On geometric interpolation of circle-like curves

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Abstract

In this paper, geometric interpolation of certain circle-like curves by parametric polynomial curves is studied. It is shown that such an interpolating curve of degree n achieves the optimal approximation order $2n$, the fact already known for particular small values of n. Furthermore, numerical experiments suggest that the error decreases exponentially with growing n.

Key words: Interpolation, Approximation, Parametric curve, Circular arc.

1 Introduction

In [1] the problem of geometric interpolation of planar data by parametric polynomial curves has been revisited. In particular, the conjecture that a parametric polynomial curve of degree $\leq n$ can interpolate $2n$ given points in $\mathbb{R}^2$ has been confirmed for $n \leq 5$ under certain natural restrictions. Furthermore, the optimal asymptotic approximation order $2n$ has been confirmed provided the interpolating polynomial curve exists. But its existence for general n has been an open challenge for quite a while since the pioneering work on geometric interpolation has appeared ([2]).

Among planar parametric curves special attention has always been given to circular arcs, probably the most important geometric objects in practice. There are several papers dealing with good approximation of circular segments with

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radial error as the parametric distance. In [3] the authors study existence of the cubic Bézier Hermite type interpolant which is sixth-order accurate, and in [4] a similar problem with various boundary conditions is presented. In [5], the problem of approximation of circle segments by quadratic Bézier curves is considered. Probably the most general results on Hermite type polynomial approximation of conic sections by parametric polynomial curves of odd degree are given in [6] and [7]. Several new special types of Hermite interpolation schemes are also given in [8] and [9]. All these results include odd degree interpolating curves as a rule and do not extend to Lagrange type of interpolation directly.

In this paper, we establish the existence of an interpolating curve of Lagrange type for general n, provided the data are sampled from a smooth curve sufficiently close to a circular arc. More precisely, let us assume that $\boldsymbol{\mathcal{A}}$ is a circular arc of an arclength $h > 0$. Since the term circle-like necessarily involves a comparison of two curves, the arclength parametric representation $\boldsymbol{\mathcal{A}}: [0,h] \mapsto \mathbb{R}^2$ is perhaps the most convenient tool. Suppose that a convex curve $\boldsymbol{f} \approx \boldsymbol{\mathcal{A}}$ is parameterized by the same parameter as $\boldsymbol{\mathcal{A}}$. The curve $\boldsymbol{f}$ will be called *circle-like*, if it agrees twice with $\boldsymbol{\mathcal{A}}$ at 0, has the curvature of the same sign at 0 as well, i.e.,

$$\boldsymbol{f}(0) = \boldsymbol{\mathcal{A}}(0),\ \boldsymbol{f}'(0) = \boldsymbol{\mathcal{A}}'(0),\ \det\left(\boldsymbol{f}'(0), \boldsymbol{f}''(0)\right) \det\left(\boldsymbol{\mathcal{A}}'(0), \boldsymbol{\mathcal{A}}''(0)\right) > 0, \quad (1)$$

and its smooth correction $\boldsymbol{g} := \boldsymbol{f} - \boldsymbol{\mathcal{A}}$ expands as

$$\boldsymbol{g}(s) = \frac{1}{2!}\,\boldsymbol{g}''(0)\,s^2 + \frac{1}{3!}\,\boldsymbol{g}^{(3)}(0)\,s^3 + \dots \quad (2)$$

In order to make a distinction among the circle-like curves, we introduce a constant M,

$$\max_{2 \le r \le 2n-1} \left\| \boldsymbol{g}^{(r)}(0) \right\|_\infty \le M, \quad (3)$$

that bounds the magnitudes of derivatives at 0. For any particular M, the corresponding set of circle-like curves will be denoted by $\mathbb{F}_M$.

As a motivation, let us consider the following numerical example. Let

$$\boldsymbol{f}(t) = \exp(t/4) \begin{pmatrix} \sin t \\ 1 - \cos t \end{pmatrix}, \quad t \in \left[0, \frac{\pi}{2}\right], \quad (4)$$

be a particular exponential spiral, geometrically interpolated by polynomial parametric curves of degrees $n = 6$ and $n = 7$, respectively, at $2n$ points obtained by the equidistant splitting of the parameter interval. The curve (4) is clearly a circle-like one. Fig. 1 (left) shows the curve (4) and the circular arc, and Table 1 gives numerical evidence of the approximation error measured as a parametric distance between the curve (4) and its geometric interpolant.

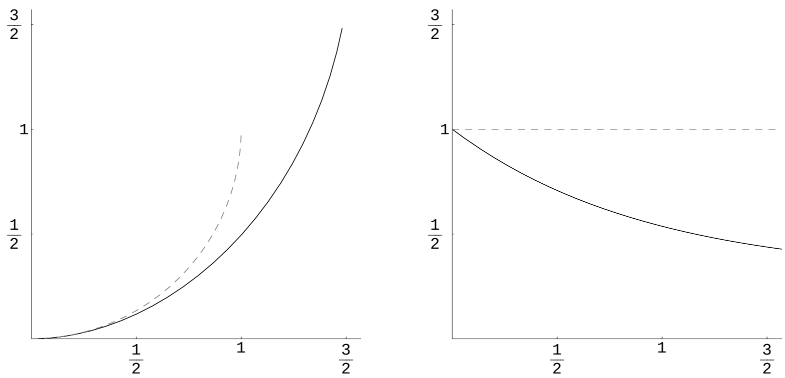

Fig. 1. The exponential spiral (4) and the circular arc (left) and their curvatures (right).

Table 1
The error of geometric interpolation of the exponential spiral (4).

Interval	Approximation error		Decay exponent	
	$n = 6$	$n = 7$	$n = 6$	$n = 7$
$[0, \frac{\pi}{2}]$	1.7783×10^{-11}	2.4704×10^{-13}	—	—
$[0, \frac{7\pi}{16}]$	2.9789×10^{-11}	3.6024×10^{-14}	3.86	- 14.42
$[0, \frac{6\pi}{16}]$	4.3754×10^{-12}	3.9342×10^{-15}	- 12.44	- 14.37
$[0, \frac{5\pi}{16}]$	4.5808×10^{-13}	2.8953×10^{-16}	- 12.38	- 14.31
$[0, \frac{4\pi}{16}]$	2.9377×10^{-14}	2.4957×10^{-18}	- 12.31	- 21.30
$[0, \frac{3\pi}{16}]$	8.6811×10^{-16}	4.4201×10^{-20}	- 12.24	- 14.02
$[0, \frac{2\pi}{16}]$	6.2403×10^{-18}	6.5502×10^{-22}	- 12.17	- 10.39
$[0, \frac{\pi}{16}]$	1.4208×10^{-21}	3.7763×10^{-26}	- 12.10	- 14.08

A simple error analysis indicates that the asymptotic approximation order is $\mathcal{O}(h^{2n})$, at least for $n = 6$, 7, and suggests the following claim.

Theorem 1 *Let $\boldsymbol{\mathcal{A}} : [0, h] \mapsto \mathbb{R}^2$ be a circular arc, parameterized by the arc-length. There exist positive constants M and h_0 with $h_0 \leq h$, such that for any $h_1 \leq h_0$, any circle-like curve $\boldsymbol{f} = \boldsymbol{\mathcal{A}} + \boldsymbol{g} \in \mathbb{F}_M$ can be geometrically interpolated by a polynomial parametric curve of degree $\leq\ n$ at $2n$ distinct points $\boldsymbol{f}(s_i)$, $s_i \in [0, h_1]$. The asymptotic approximation order is optimal, i.e., equal to $2n$.*

The proof of Theorem 1 is based upon [1] and two simple, but not quite obvious, observations. The first is the fact that one can always find two nonconstant polynomials $x_n,\ y_n \in \mathbb{R}[t]$ of degree $\leq n$ such that

$$x_n^2(t) + y_n^2(t) = 1 + t^{2n}, \quad x_n(0) = 0, \tag{5}$$

and the second is stated in Theorem 3. Let

$$z_n(t) := x_n^2(t) + y_n^2(t) - \left(1 + t^{2n}\right). \tag{6}$$

The relation (5) can also be considered as a system of nonlinear equations for the coefficients of the polynomials

$$x_n(t) = \sum_{j=1}^{n} \alpha_j\, t^j, \quad y_n(t) = \sum_{j=0}^{n} \beta_j\, t^j, \tag{7}$$

i.e.,

$$\frac{d^j}{dt^j} z_n(t)\Big|_{t=0} = 0, \quad j = 0, 1, \ldots, 2n. \tag{8}$$

The importance of equation (5) has already been noted in [10] considering a slightly different approximation problem, and the existence of a solution has been established for odd n. However, equation (5) has at least one real solution for all $n \in \mathbb{N}$. It is based upon a particular rational parameterization of the circle,

$$\frac{1}{1 - 2\,c\,t + t^2} \begin{pmatrix} 2\sqrt{1-c^2}\,t\ (1 - c\,t) \\ 1 - 2\,c\,t + (2\,c^2 - 1)\,t^2 \end{pmatrix}, \qquad c \in [0, 1), \quad t \in (-\infty, \infty). \tag{9}$$

The polynomials (7) depend heavily on the degree n of the interpolating curve, and throughout the paper it will be assumed that the integers $n,\ k$, and r are related as

$$n = 2^k\,(2\,r - 1), \quad k \geq 0, \quad r \geq 1. \tag{10}$$

Each k determines a family of polynomials that satisfy (5), and $k = 0$ is a very particular case. In addition, the coefficients of the polynomials x_n and y_n can be given in closed form with the help of Chebyshev polynomials of the first and the second kind, T_n and U_n.

Theorem 2 *Suppose that $n,\ k$, and r satisfy (10), and let the constants $c_k,\ s_k$ be given as*

$$c_k := \cos\left(\frac{\pi}{2^{k+1}}\right), \quad s_k := \sin\left(\frac{\pi}{2^{k+1}}\right). \tag{11}$$

Further, suppose that q_i are polynomials of degree ≤ 2, defined as

$$\begin{aligned} q_0(t) &:= q_0(t;k) := 1 - 2\,c_k\,t + t^2, \\ q_1(t) &:= q_1(t;k) := 2\,s_k\,t\ (1 - c_k\,t), \\ q_2(t) &:= q_2(t;k) := 1 - 2\,c_k\,t + (2\,c_k^2 - 1)\,t^2. \end{aligned} \tag{12}$$

Then, the functions x_n and y_n, defined by

$$\begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix} := \frac{1}{q_0(t)} \begin{pmatrix} 1 & (-1)^r \, t^n \\ -(-1)^r \, t^n & 1 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}, \tag{13}$$

are polynomials of degree $\leq n$ that satisfy (5)*. Furthermore, their coefficients are given as*

$$\alpha_j = 2\, s_k \cos\left((j-1)\frac{\pi}{2^{k+1}}\right) = 2\, s_k\, T_{j-1}(c_k), \quad j = 1, 2, \ldots, n-1, \tag{14}$$

$$\alpha_n = 2\, s_k \cos\left((n-1)\frac{\pi}{2^{k+1}}\right) + (-1)^r = 2\, s_k\, T_{n-1}(c_k) + (-1)^r, \tag{15}$$

and

$$\beta_0 = 1,\ \beta_1 = 0, \tag{16}$$

$$\beta_j = -2\, s_k \sin\left((j-1)\frac{\pi}{2^{k+1}}\right) = -2\, s_k^2\, U_{j-2}(c_k), \quad j = 2, 3, \ldots, n. \tag{17}$$

Theorem 2 actually proves the optimal approximation order for the circular arcs as studied in [10]. But Theorem 1 extends the conclusion to Lagrange interpolation of circular arcs and of the circle-like curves of degree n. The simplest way to confirm this is to prove that the Jacobian of the system of equations (8) with respect to the variables α_j, β_j at values provided by Theorem 2 is nonsingular, and to apply the Implicit Function Theorem. A surprisingly simple closed form of the determinant of the Jacobian that confirms this fact is given in the next theorem.

Theorem 3 *With n, k, and r as in* (10)*, and α_j, β_j given by Theorem 2, the determinant of the Jacobian of the system* (8) *is*

$$\det J = (-1)^{nr+1}\, 2^{2n+1}\, n^2\, s_k^2.$$

The asymptotic conclusion of Theorem 1 seems to be rather pessimistic since the parameter interval is supposed to be small. As an impetus, consider the approximation of a complete circle. A quick numerical test in Table 2 shows that the circle

$$\begin{pmatrix} \sin s \\ \cos s \end{pmatrix}, \quad s \in [-\pi, \pi], \tag{18}$$

can be geometrically interpolated at points corresponding to the parameter values

$$-\pi + \frac{2\pi}{2n-1}\ell, \quad \ell = 0, 1, \ldots, 2n-1,$$

by a polynomial curve of small degree n quite accurately. The error seems to

Table 2
The error of geometric interpolation of the circle measured as radial distance.

n	Approximation error	τ	n	Approximation error	τ
3	2.85951×10^{-1}	—	10	7.28389×10^{-11}	-1.21
4	2.32476×10^{-2}	-1.11	11	1.14441×10^{-12}	-1.24
5	2.08441×10^{-3}	-0.96	12	1.49890×10^{-14}	-1.26
6	1.22589×10^{-4}	-1.05	13	1.66223×10^{-16}	-1.28
7	5.09328×10^{-6}	-1.11	14	1.58128×10^{-18}	-1.29
8	1.57805×10^{-7}	-1.15	15	1.30483×10^{-20}	-1.30
9	3.79252×10^{-9}	-1.19	16	9.42975×10^{-23}	-1.31

decrease exponentially with n, like $\mathcal{O}(n^{\tau n}), \tau \approx -1.30$. Fig. 2 shows that the curvatures of geometric interpolants are close to 1 (the curvature of the circle). As can be seen from Table 3, they approach 1 with raising n.

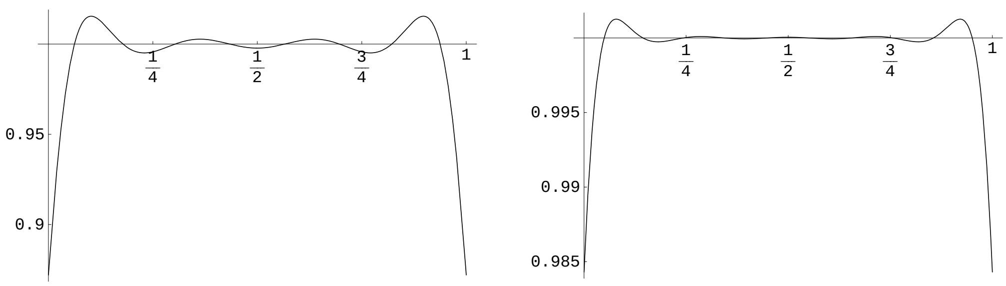

Fig. 2. The curvatures of geometric interpolants of the complete circle for $n = 5, 6$.

Table 3
Maximal deviation from 1 of the curvatures κ_n of degree n geometric interpolants of the complete circle.

n	3	4	5	6	7	8
$\|1 - \kappa_n\|_{\infty,[0,1]}$	1.59739	0.48393	0.12798	0.01569	0.00110	0.00005

The outline of the paper is as follows. In Section 2 the geometric interpolation problem is precisely defined. Section 3 provides the system of nonlinear equations that has to be studied. The last section gives the proofs of theorems, listed in the introduction.

2 Interpolation problem

Let us recall the geometric interpolation problem in its simplest, Lagrange form. Suppose that a sequence of $2n$ points $\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_{2n-1}$, $\boldsymbol{T}_j \neq \boldsymbol{T}_{j+1}$, in the plane $\mathbb{R}^2$ is given. Find a parametric polynomial curve

$$\boldsymbol{P}_n : [0,1] \to \mathbb{R}^2$$

of degree $\leq n$ that interpolates the given points at some values $t_\ell \in [0,1]$ in an increasing order, i.e.,

$$\boldsymbol{P}_n(t_j) = \boldsymbol{T}_j, \quad j = 0, 1, \ldots, 2n-1. \tag{19}$$

One can assume $t_0 := 0$ and $t_{2n-1} := 1$, but the remaining parameters

$$\boldsymbol{t} := (t_\ell)_{\ell=1}^{2n-2}$$

are unknown, ordered as

$$t_0 = 0 < t_1 < \cdots < t_{2n-2} < t_{2n-1} = 1. \tag{20}$$

The system of equations (19) should determine the unknown $\boldsymbol{P}_n$ as well as the parameters $\boldsymbol{t}$. But the two tasks can be separated as in [1] and the nonlinear system for the unknown parameters written as

$$\sum_{\ell=j-1}^{n+j} \frac{1}{\prod\limits_{\substack{m=j-1\\ m\neq \ell}}^{n+j} (t_\ell - t_m)} \cdot \boldsymbol{T}_\ell = 0, \quad j = 1, 2, \ldots, n-1. \tag{21}$$

Once the parameters (20) are determined, it is straightforward to obtain the polynomial curve $\boldsymbol{P}_n$. One only has to take any $n+1$ distinct interpolating conditions in (19), and apply any standard interpolation scheme to $\boldsymbol{P}_n$ componentwise.

Here, the data points $\boldsymbol{T}_j$ are sampled from a smooth circle-like curve $\boldsymbol{f} = \boldsymbol{\mathcal{A}} + \boldsymbol{g} : [0,h] \to \mathbb{R}^2$. Since affine transformations of data points do not change the equations (21), one can place the origin of a coordinate system at $\boldsymbol{f}(0) = \boldsymbol{\mathcal{A}}(0)$, and choose $\boldsymbol{\mathcal{A}}$ to be the unit circle, centered at $(0,1)^T$. Thus $\boldsymbol{\mathcal{A}}(t) := (\sin t, 1-\cos t)^T$. But then the expansion (2) implies $\boldsymbol{f}'(0) = \boldsymbol{\mathcal{A}}'(0) = (1,0)^T$. Therefore, for h small enough, $\boldsymbol{f} = (f_i)_{i=1}^2$ can be reparameterized as

$$\boldsymbol{f}(s) := \begin{pmatrix} s \\ u(s) \end{pmatrix} := \begin{pmatrix} s \\ \alpha(s) + \gamma(s) \end{pmatrix}, \tag{22}$$

where

$$\alpha(s) := 1 - \sqrt{1-s^2}$$

is a circular arc, parameterized by the first component, and

$$\gamma(s) = f_2\left(f_1^{-1}(s)\right) - \alpha(s) = \frac{\gamma''(0)}{2!}s^2 + \frac{\gamma^{(3)}(0)}{3!}s^3 + \dots. \tag{23}$$

The coefficients $\gamma^{(i)}(0)$ in (23) are polynomials in the components of $\boldsymbol{g}^{(r)}(0)$, but with the constant term equal to 0. Indeed, if $\nu(s) := f_1^{-1}(s) - \arcsin s$, then obviously $\nu(0) = 0$, $\nu'(0) = 0$, since

$$f_1^{-1}(0) = 0, \quad \frac{d}{ds}f_1^{-1}(s)\bigg|_{s=0} = \frac{1}{f_1'(0)} = 1.$$

Thus we obtain

$$\begin{aligned}\gamma(s) &= 1 - \cos(f_1^{-1}(s)) + g_2(f_1^{-1}(s)) - \alpha(s)\\ &= \sqrt{1-s^2}\,(1-\cos(\nu(s))) + s\sin(\nu(s)) + g_2(f_1^{-1}(s))\\ &= \frac{1}{2!}\,g_2''(0)s^2 + \left(\frac{1}{2!}\,(1+g_2''(0))\,\nu''(0) + \frac{1}{3!}g_2^{(3)}(0)\right)s^3 + \dots,\end{aligned}$$

and the claim will be confirmed if $\nu^{(i)}(0)$ are polynomials in $g_1^{(r)}(0)$ without the constant term. This fact could be formally verified by an application of Faa di Bruno's formula to the implicit definition of ν, i.e., $f_1(\nu(s) + \arcsin s) - s = 0$ and the induction, but the following expansion is even more convincing,

$$\begin{aligned}0 = f_1(\nu(s) + \arcsin s) - s =& \frac{1}{2!}\,(g_1''(0) + \nu''(0))\,s^2\\ &+ \frac{1}{3!}\left(g_1^{(3)}(0) + 3g_1''(0)\nu''(0) + \nu^{(3)}(0)\right)s^3 + \dots\end{aligned}$$

So one can find a bound $c(M)$,

$$|\gamma^{(i)}(0)| \le c(M), \quad i = 2, 3, \dots, 2n-1,$$

depending only on M that was introduced in (3). This bound can be chosen as a nondecreasing continuous function of M, starting with $c(0) = 0$, since $\boldsymbol{g} \equiv \mathbf{0}$ implies $\gamma = 0$. This proves the following lemma.

Lemma 4 *With a proper choice of M, a circle-like curve $\boldsymbol{f} \in \mathbb{F}_M$ has the correction γ and its derivatives arbitrary small.*

Now, since $\boldsymbol{f}$ is of the form (22), the expansion (23) and assumptions (1) imply

$$u(0) = u'(0) = 0, \quad u''(0) > 0.$$

With this assumption, a careful, but technically quite tedious analysis carried out in [1] shows that the asymptotic existence of the solution of the interpolation problem (19) in general is equivalent to a fact that a certain system of nonlinear equations

$$C_{n+j}(\boldsymbol{a}) + \mathcal{O}(h) = 0, \quad j = 1, 2, \ldots, n-1, \tag{24}$$

has a real solution $\boldsymbol{a} := (a_\ell)_{\ell=1}^{n-1}$ for all h small enough.

3 Nonlinear equations

For circle-like curves (22), the functions C_{n+j} ([1], Thm. 4.5) simplify to

$$C_{n+j}(\boldsymbol{a}) := \frac{1}{(n+j)!} \frac{d^{n+j}}{dt^{n+j}} \left(\alpha \left(t + \sum_{\ell=1}^{n-1} a_\ell \, t^{\ell+1} \right) + \gamma \left(t + \sum_{\ell=1}^{n-1} a_\ell \, t^{\ell+1} \right) \right) \Bigg|_{t=0}.$$

Further discussion will prove that the system

$$\frac{1}{(n+j)!} \frac{d^{n+j}}{dt^{n+j}} \left(\alpha \left(t + \sum_{\ell=1}^{n-1} a_\ell \, t^{\ell+1} \right) \right) \Bigg|_{t=0} = 0, \quad j = 1, 2, \ldots, n-1, \tag{25}$$

has a real solution and the Jacobian at that solution is nonsingular. So the Implicit Function Theorem implies the existence of constant $c(M)$ for M small enough such that the equations

$$C_{n+j}(\boldsymbol{a}) = 0, \quad j = 1, 2, \ldots, n-1,$$

for circle-like curves that satisfy (3) for this particular M also have a real solution with a nonsingular Jacobian, by Lemma 4. But then, again by the Implicit Function Theorem, the system (24) has a real solution for h small enough too, and Theorem 1 is confirmed.

We are thus left to show the existence of the solution of (25) and to prove the nonsingularity of the Jacobian. The expansion

$$1 - \alpha \left(t + \sum_{\ell=1}^{n-1} a_\ell \, t^{\ell+1} \right) = \sqrt{1 - \left(t + \sum_{\ell=1}^{n-1} a_\ell \, t^{\ell+1} \right)^2} =: 1 + \sum_{\ell=1}^{\infty} b_\ell \, t^\ell \tag{26}$$

yields

$$\left(t + \sum_{\ell=1}^{n-1} a_\ell \, t^{\ell+1} \right)^2 + \left(1 + \sum_{\ell=1}^{\infty} b_\ell \, t^\ell \right)^2 = 1. \tag{27}$$

Since the equations (25) for $h \to 0$ are equivalent to the fact that the expansion (26) does not contain the powers $n+1, n+2, \ldots, 2n-1$, the relation (27) implies

$$\left(t + \sum_{\ell=1}^{n-1} a_\ell \, t^{\ell+1} \right)^2 + \left(1 + \sum_{\ell=1}^{n} b_\ell \, t^\ell \right)^2 = 1 + (a_{n-1}^2 + b_n^2) \, t^{2n}. \tag{28}$$

One is now left with $2n-1$ equations for $2n-1$ unknowns $\boldsymbol{a}$ and $\boldsymbol{b} := (b_\ell)_{\ell=1}^n$. Once $\boldsymbol{a}$ is obtained, linear relations determine $\boldsymbol{b}$ and vice-versa.

Let
$$\alpha_1 := \frac{1}{\sqrt[2n]{a_{n-1}^2 + b_n^2}}.$$
The regular reparameterization $t \to \alpha_1 \cdot t$, and new variables
$$\begin{aligned} &\alpha_j := (\alpha_1)^j \ a_{j-1}, \quad j = 2, 3, \ldots, n, \\ &\beta_0 := 1, \quad \beta_j := (\alpha_1)^j \ b_j, \quad j = 1, 2, \ldots, n, \end{aligned}$$
simplify (28) to the relation (5), familiar from the introduction. Therefore, the system (25) in the limit as $h \to 0$ is equivalent to the system (8).

4 Proofs

From the previous discussion it is obvious that it suffices to prove Theorem 2 and Theorem 3 only. Theorem 1 then follows as a corollary. Consider the proof of Theorem 2 first. Equation (13) yields
$$x_n(t) = \frac{q_1(t) + (-1)^r\, t^n\, q_2(t)}{q_0(t)}, \quad y_n(t) = \frac{q_2(t) - (-1)^r\, t^n\, q_1(t)}{q_0(t)}, \tag{29}$$
where q_i, $i = 0, 1, 2$, are defined by (12). In order to verify that the function x_n is actually a polynomial of the form (7), by (29) it is sufficient to check that
$$\begin{aligned} q_0(t) \sum_{j=1}^n \alpha_j\, t^j = \alpha_1\, t + (\alpha_2 - 2\, c_k\, \alpha_1)\, t^2 + \sum_{j=3}^n (\alpha_j - 2\, c_k\, \alpha_{j-1} + \alpha_{j-2})\, t^j \\ + (-2\, c_k\, \alpha_n + \alpha_{n-1})\, t^{n+1} + \alpha_n\, t^{n+2} = q_1(t) + (-1)^r\, t^n\, q_2(t). \end{aligned}$$
A comparison of the coefficients implies the linear recurrence
$$\alpha_1 = 2s_k, \ \alpha_2 = c_k\, \alpha_1, \ \alpha_j - 2\, c_k\, \alpha_{j-1} + \alpha_{j-2} = 0, \quad j = 3, 4, \ldots n-1, \tag{30}$$
with additional conditions
$$\begin{aligned} \alpha_n - 2\, c_k\, \alpha_{n-1} + \alpha_{n-2} =& (-1)^r, \\ 2\, c_k\, \alpha_n - \alpha_{n-1} =& (-1)^r\, 2\, c_k, \\ \alpha_n =& (-1)^r\, (2\, c_k^2 - 1). \end{aligned} \tag{31}$$
A straightforward calculation confirms that (14) and (15) give a solution of (30) and (31). The proof for the function y_n is similar and will be omitted.

Since by Theorem 2 the function (6) vanishes identically, the limit solution of the system of equations (24) is obtained. The existence of a real solution is verified if the Jacobian at the limit solution is nonsingular. From

$$\begin{aligned}
\frac{\partial}{\partial \alpha_j} z_n(t) &= \frac{\partial}{\partial \alpha_j} x_n^2(t) = 2\, t^j\, x_n(t), \quad j = 1, 2, \ldots, n, \\
\frac{\partial}{\partial \beta_j} z_n(t) &= \frac{\partial}{\partial \beta_j} y_n^2(t) = 2\, t^j\, y_n(t), \quad j = 0, 1, \ldots, n,
\end{aligned}$$

it is straightforward to compute the Jacobian $J := 2\, D$, where

$$D := \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 & \beta_0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & \beta_1 & \beta_0 & 0 & & \vdots & 0 \\
\alpha_1 & 0 & \ddots & & \vdots & \vdots & \beta_1 & \beta_0 & \ddots & \vdots & \vdots \\
\alpha_2 & \alpha_1 & \ddots & \ddots & \vdots & \vdots & \vdots & \beta_1 & \ddots & 0 & \vdots \\
\vdots & \alpha_2 & \ddots & 0 & 0 & \vdots & \vdots & \vdots & \ddots & \beta_0 & 0 \\
\vdots & \vdots & \ddots & \alpha_1 & 0 & \beta_n & \vdots & \vdots & \ddots & \beta_1 & \beta_0 \\
\alpha_n & \vdots & & \alpha_2 & \alpha_1 & 0 & \beta_n & \vdots & & \vdots & \beta_1 \\
0 & \alpha_n & & \vdots & \alpha_2 & \vdots & 0 & \beta_n & & \vdots & \vdots \\
\vdots & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \alpha_n & \vdots & \vdots & \vdots & \vdots & \ddots & \beta_n & \vdots \\
0 & 0 & \cdots & 0 & \alpha_n & 0 & 0 & 0 & \cdots & 0 & \beta_n
\end{pmatrix}.$$

Unfortunately, obtaining the explicit formula for $\det D$ is not an easy task, since its entries are given by (14)–(17). But the columns of D are simply the shifts of the coefficients of x_n and y_n, which leads to the following observation. If

$$u_0 := 0, u_1, u_2, \ldots, u_{2n} \in \mathbb{C}$$

are $2n+1$ pairwise distinct values, and

$$V(u_0, u_1, \ldots, u_{2n}) := \left(u_{j-1}^{\ell-1}\right)_{j,\ell=1}^{2n+1}$$

is the corresponding Vandermonde matrix, then the rows of the product VD are given by

$$u_j\, x_n(u_j),\ u_j^2\, x_n(u_j), \ldots,\ u_j^n\, x_n(u_j),\ y_n(u_j),\ u_j\, y_n(u_j), \ldots,\ u_j^n\, y_n(u_j), \tag{32}$$

where $j = 0, 1, \ldots, 2\, n$. Now (5) suggests how to choose u_j, i.e., to define u_j as $2\, n$ different solutions of the equation

$$t^{2\, n} + 1 = 0. \tag{33}$$

Then (5) implies $x_n(u_j) = \pm i\, y_n(u_j)$. Here, and throughout the rest of the paper, i will denote the imaginary unit, i.e., $i^2 = -1$. If

$$\begin{aligned}
u_j &:= \exp\left((-1)^r \frac{i\,\pi}{2\,n}(4\,j-3)\right), \quad j = 1, 2\ldots, n, \quad j \neq j_0, \\
u_{j_0} &:= \exp\left(\frac{i\,\pi}{2\,n}(2\,r-1)\right), \\
u_{n+j} &:= u_j^{-1}, \quad j = 1, 2, \ldots, n,
\end{aligned}$$

where

$$j_0 = \begin{cases} \frac{2n-r+2}{2}, & r \text{ even}, \\ \frac{r+1}{2}, & r \text{ odd}, \end{cases}$$

then

$$\begin{aligned}
y_n(u_j) &= -i\, x_n(u_j), \quad j = 1, 2, \ldots, n, \\
y_n(u_j) &= i\, x_n(u_j), \quad j = n+1, n+2, \ldots, 2\,n,
\end{aligned} \tag{34}$$

as can easily be verified. Recall that $u_0 = 0$, $y_n(0) = 1$, and use (32) and (34) to see that

$$\det(V\, D) = i^n \prod_{j=1}^{2\,n} x_n(u_j) \det C,$$

where

$$C := \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
u_1 & u_1^2 & \cdots & u_1^n & -1 & -u_1 & \cdots & -u_1^n \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
u_n & u_n^2 & \cdots & u_n^n & -1 & -u_n & \cdots & -u_n^n \\
u_{n+1} & u_{n+1}^2 & \cdots & u_{n+1}^n & 1 & u_{n+1} & \cdots & u_{n+1}^n \\
u_{n+2} & u_{n+2}^2 & \cdots & u_{n+2}^n & 1 & u_{n+2} & \cdots & u_{n+2}^n \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
u_{2\,n} & u_{2\,n}^2 & \cdots & u_{2\,n}^n & 1 & u_{2\,n} & \cdots & u_{2\,n}^n
\end{pmatrix}.$$

Since $J = 2D$,

$$\det J = 2^{2\,n+1}\, i^n \prod_{j=1}^{2\,n} x_n(u_j) \frac{\det C}{\det V}. \tag{35}$$

Lemma 5 *If s_k is given by (11), then*

$$\prod_{j=1}^{2\,n} x_n(u_j) = n^4\, s_k^4.$$

PROOF. Some straightforward computation and (29) yield

$$x_n(u_j)x_n(u_{n+j}) = \frac{\sin^2\left(\frac{\pi}{4n}((-1)^r(4j-3)+2r-1)\right)}{\sin^2\left(\frac{\pi}{4n}((-1)^r(4j-3)-2r+1)\right)},\ j=1,2,\ldots,n,\ j\neq j_0,$$

and, by L'Hôspital rule,

$$x_n(u_{j_0})\,x_n(u_{n+j_0}) = n^2\,s_k^2. \tag{36}$$

The formulae (see [11], e.g.)

$$\begin{aligned}\prod_{j=1}^{n}\sin\left(\frac{\pi}{2n}(2j-2m-1)\right) &= \frac{(-1)^m}{2^{n-1}}, \quad m\in\mathbb{Z},\\ \prod_{\substack{j=1\\ j\neq m}}^{n}\sin\left(\frac{\pi}{2n}(2j-2m)\right) &= \frac{(-1)^{m+1}n}{2^{n-1}}, \quad 1\le m\le n,\end{aligned}$$

imply

$$\prod_{\substack{j=1\\ j\neq j_0,\, n+j_0}}^{2n} x_n(u_j) = n^2\,s_k^2,$$

which, together with (36), completes the proof of the lemma.

Lemma 6 *The quotient of determinants in (35) is*

$$\frac{\det C}{\det V} = \frac{(-1)^{n\,r+1}}{i^n\,n^2\,s_k^2}.$$

PROOF. The determinant of C can be reduced to

$$\det C = (-1)^n \prod_{j=1}^{2n} u_j\ \det\begin{pmatrix} V_1 & -V_1\\ V_2 & V_2\end{pmatrix},$$

where $V_1 := V(u_1,u_2,\ldots,u_n)$ and $V_2 := V(u_{n+1},u_{n+2},\ldots,u_{2n})$ are the corresponding Vandermonde matrices. Since u_j are the roots of (33), $\prod_{j=1}^{2n} u_j = 1$. Further, a simple columnwise reduction implies that

$$\det C = (-1)^n\,2^n \det V_1 \det V_2,$$

which finally gives

$$\frac{\det C}{\det V} = (-1)^n\,2^n \frac{1}{\prod_{\ell=1}^{n}\prod_{j=1}^{n}(u_{n+\ell}-u_j)}.$$

If

$$p_1 := \prod_{\substack{\ell=1\\ \ell\neq j_0}}^{n} \prod_{\substack{j=1\\ j\neq j_0}}^{n} (u_{n+\ell} - u_j),$$

$$p_2 := (u_{n+j_0} - u_{j_0}) \prod_{\substack{j=1\\ j\neq j_0}}^{n} (u_{n+j_0} - u_j) \prod_{\substack{\ell=1\\ \ell\neq j_0}}^{n} (u_{n+\ell} - u_{j_0}),$$

then obviously

$$\prod_{\ell=1}^{n} \prod_{j=1}^{n} (u_{n+\ell} - u_j) = p_1\, p_2.$$

A straightforward computation yields

$$p_1 = (-1)^{n\,r}\, i^{n+1}\, 2^{n-1}\, s_k,$$
$$p_2 = (-1)^{n}\, n^2\, 2\, i\, s_k,$$

thus

$$\frac{\det C}{\det V} = \frac{(-1)^{n\,r+1}}{i^n\, n^2\, s_k^2}.$$

Lemma 6 and (35) confirm the result of Theorem 3 which concludes this section.

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