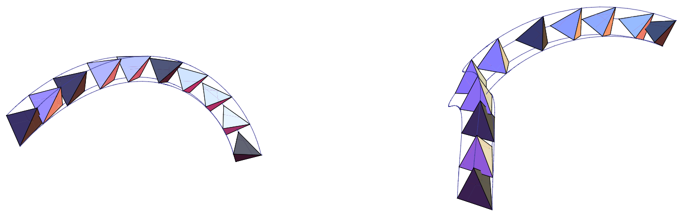

Fig. 7 Spherical parts of the motion of a tetrahedron with four interpolated positions (dark gray) given by (47). The left figure is induced by the plus, and the right one by the minus solution.

are to be interpolated by a quintic PH curve. The equations (3) give four admissible solutions

i	t_1	t_2	t_3	t_4
1	0.0798998	0.44531	0.588504	0.682577
2	0.120688	0.383729	0.620533	0.771804
3	0.0762358	0.462118	0.604523	0.946105
4	0.141656	0.303587	0.470881	0.942122

that determine four interpolatory curves. They are shown in Fig. 8 together with their parametric speeds.

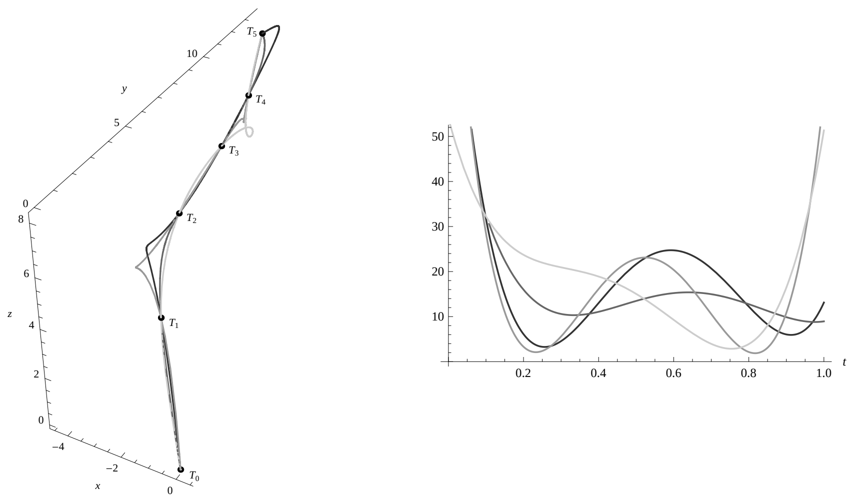

Fig. 8 Four quintic PH interpolatory curves (left) and corresponding parametric speeds (right).

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