Three-pencil lattices on triangulations

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Abstract. In this paper, three-pencil lattices on triangulations are studied. The explicit representation of a lattice, based upon barycentric coordinates, enables us to construct lattice points in a simple and numerically stable way. Further, this representation carries over to triangulations in a natural way. The construction is based upon group action of S_3 on triangle vertices, and it is shown that the number of degrees of freedom is equal to the number of vertices of the triangulation.

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1. Introduction

In contrast to the univariate case, uniqueness of the solution of a multivariate Lagrange polynomial interpolation problem depends not only on the fact that interpolation points should be distinct but also on their actual positions. Consequently, a more detailed study of the geometry of interpolation points is needed. Though it is well-known that the Lagrange interpolation problem at $\binom{n+d}{d}$ interpolation points is correct in Π_n^d (the space of polynomials in d variables of total degree $\leq n$) if and only if the points do not lie on an algebraic curve of degree $\leq n$, this is hard to verify in practical computations.

Lattices are perhaps the most often used configurations of prescribed interpolation points and can be traced back to [2]. They are constructed by intersections of hyperplanes. Several generalizations followed. Among them, *principal lattices* (cf. e.g. [2], [4]) are probably the most widely met in practice. They are generated as intersections of $d+1$ pencils of parallel hyperplanes. In [5], these lattices have been generalized to the case of not necessarily parallel hyperplanes intersecting in so called *centers*. These lattices are known as $(d+1)$*-pencil lattices of order* n. Further generalizations can be found in [1]. It is well-known that all these lattices admit correct interpolation in Π_n^d since they satisfy the GC condition (cf. [2]).

Here we study a particular case, *three-pencil lattices of order* n but extended to triangulations, the most natural subdivisions of a complex (polygonal) domain. Lattices on regular triangulations, where the points on adjacent edges coincide, are important since they provide continuous piecewise polynomial interpolants. Recall that a triangulation is *regular* if every pair of adjacent triangles has only a vertex or the whole edge in common.

Although an explicit construction of a lattice on a single triangle can already be found in [5] and later in an excellent presentation [7, pp. 195–214], in Section 2 we derived it in terms of barycentric coordinates rather than of homogeneous ones. Namely, barycentric form turns out to be a natural tool to extend lattices from a single triangle to a regular triangulation since it keeps clear track of geometric properties. There is of course a shortcut from homogeneous to barycentric form but a straightforward approach, starting with a prescribed triangle, gives a new insight to pencil lattices and deserves its place in the paper. In particular, the determination of the points of a lattice reduces to a simple and very well-known equation $z^n - 1 = 0$. Also, the barycentric form provides an efficient and stable way to compute the points of the lattice.

Section 3 is devoted to lattices on regular triangulations. Explicit conditions, which allow construction of lattices on adjacent triangles are given. They are further extended to cells and finally to a whole triangulation. Even more, the proof of Theorem 3 can be used as a basis for an efficient computer algorithm. The number of free parameters is explained in the most natural manner, by the number of vertices and edges of a triangulation.

2. Three-pencil lattices

Suppose that $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ are vertices of a given triangle $\triangle$. If one is looking for a three-pencil lattice on $\triangle$, the answer will undoubtedly depend on the coordinates of $\boldsymbol{P}_i$. But a general approach should work for any given triangle. So it is natural to switch to barycentric coordinates corresponding to the vertices of the triangle $\triangle$, and apply a simple transformation of coordinates for each particular case separately. Also, the barycentric coordinates on the common edge of adjacent triangles coincide, an important fact when dealing with triangulations. The points to be determined will be indexed with respect to the barycentric coordinates as

$$\boldsymbol{T}_{n-k-j,k,j}, \quad k, j \geq 0, \quad k + j \leq n,$$

with known triangle vertices $\boldsymbol{T}_{n,0,0} = \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right]^T$, $\boldsymbol{T}_{0,n,0} = \left[\begin{array}{ccc} 0 & 1 & 0 \end{array}\right]^T$, $\boldsymbol{T}_{0,0,n} = \left[\begin{array}{ccc} 0 & 0 & 1 \end{array}\right]^T$. Let us write the centers in barycentric form as

$$\boldsymbol{C}_0 = \begin{bmatrix} \dfrac{1}{1-\xi_0} \\ -\dfrac{\xi_0}{1-\xi_0} \\ 0 \end{bmatrix}, \quad \boldsymbol{C}_1 = \begin{bmatrix} 0 \\ \dfrac{1}{1-\xi_1} \\ -\dfrac{\xi_1}{1-\xi_1} \end{bmatrix}, \quad \boldsymbol{C}_2 = \begin{bmatrix} -\dfrac{\xi_2}{1-\xi_2} \\ 0 \\ \dfrac{1}{1-\xi_2} \end{bmatrix}, \quad (1)$$

where $\xi_i > 0$ are free parameters (Figure 1). Of a particular importance

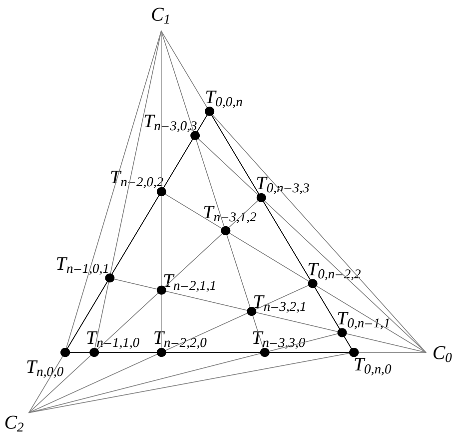

Figure 1. A three-pencil lattice of order n.

will be a constant $\alpha > 0$, defined as

$$\alpha := \alpha\left(\xi_0, \xi_1, \xi_2\right) := \sqrt[n]{\xi_0\, \xi_1\, \xi_2}. \qquad (2)$$

In [5], a three-pencil lattice is determined by 3 centers, two lines that define a vertex of a triangle, and an additional line that completely determines the geometric construction. One can assume that this construction starts at $\boldsymbol{T}_{n,0,0}$, with $\boldsymbol{T}_{n-1,0,1}\boldsymbol{C}_0$ as the chosen additional line. The first step to determine the points $\boldsymbol{T}_{n-k-j,k,j}$ is the following.

Lemma 1. Let $\boldsymbol{T}_{n-k-j,k,j}$, $k, j \geq 0$, $k + j \leq n$, be points of a three-pencil lattice, generated by centers $\boldsymbol{C}_i$ given in (1). Then

$$\boldsymbol{T}_{n-k,k,0} = \begin{bmatrix} \tau_k \\ 1-\tau_k \\ 0 \end{bmatrix}, \quad k = 0, 1, \ldots, n,$$

where

$$\tau_k := \tau_k(\xi_0) := \frac{\alpha^n - \alpha^k}{\alpha^n - \alpha^k + (\alpha^k - 1)\,\xi_0}, \quad k = 0, 1, \ldots, n, \tag{3}$$

and α is defined in (2).

Proof. Suppose that all centers are finite, $\xi_i \neq 1$ and $\alpha \neq 1$. Let us choose $\omega \in (0,1)$, so that the point

$$\boldsymbol{U}_1 := \begin{bmatrix} 1-\omega \\ 0 \\ \omega \end{bmatrix}$$

is on the edge $\boldsymbol{T}_{n,0,0}\boldsymbol{T}_{0,0,n}$, and let $\boldsymbol{\ell}$ denote the line connecting $\boldsymbol{U}_1$ and $\boldsymbol{C}_0$. To start with, let us assume $\boldsymbol{L}_0 := \boldsymbol{T}_{n,0,0}$, and consider the

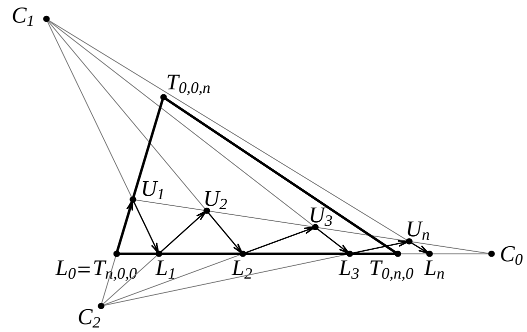

Figure 2. The "zig-zag" construction with $\xi_0 > 1$.

following geometric construction for $k = 1, 2, \ldots, n$ (Figure 2):

- *Forward step:* determine a point $\boldsymbol{U}_k$ as the intersection of the lines $\boldsymbol{C}_2\boldsymbol{L}_{k-1}$ and $\boldsymbol{\ell}$.
- *Backward step:* determine a point $\boldsymbol{L}_k$ as the intersection of the edge $\boldsymbol{T}_{n,0,0}\boldsymbol{T}_{0,n,0}$ and the line $\boldsymbol{C}_1\boldsymbol{U}_k$.

This "zig-zag" procedure produces points

$$\boldsymbol{L}_0, \boldsymbol{U}_1, \boldsymbol{L}_1, \ldots, \boldsymbol{U}_n, \boldsymbol{L}_n, \tag{4}$$

that are clearly part of a three-pencil lattice for some triangle since each is determined as an intersection of lines from all the centers. We proceed to find a unique $\omega \in (0,1)$ such that the points (4) are the lattice points for the given $\triangle$, i.e., the equation

$$\boldsymbol{L}_n = \boldsymbol{T}_{0,n,0} \tag{5}$$

is satisfied. This is the key point of our algebraic construction. Since the points $\boldsymbol{L}_k$ lie on the edge $\boldsymbol{T}_{n,0,0}\boldsymbol{T}_{0,n,0}$, their barycentric coordinates are

$$\boldsymbol{L}_k = \begin{bmatrix} \phi_k(\omega) \\ 1-\phi_k(\omega) \\ 0 \end{bmatrix}, \quad k = 0, 1, \ldots, n,$$

with $\phi_0(\omega) := 1$. The forward step computes the intersection $\boldsymbol{U}_k$ as

$$\boldsymbol{U}_k = \mu_k \boldsymbol{C}_2 + (1-\mu_k)\boldsymbol{L}_{k-1} = \rho_k \boldsymbol{U}_1 + (1-\rho_k)\boldsymbol{C}_0, \tag{6}$$

and an elimination from the right-hand side equation yields

$$\begin{aligned} \mu_k &= \frac{\omega\,(1-\xi_2)\,((\xi_0-1)\,\phi_{k-1}(\omega)+1)}{\omega\,(1-\xi_2)\,(\xi_0-1)\,(\phi_{k-1}(\omega)-1)+\xi_0}, \\ \rho_k &= \frac{1}{\omega\,(1-\xi_2)}\,\mu_k. \end{aligned} \tag{7}$$

Similarly, the backward step determines $\boldsymbol{L}_k$ as

$$\boldsymbol{L}_k = \gamma_k \boldsymbol{C}_1 + (1-\gamma_k)\boldsymbol{U}_k, \tag{8}$$

with $\boldsymbol{U}_k$ given by (6), and ρ_k by (7). However, the third component of $\boldsymbol{L}_k$ is 0, which implies

$$\gamma_k = \frac{(\xi_1-1)\,\mu_k}{(\xi_1-1)\,\mu_k + \xi_1\,(\xi_2-1)}.$$

But then the first component of (8) reveals the recurrence relation for $\phi_k(\omega)$,

$$\phi_k(\omega) = \frac{a\,\phi_{k-1}(\omega)+b}{c\,\phi_{k-1}(\omega)+d}, \quad \phi_0(\omega) := 1, \tag{9}$$

where the coefficient matrix is obtained as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{bmatrix} \xi_1\,(\omega\xi_2-(\omega-1)\xi_0) & -\omega\xi_1\xi_2 \\ \omega\,(\xi_0-1)\,(1-\xi_1\xi_2) & \omega+\xi_1\,(\omega\xi_2\,(\xi_0-1)-(\omega-1)\xi_0) \end{bmatrix}.$$

The difference equation (9) admits a closed form solution (cf. [6, p. 146])

$$\phi_k(\omega) = \frac{\psi(\omega)^k - \xi_0\xi_1\xi_2}{(1-\xi_0)\,\psi(\omega)^k + \xi_0\,(1-\xi_1\xi_2)}, \tag{10}$$

where

$$\psi(\omega) := \frac{(1-\omega+\omega\xi_2)\,\xi_0\xi_1}{\omega+(1-\omega)\xi_1\xi_3}. \tag{11}$$

The numerator and the denominator in (10) have clearly no common root $\psi(\omega)^k$, and the equation (5) simplifies to

$$\psi(\omega)^n - \xi_0\xi_1\xi_2 = \psi(\omega)^n - \alpha^n = 0.$$

But this is a well-known equation with solutions proportional to the roots of unity,

$$\psi(\omega) = \alpha\, e^{\frac{2\pi i}{n}k}, \quad k = 1, 2, \ldots, n,$$

and with precisely one positive real root $\psi(\omega) = \alpha \neq 1$. From (11), it is now straightforward to derive

$$\omega = \psi^{-1}(\alpha) = \frac{1}{1 + \dfrac{1}{\xi_0 \xi_1} \dfrac{\alpha^n - \alpha}{\alpha - 1}}.$$

Obviously, $0 < \omega < 1$, even if $\alpha \to 1$, since then

$$\frac{\alpha^n - \alpha}{\alpha - 1} = \sum_{j=0}^{n-2} \alpha^{j+1} \to n - 1, \quad \omega \to \frac{1}{1 + (n-1)\,\xi_2}.$$

Finally, the claim (3) is confirmed by simplifying $\tau_k = \phi_k\left(\psi^{-1}(\alpha)\right)$. Note that the expression (3) makes sense as $\alpha \to 1$ too, namely

$$\frac{\alpha^k - 1}{\alpha^n - \alpha^k} = \frac{\sum\limits_{j=0}^{k-1} \alpha^j}{\sum\limits_{j=0}^{n-1-k} \alpha^{n-1-j}} \to \frac{k}{n-k}, \quad 0 \le k \le n-1,$$

and

$$\tau_k \to \frac{n-k}{n-k+k\,\xi_0}, \quad k = 0, 1, \ldots, n.$$

This concludes the proof of the lemma. □

The following theorem reveals the whole lattice.

Theorem 1. Suppose that the centers $\boldsymbol{C}_i$ of a three-pencil lattice are prescribed by ξ_i as in (1), and that the corresponding α is determined by (2). Let $v_i := \alpha^i$, $w_i := \sum_{j=0}^{i-1} v_j$, $i = 0, 1, \ldots, n$. The points of a three-pencil lattice of order n are given as

$$\boldsymbol{T}_{n-k-j,k,j} = \begin{bmatrix} \dfrac{v_{k+j} w_{n-k-j}}{v_{k+j} w_{n-k-j} + (v_j w_k + w_j \xi_1)\,\xi_0} \\ \dfrac{v_{n-k} w_k}{v_{n-k} w_k + (v_{n-k-j} w_j + w_{n-k-j} \xi_2)\,\xi_1} \\ \dfrac{v_{n-j} w_j}{v_{n-j} w_j + (v_k w_{n-k-j} + w_k \xi_0)\,\xi_2} \end{bmatrix}. \qquad (12)$$

Proof. Let us prove (12) constructively. Consider Figure 1. The point $\boldsymbol{T}_{0,n-j,j}$ is clearly determined by the line $\boldsymbol{C}_2\boldsymbol{T}_{j,n-j,0}$, and it is enough to compute the second component only. With the help of Lemma 1 and (3), it turns out that

$$\frac{\left(\alpha^n-\alpha^j\right)\xi_0\xi_2}{\alpha^n\left(\alpha^j-1\right)+\left(\alpha^n-\alpha^j\right)\xi_0\xi_2}=\frac{\alpha^n-\alpha^j}{\alpha^n-\alpha^j+\left(\alpha^j-1\right)\xi_1}=\tau_j\left(\xi_1\right),$$

since $\xi_0\xi_1\xi_2=\alpha^n$. Similarly, the third component of the point $\boldsymbol{T}_{n-k-j,0,k+j}$, determined by the line $\boldsymbol{C}_1\boldsymbol{T}_{n-k-j,k+j,0}$, is

$$\begin{aligned}\frac{\left(\alpha^{k+j}-1\right)\xi_0\xi_1}{\alpha^n-\alpha^{k+j}+\left(\alpha^{k+j}-1\right)\xi_0\xi_1} &= \\ \frac{\alpha^n-\alpha^{n-k-j}}{\alpha^n-\alpha^{n-k-j}+\left(\alpha^{n-k-j}-1\right)\xi_2} &= \tau_{n-k-j}\left(\xi_2\right).\end{aligned}$$

The lines $\boldsymbol{C}_2\boldsymbol{T}_{n-k,k,0}$ and $\boldsymbol{C}_1\boldsymbol{T}_{n-k-j,0,k+j}$ are not parallel, so they meet at some point which is the lattice point $\boldsymbol{T}_{n-k-j,k,j}$,

$$\boldsymbol{T}_{n-k-j,k,j}=\begin{bmatrix}\dfrac{\alpha^n-\alpha^{k+j}}{\alpha^n-\alpha^{k+j}+\left(\alpha^{k+j}-\alpha^j+\left(\alpha^j-1\right)\,\xi_1\right)\,\xi_0}\\ \dfrac{\alpha^n-\alpha^{n-k}}{\alpha^n-\alpha^{n-k}+\left(\alpha^{n-k}-\alpha^{n-k-j}+\left(\alpha^{n-k-j}-1\right)\,\xi_2\right)\,\xi_1}\\ \dfrac{\alpha^n-\alpha^{n-j}}{\alpha^n-\alpha^{n-j}+\left(\alpha^{n-j}-\alpha^k+\left(\alpha^k-1\right)\xi_0\right)\,\xi_2}\end{bmatrix}, \tag{13}$$

iff it lies on the line $\boldsymbol{C}_0\boldsymbol{T}_{0,n-j,j}$ too. In order to show this, one may base the argument on Pappus' theorem (cf. e.g. [3, Axiom 14.15]). The substitution stated in the theorem and (13) conclude the proof. □

The points $\boldsymbol{T}_{n-k-j,k,j}$ can be computed efficiently and stably, avoiding any cancellations. Indeed, one is able to obtain $v_i, w_i,\ i=0,1,\ldots,n$, in $2n+\mathcal{O}(1)$ floating point operations only.

3. Lattices on triangulations

The three-pencil lattice, given in Theorem 1, can easily be extended to a regular triangulation in a continuous way (see Figure 6). Namely, every two adjoining triangles have to share all the lattice points on the common edge. This implies some relations between the center positions which are revealed in the following theorem.

Theorem 2. Let $\triangle$ and $\triangle'$ be given triangles, and let the corresponding three-pencil lattices be determined by parameters ξ_i and ξ_i', respectively. Barycentric coordinates of lattice points at edges $\boldsymbol{T}_{n,0,0}\boldsymbol{T}_{0,n,0}$ and $\boldsymbol{T}'_{n,0,0}\boldsymbol{T}'_{0,n,0}$ agree iff

$$\xi_0\xi_1'\xi_2' = \xi_0'\xi_1\xi_2,$$

in the case $n = 2$, and

$$\xi_1\xi_2 = \xi_1'\xi_2',\ \xi_0 = \xi_0',\ (\alpha' = \alpha),\ \text{or}\ \xi_0\xi_1'\xi_2' = 1,\ \xi_0'\xi_1\xi_2 = 1,\ \Big(\alpha' = \frac{1}{\alpha}\Big),$$

for $n \geq 3$.

Proof. One has to verify

$$\boldsymbol{T}_{n-k,k,0} = \boldsymbol{T}'_{n-k,k,0},\ k = 1, 2, \ldots, n-1, \tag{14}$$

only. But then (3) simplifies (14) to

$$\frac{\alpha^k - 1}{\alpha^n - \alpha^k}\,\xi_0 = \frac{\alpha'^k - 1}{\alpha'^n - \alpha'^k}\,\xi_0', \quad k = 1, 2, \ldots, n-1. \tag{15}$$

Since the case $n = 2$ is straightforward, let $n \geq 3$. Dividing equations in (15) for $k = 1$ and $k = 2$ leads to $f(\alpha) = f(\alpha')$, where

$$f(\alpha) = \frac{\alpha\,(\alpha^{n-2} - 1)}{(\alpha + 1)\,(\alpha^{n-1} - 1)} = \frac{\sum_{j=1}^{n-2}\alpha^j}{\sum_{j=1}^{n-1}(\alpha^j + \alpha^{j-1})}.$$

It is easy to verify that $f(\alpha) = f(1/\alpha)$, thus $\alpha' = \alpha$ or $\alpha' = 1/\alpha$. We have to see that there are no other positive solutions. Since f is nonnegative, it is enough to prove that $f(\alpha) = c$, $c \geq 0$, has at most two positive solutions. Since $\alpha > 0$, the equation $c - f(\alpha) = 0$ is equivalent to

$$\left(\sum_{j=1}^{n-1}\Big(\alpha^j + \alpha^{j-1}\Big)\right)(c - f(\alpha)) = c + \sum_{j=1}^{n-2}(2\,c - 1)\ \alpha^j + c\,\alpha^{n-1} = 0.$$

Then the Descartes' rule of signs shows that there are at most two roots in $[0, \infty)$. The relation (15) for $k = 1$ now gives the desired results stated in the theorem. It can also be easily checked that all solutions satisfy (15) for any suitable k as well. $\square$

Let us simplify the notation and denote the vertices $\boldsymbol{T}_{n,0,0}$, $\boldsymbol{T}_{0,n,0}$, $\boldsymbol{T}_{0,0,n}$, by $1, 2, 3$, respectively. Consider the following example. Let $\xi_0 =$

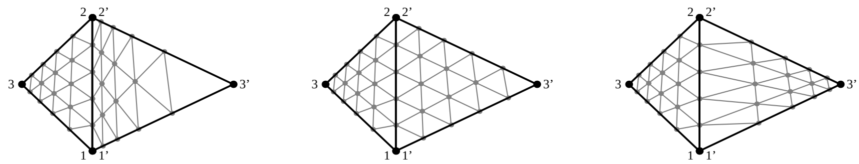

Figure 3. The lattices obtained for parameters $\xi_1' = \frac{1}{3}, 1, 3$ and the transformation $\alpha \to \alpha$.

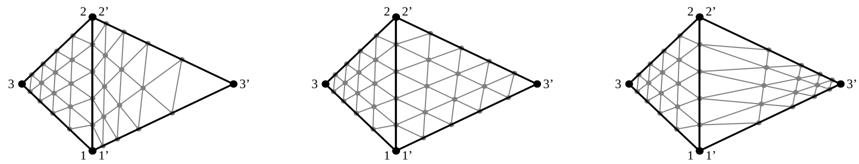

Figure 4. The lattices obtained for parameters $\xi_1' = \frac{1}{3}, 1, 3$ and the transformation $\alpha \to 1/\alpha$.

5/4, $\xi_1 = 2$ and $\xi_2 = 2/3$. The relations between ξ_1' and ξ_2', outlined in Theorem 2, for the transformations $\alpha \to \alpha$ and $\alpha \to 1/\alpha$, are $\xi_2' = \frac{4}{3\xi_1'}$, $\xi_2' = \frac{4}{5\xi_1'}$, respectively. In Figure 3 and Figure 4, the lattices that correspond to points $\xi_1' = \frac{1}{3}, 1, 3$ are presented for both cases.

Theorem 2 gives relations which assure that the lattice points agree on the common edge for a particular labeling of triangle vertices. But, in order to construct a lattice on the whole triangulation, similar results for every possible pair of edges would be needed. Instead, only Theorem 2 together with rotations and mirror maps on the labels of the triangle vertices can be used. It is well-known that these transformations form the symmetric group S_3. Reflection of the triangle around one of the angle bisectors gives the permutations (1 2), (1 3), (2 3), and the rotations are given by (1 2 3), (1 3 2), (1)(2)(3). The question is how the group transformations translate the center parameters ξ_i. By (1), it is easy to verify that the rotation (1 2 3) yields

$$\xi_0 \to \xi_1, \quad \xi_1 \to \xi_2, \quad \xi_2 \to \xi_0, \quad \alpha \to \alpha, \tag{16}$$

and the mirror map (1 2) gives

$$\xi_0 \to \xi_0^{-1}, \quad \xi_1 \to \xi_2^{-1}, \quad \xi_2 \to \xi_1^{-1}, \quad \alpha \to \alpha^{-1}. \tag{17}$$

Since S_3 is generated by (1 2 3) and (1 2), other transformations of centers can be obtained by compositions of (16) and (17). A lattice on the given regular triangulation can now be constructed in the following way. First, choose an arbitrary triangle and use Theorem 1 to obtain the lattice. Then repeat the following steps until the whole triangulation is covered:

- add a triangle at a time, in such a way that the current subtriangulation is simply connected,
- use transformations from group S_3 and Theorem 2 to construct the lattice on the new triangle.

Now, new triangles can be added in various ways. Here they will be added so that the cells at the boundary of the current subtriangulation will be completed in the positive direction around the cell's inner vertex. Note that a *cell of degree* m is a triangulation with exactly one inner vertex (of the degree m).

Theorem 2 points out that each triangle added to a regular simply connected triangulation brings in an additional free center position unless the lattice points have already been prescribed on two edges. This happens when a cell around an inner vertex is completed. In this case there are two additional equations to be fulfilled. The first chosen triangle brings 3 degrees of freedom, every other triangle adds one, and every cell diminishes the degree by one as will be shown later on. With the help of Euler's formula one can conclude that a three-pencil lattice, extended to a regular simply connected triangulation with V vertices, has V degrees of freedom. A more detailed analysis is given in the proof of the following theorem.

Theorem 3. Let $n > 2$. A three-pencil lattice on a regular simply connected triangulation T with V vertices can be constructed by using Theorem 2 and transformations from group S_3. There are V degrees of freedom.

Proof. By Theorem 2 the result obviously holds for two triangles. Consider now a cell of degree m. Let the starting triangle be chosen arbitrarily and the rest of triangles numbered consecutively in the positive direction around the cell's inner vertex. Suppose that on the i-th triangle a lattice is given by parameters $\xi_0^{(i)}, \xi_1^{(i)}, \xi_2^{(i)}$, $i = 1, 2, \ldots, m$, where each $\xi_j^{(i)}$ defines a center $\boldsymbol{C}_j^{(i)}$ as in the previous section. Similarly, let each triangle be labeled in the positive direction starting with the inner point of the cell (see Figure 5). The connections between parameters $\xi_j^{(i)}$ must be found so that the lattice points on common edges will agree.

Let us choose the parameters for the first triangle as $\xi_j^{(1)} := \xi_j$, $j = 0, 1, 2$, and consider the i-th and $(i+1)$-th triangle (see Figure 5). In order to use Theorem 2 vertices of the common edge of the triangles considered must be labeled by 1 and 2. Therefore the transformation

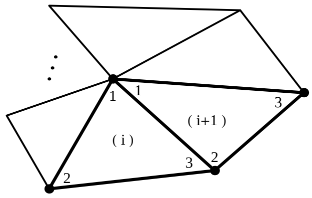

Figure 5. Labeling of the i-th and $(i+1)$-th triangle before the transformation.

$(2\ 3) = (1\ 2)(1\ 2\ 3)$ that maps

$$\xi_0^{(i)} \to \frac{1}{\xi_2^{(i)}} =: \tilde{\xi}_0, \quad \xi_1^{(i)} \to \frac{1}{\xi_1^{(i)}} =: \tilde{\xi}_1, \quad \xi_2^{(i)} \to \frac{1}{\xi_0^{(i)}} =: \tilde{\xi}_2,$$

must first be used on the labels of the vertices of the i-th triangle. Now, Theorem 2 gives two options, $\alpha \to \frac{1}{\alpha}$ and $\alpha \to \alpha$. In the first case the required equations are fulfilled iff

$$\xi_0^{(i+1)} = \frac{1}{\tilde{\xi}_1\tilde{\xi}_2} = \xi_0^{(i)}\xi_1^{(i)}, \quad \xi_1^{(i+1)} = \sigma_i, \quad \xi_2^{(i+1)} = \frac{1}{\sigma_i\tilde{\xi}_0} = \frac{\xi_2^{(i)}}{\sigma_i}, \tag{18}$$

and in the second case iff

$$\xi_0^{(i+1)} = \tilde{\xi}_0 = \frac{1}{\xi_2^{(i)}}, \quad \xi_1^{(i+1)} = \sigma_i\tilde{\xi}_1 = \frac{\sigma_i}{\xi_1^{(i)}}, \quad \xi_2^{(i+1)} = \frac{\tilde{\xi}_2}{\sigma_i} = \frac{1}{\sigma_i\xi_0^{(i)}}, \tag{19}$$

where a new free parameter σ_i follows from Theorem 2. In the case (18) induction shows that

$$\xi_0^{(i)} = \xi_0\xi_1\prod_{j=1}^{i-2}\sigma_j, \quad \xi_1^{(i)} = \sigma_{i-1}, \quad \xi_2^{(i)} = \xi_2\prod_{j=1}^{i-1}\sigma_j^{-1}, \quad i = 2, 3, \ldots, m.$$

Since the lattice points on the edge between the first and the last triangle must also agree, one more step gives the restriction $\prod_{i=1}^{m-1}\sigma_i\xi_1 = 1$ on the choice of parameters σ_i. Therefore it is clear that in this case the lattice on the cell is determined by $m+1$ free parameters. In the second case expressions (19) imply a distinction between odd and even m. For even numbered triangles, i.e., $i = 2k$, one obtains

$$\xi_0^{(2k)} = \frac{1}{\xi_2}\prod_{j=1}^{k-1}\sigma_{2j},\ \xi_1^{(2k)} = \frac{1}{\xi_1}\prod_{j=1}^{k-1}\sigma_{2j}^{-1}\prod_{j=1}^{k}\sigma_{2j-1},\ \xi_2^{(2k)} = \frac{1}{\xi_0}\prod_{j=1}^{k}\sigma_{2j-1}^{-1},$$

and for odd numbered triangles, $i = 2k+1$,

$$\xi_0^{(2k+1)} = \xi_0\prod_{j=1}^{k}\sigma_{2j-1},\ \xi_1^{(2k+1)} = \xi_1\prod_{j=1}^{k}\sigma_{2j}\,\sigma_{2j-1}^{-1},\ \xi_2^{(2k+1)} = \xi_2\prod_{j=1}^{k}\sigma_{2j}^{-1},$$

for $i = 1, 2, \ldots, m$. Now, in the case when a degree of the cell is even, $m = 2k$, the lattice points on the common edge between the first and the last triangle agree if $\prod_{i=1}^{k} \sigma_{2i-1} = 1$, which gives $m+1$ degrees of freedom. For odd degrees, $m = 2k-1$, this is true if

$$\prod_{i=1}^{k-1} \sigma_{2i} = \xi_0 \xi_2 \quad \text{and} \quad \xi_0 \xi_1 \xi_2 = 1,$$

therefore the lattice is determined by one less degree of freedom, because the equation $\alpha = 1$ must be fulfilled.

Since methods $\alpha \to \alpha$ and $\alpha \to \frac{1}{\alpha}$ can also be combined, the conclusions obtained above yield the only restriction: the method $\alpha \to \alpha$ in Theorem 2 must be used even number of times for this particular labeling of triangle vertices. But triangle vertices can be labeled arbitrarily. Each particular labeling determines the number of group transformations that give $\alpha \to \frac{1}{\alpha}$ so that Theorem 2 can be used as explained before. If this number is odd (even), the method $\alpha \to \frac{1}{\alpha}$ in Theorem 2 must be used odd (even) number of times. That assures that the number of degrees of freedom is $m+1$.

Suppose now that the lattice has already been constructed on a simply connected subtriangulation T' of the triangulation T. In the next step of the algorithm pick a vertex P at the boundary of T' and continue with the construction of the lattice on the cell C around P in the positive direction. The lattice on a subtriangulation C' of C has already been computed. Let the triangle in C' that is adjacent to the starting triangle in $C \setminus C'$ be denoted by $\triangle^F$, and the triangle in C' adjacent to the last triangle in $C \setminus C'$ by $\triangle^L$. Let each triangle in $C \setminus C'$ be oriented in the positive direction with the point P corresponding to $\boldsymbol{T}_{n,0,0}$. The same must be done for triangles $\triangle^F$ and $\triangle^L$ by using transformations from S_3. The problem of determining the lattice parameters for triangles in $C \setminus C'$ is now the same as for the cell. As shown before, each new triangle brings an additional degree of freedom, and the last triangle reduces the degree by one. Therefore the number of degrees of freedom increases by the number of points added to a triangulation. This concludes the proof of the theorem. □

Note that the results of Theorem 3 can be easily extended to s-connected triangulated polygonal domains. One simply applies generalized Euler's formula and the same construction, given in the proof of Theorem 3. The number of free parameters is $V + s - 1$.

Since the case $n = 2$ is exceptional in Theorem 2, the number of degrees of freedom in this case can be larger than the number of vertices of the triangulation. More precisely, the first chosen triangle brings 3 degrees of freedom, every other triangle adds two, and every cell diminishes the degree by one. Therefore, for a regular s-connected triangulation with E edges there are E degrees of freedom.

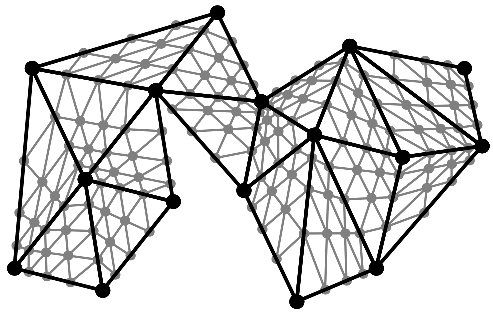

Figure 6. Lattice on a regular triangulation.

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