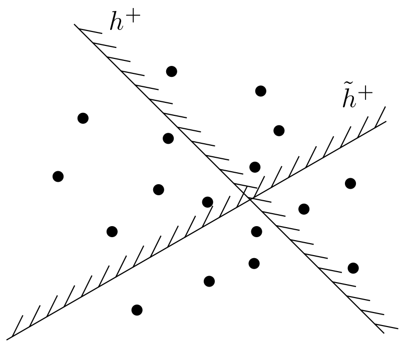
Contents

1 Motivation I

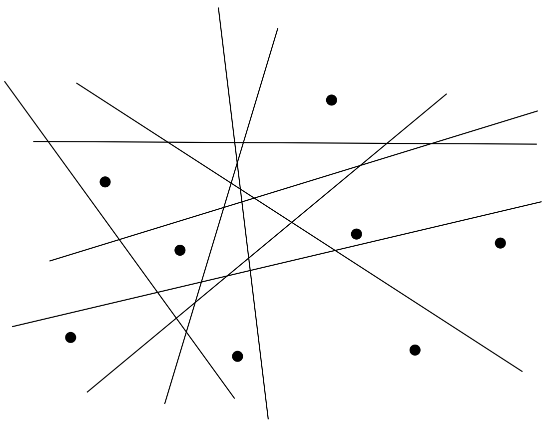
Let S be a set of n points.
Data structure for counting points in halfplanes:

$$h^+ = \{ax + by \geq c\} \longrightarrow |h^+ \cap S|$$



1 Motivation II

Given n points and n lines:
does any point belong to any line?



$$P_1 = (x_1, y_1) \qquad a_1x + b_1y = c_1$$
$$\dots \qquad \dots$$
$$P_n = (x_n, y_n) \qquad a_nx + b_ny = c_n$$

Are there i, j such that $a_jx_i + b_jy_i = c_j$?

2.a Crossing numbers

S set of n points in the plane.
$G = (S, E)$ a graph embedded with segments.
h a line.
Definition: The *crossing number of* h in G is

$$cr(h, G) := |h \cap E|.$$

The *crossing number* of G is

$$cr(G) := \max_h \{cr(h, G)\}.$$

So, no line intersects G more than $cr(G)$ times.

Problem: Find a spanning tree T of S with low crossing number.
Objective: Find T with $cr(T) = O(\sqrt{n})$

2.b Lower bound

Scenario with $cr(T) = \Omega(\sqrt{n})$ for all spanning trees T:

1. choose a set H of $\sqrt{2n}$ lines in general position;
2. k lines in general position determine $\frac{k^2+k+2}{2}$ regions;
3. regions determined by H: $\frac{\sqrt{2n}^2+\sqrt{2n}+2}{2} = n + \sqrt{n/2} + 1 \geq n$;
4. place n points, no two points sharing region;
5. fix a spanning tree T;
6. each edge of T crosses some line;
7. $|T \cap H| \geq n - 1$;
8. pigeonhole: $\exists$ line $h \in H$ with $\frac{n-1}{|H|} = \frac{n-1}{\sqrt{2n}} = \Theta(\sqrt{n})$ intersections.
9. $cr(T) = \Omega(\sqrt{n})$.

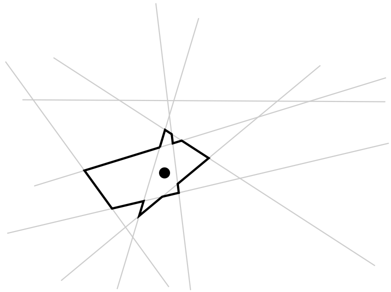
2.c Pseudodistance

- H a set of lines (general position)
- pseudodistance for $p, q \in \mathbb{R}^2$:

$$\delta_H(p,q) := |H \cap \bar{pq}|$$

- The idea: try to match edges crossing few 'important' lines
- disc of center p and radius r:

$$D_H(p,r) := \{h \cap h' | \delta_H(p, h \cap h') \leq r\}$$



Here $D_H(p,1)$ has 13 elements.

2.c Packing argument

Property: If $r \leq \frac{|H|}{2}$, then $|D_H(p,r)| \geq \binom{r+1}{2}$.
Proof:

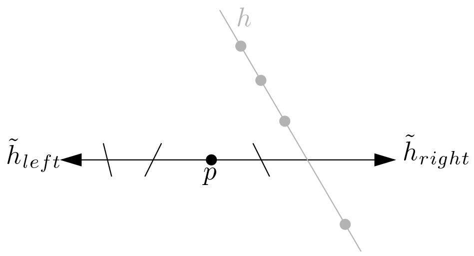
- line $\tilde{h}$ through p with

$$|\tilde{h}_{left} \cap H| = |\tilde{h}_{right} \cap H|$$

- by triangular inequality, $h \in H$ gives $r - (\delta_H(\tilde{h}_{left} \cap h)) + 1$ vertices to $D_H(p,r)$
- points that we have put in $D_H(p,r)$:

$$2(r + (r-1) + \cdots + 2 + 1) = 2\binom{r+1}{2}$$

- each point has been put ≤ 2 times.



2.c Packing argument

H set of lines. S set of n points.
Property: There are $p, q \in S$ such that

$$\delta_H(p,q) \leq \frac{2|H|}{\sqrt{n}}.$$

Proof: Discs of radius $\frac{|H|}{\sqrt{n}}$. Then

$$\sum_{p \in S} |D_H(p, \frac{|H|}{\sqrt{n}})| \geq n \binom{\frac{|H|}{\sqrt{n}} + 1}{2} > \binom{|H|}{2}$$

$\exists p, q \in S$ with $D_H(p, \frac{|H|}{\sqrt{n}}) \cap D_H(q, \frac{|H|}{\sqrt{n}}) \neq \emptyset$

Intuition:
Greedy algorithm: match closest pair.
Lines crossed by this spanning tree:

$$\sum_{i=n}^{2} \frac{2|H|}{\sqrt{i}} \leq 4|H|\sqrt{n}$$

(because $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$)
Average crossing of $h \in H$ is $4\sqrt{n}$...

2.c Weighted lines

What if line $h \in H$ has weight $w(h)$?

$$\delta_{H,w}(p,q) := \sum_{h \in H, h \cap \overline{pq} \neq \emptyset} w(h)$$

Let S as before and $W := \sum_{h \in H} w(h)$.
Property: There are $p, q \in S$ such that

$$\delta_{H,w}(p,q) \leq \frac{2W}{\sqrt{n}}.$$

Proof:

- $H_k :=$ replace $h \in H$ by $\lceil k \cdot w(h) \rceil$ lines

- H_k has $\leq kW + |H|$ lines
- perturb H_k to general position
- $\exists p, q \in S$ with $\delta_{H_k}(p,q) \leq \frac{2kW+2|H|}{\sqrt{n}}$
- $\delta_{H,w}(p,q) \leq \frac{\delta_{H_k}(p,q)}{k} = \frac{2W}{\sqrt{n}} + \frac{2|H|}{k\sqrt{n}} = \frac{2W}{\sqrt{n}} + \epsilon$

2.c Iterative reweighting

Theorem: Every set S of n points has a spanning tree T with

$$cr(T) < \frac{4}{\ln 2}\sqrt{n} + O(\log n).$$

Proof: This algorithm:

1. H a set of $\binom{n}{2}$ important lines
2. $w_0(h) := 1 \quad \forall h \in H$
3. $E := \emptyset$ and $S_0 := S$
4. for $i = 0$ to $n-2$
   (a) using distance δ_{H,w_i}, find 'closest pair' $p, q \in S_i$
   (b) $E := E \cap \{\overline{pq}\}$ and $S_{i+1} := S_i \setminus \{p\}$
   (c) $\forall h \in H$,
   if $h \cap \overline{pq} \neq \emptyset$ $w_{i+1}(h) := 2 * w_i(h)$
   otherwise $w_{i+1}(h) := w_i(h)$
5. return $T := (S, E)$ $\rightarrow$

2.c Iterative reweighting

(proof continuation) Analysis of algorithm
Let $W_i := \sum_{h \in H} w_i(h)$
In iteration i: $\delta_{H,w_i}(p,q) \leq \frac{2W_i}{\sqrt{n-i}}$
Then $W_{i+1} \leq W_i(1 + \frac{2}{\sqrt{n-i}})$, and

$$W_{n-1} \leq |H| \prod_{j=0}^{n-2} (1 + \frac{2}{\sqrt{n-j}})$$

(because $1 + x \leq e^x$, $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} < 2\sqrt{n}$)

$$\leq n^2 e^{\sum_{j=0}^{n-2} \frac{2}{\sqrt{n-j}}} \leq e^{4\sqrt{n} + 2\ln n}$$

The result follows because

$$W_{n-1} = \sum_{h \in H} 2^{cr(h,T)}$$

and then for all $h \in H$

$$cr(h,T) < \frac{4\sqrt{n} + 2\ln n}{\ln 2}.$$

2.c Bonus

Remark on the constant: If we reweight with $1 + \mu$ ($\mu > 0$) instead of 2:

$$cr(T_\mu) < \frac{1}{\ln(1+\mu)}(4\mu\sqrt{n} + 2\ln n)$$

With optimal μ:

$$cr(T) < 4\sqrt{n} + O(n^{1/4}\sqrt{\log n})$$

With better tools: $cr(T) \leq (2.31 + o(1))\sqrt{n}$
Further remarks: Construct T with optimal crossing number:

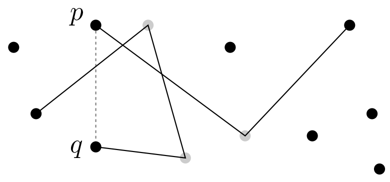
???

Nothing known...

2.c About paths

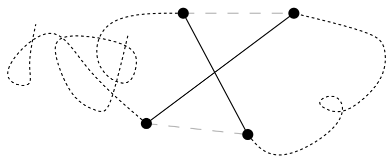
By the same price: simple spanning paths.

- Maintain S_i as extremes of connected components of $E \Rightarrow$ Non-simple spanning path.



p, q do not stay in S_{i+1}.

- Remove one intersection. Iterate



It keeps or reduces crossing number.
Reduces euclidean distance $\Rightarrow$ Converges

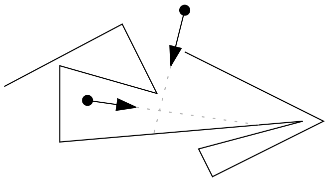
2.d Halfplane queries

$S = \{p_1, \ldots, p_n\}$ a set of points.
Each $p_i \in S$ with weight $w(p_i)$.
DS: for a query halfplane h^+, report

$$\sum_{p_i \in S \cap h^+} w(p_i)$$

- reordering, simple spanning path $P = p_1, p_2, \ldots, p_n$ with $cr(P) = O(\sqrt{n})$
- Accumulative weight: $W(p_i) := \sum_{j=1}^{i} w(p_j)$
- ray shooting DS for path P

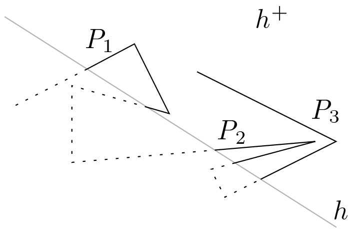
Theorem [static ray shooting]: For a simple path with n edges, there is a DS of size $O(n)$ with $O(\log n)$ time for ray shooting queries.



2.d Halfplane queries

For a halfplane h^+:

- find $P \cap h$ in $O(\sqrt{n} \log n)$
- we have $P_1, \ldots, P_j$, $j = O(\sqrt{n})$ subpaths inside h^+



- if $P_i = p_{i_1}, \ldots, p_{i_k}$, then its weight is

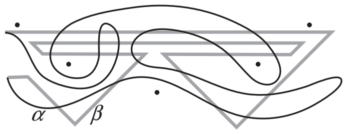
$$w(P_i) = W(p_{i_k}) - W(p_{i_1})$$

- report $\sum_{i=1}^{j} w(P_i)$ in $O(\sqrt{n})$ operations

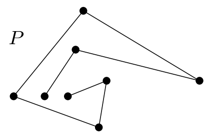
Theorem: Given n points, DS of size $O(n)$ answering in $O(\sqrt{n} \log n)$ time both halfplane and triangle (!!) queries.
(triangle $\equiv \cap$ 3 halfplane)

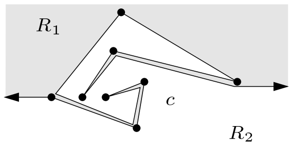
2.e Homotopic paths

$S = \{p_1, \ldots, p_n\}$ a set of points.
Are curves α and β homotopic in $\mathbb{R}^2 \setminus S$?



- take a simple spanning path P with $cr(P) = O(\sqrt{n})$
- doubling edges of P, find a simple 'path' c of size $\leq 2n$, going through S, and dividing $\mathbb{R}^2 \setminus S$ into simply connected regions R_1, R_2





(continuation)

- $cr(c) = O(\sqrt{n})$, and if $c \equiv s_1, s_2, \ldots, s_j$, then $j = O(n)$ (s_i a segment)
- gluing infinite copies of R_1, R_2 through copies of s_i, we make the universal cover of $\mathbb{R}^2 \setminus S$
- the fundamental group of $\mathbb{R}^2 \setminus S$ is 'similar' to

$$\pi \equiv \{s_i \mid s_i^2 = 1\}$$

- the expression of a curve α in π has $O(|\alpha|\sqrt{n})$ elements
- algorithm to find intersections I among n segments takes $O(|I| + n \log n)$

Theorem: In time $O((|\alpha| + |\beta|)\sqrt{n} + n \log n)$ we can test if α and β are homotopic in $\mathbb{R}^2 \setminus S$.

Remark: if α, β are simple, it can be done better!!

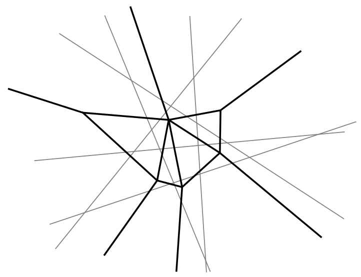
3.a Cuttings

H a set of n planes and $\Sigma = \{\Delta_1, \ldots, \Delta_k\}$ a disjoint decomposition of $\mathbb{R}^2$ into simplices. For $\Delta \in \Sigma$, let

$$L(\Delta) := \{h \in H \mid h \cap \Delta \neq \emptyset\}.$$

Definition: Σ is a $\frac{1}{r}$−cutting for H iff

$$|L(\Delta)| \leq \frac{|H|}{r} = \frac{n}{r} \qquad \forall \Delta \in \Sigma$$



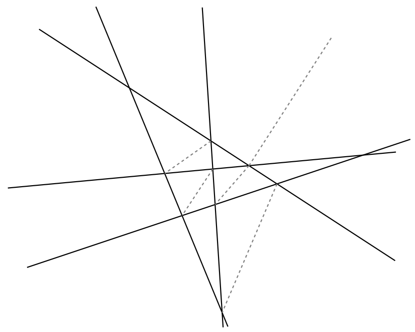
A $(1/2)$−cutting of size ten for a six lines.

The size of Σ is critical for applications.

3.b Randomized construction

Lemma: Given n lines, there is a $\frac{\ln r}{r}$−cutting of size $O(r^2)$.
Proof:

- Take random $R \subset H$ with $|R| = r$
- $\Sigma \equiv$ canonical triangulation arrangement R



- $|\Sigma| = O(r^2)$ by Euler
- $\mathrm{E}_{\Delta \in \Sigma}[|L(\Delta)|] = \frac{n}{r}$ is weak result

3.b Randomized construction

(proof continuation) We will see

$$\Pr\Big[\max_{\Delta \in \Sigma} |L(\Delta)| < O(\frac{n \ln r}{r})\Big] > \frac{1}{2}$$

If $p(\Delta) := \Pr[\text{endpoints of } \Delta \text{determined by } R]$,

$$\Rightarrow \sum_{\text{possible } \Delta} p(\Delta) = O(r^6)$$

If $q(\Delta) := \Pr[\Delta \text{not splited by } R]$,

$$\Rightarrow q(\Delta) \leq \prod_{j=1}^{r-6} (1 - \frac{|L(\Delta)|}{n-j})$$

$$\leq \big(1 - \frac{|L(\Delta)|}{n}\big)^r \leq e^{-\frac{|L(\Delta)| r}{n}}$$

$\Pr\big[\exists \Delta \in \Sigma \text{ s.t. } |L(\Delta)| > \frac{cn \ln r}{r}\big]$ bounded by

$$\sum_{|L(\Delta)| > \frac{cn \ln r}{r}} p(\Delta) q(\Delta) \leq \sum_{|L(\Delta)| > \frac{cn \ln r}{r}} p(\Delta) r^{-c} =$$

$$= r^{-c} \cdot O(r^6) \qquad \big(< \frac{1}{2} \text{ if } c \text{ big enough}\big)$$

3.b Randomized construction

Lemma: Given n lines, there is a $\frac{1}{r}$–cutting of size $O(r^2)$.
Proof: Compute Σ as before and:

- $\forall \Delta \in \Sigma$, compute $s(\Delta) := \frac{|L(\Delta)|r}{n}$
- further decompose Δ with random sample $R(\Delta) \subset L(\Delta), |R(\Delta)| = k \cdot s(\Delta) \ln s(\Delta)$
- again canonical triangulation $\Rightarrow \Sigma'$
- k big, $\Sigma' \ni \Delta' \subset \Delta \in \Sigma$, high probability of
$$|L(\Delta')| = O\big(\frac{|L(\Delta)| \ln |R(\Delta)|}{|R(\Delta)|}\big) \leq \frac{n}{r}$$
- the expected $|\Sigma'|$ is
$$O\Bigg(\sum_{\Delta\in\Sigma} |R(\Delta)|^2\Bigg) = O\Bigg(\sum_{\Delta\in\Sigma} \frac{r^4|L(\Delta)|^4}{n^4}\Bigg)$$
- Applying next result we are done
Th: $\sum_{\Delta\in\Sigma} |L(\Delta)|^k = \frac{n^k}{r^k} O(r^2)$

3.c Deterministic algorithms

Our results give a randomized algorithm in expected $O(nr + n\log n)$ time.

Iterate (random sample + check)

Remark: We have to be careful when composing cuttings: the constant blows up if iteratively compute $\frac{1}{r^k}$–cutting

$\frac{1}{r^j}$ – cutting $\rightarrow$ $\frac{1}{r^{j+1}}$ – cutting

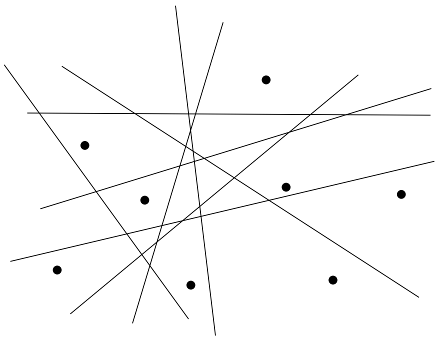
Theorem: For n lines, given r, there is $\rho > 0$ and sequence $\Sigma_1, \ldots \Sigma_k$:

- Σ_{i+1} is refinement of Σ_i
- Σ_i is $\frac{1}{\rho^i}$–cutting
- $k = \log r$
- computed in $O(nr)$ deterministic time

Theorem: If $r \leq n^{1/3}$, we can do in $O(n\log n)$. Quicker than to check a candidate to cutting!! Not known how to do it when $r > n^{1/3}$.

3.d Hopcroft's problem

Given S (n points) and H (n lines): does any point belong to any line?



Approach:

- divide in regions
- place points in subdivision
- group points in same regions
- consider each group separately

3.d Hopcroft's problem

- Compute a $\frac{1}{n}$–cutting $\Sigma_1, \ldots, \Sigma_{O(\log n)}$. Point location in arrangement of H takes $O(\log n)$ time
- n lines can be stored in $O(n^2)$ size s.t. in $O(\log n)$ we can decide if query point $p \in \mathbb{R}^2$ lies in some line
- Dual: t points stored in $O(t^2)$ size s.t. query incidence with line takes $O(\log t)$.

Theorem: Hopcroft's problem is solvable in $O(n^{4/3}\log n)$ time.
Proof:

1. compute $\frac{1}{n^{1/3}}$–cutting Σ of H
2. place each point in Σ
3. groups S_i of $n^{1/3}$ points in the same $\Delta \in \Sigma$
4. with duality, check if any $h \in L(\Delta)$ lies on some point of S_i, for each group $S_i \subset L(\Delta)$.

3.e Spanning trees

S a set of n points. Snapshot of construction:

1. consider Σ, a $\frac{1}{\sqrt{n}}$−cutting of its dual
2. select one point in each region of Σ
3. its dual, H, has $O(n)$ lines
4. we have, for any spanning path T,

$$cr(T) = O(\max_{h \in H} cr(h, T))$$

5. consider cuttings of weighted lines
6. using them, we can find 'closest pairs'
7. paying in the crossing constants, we find several 'close pairs' to match

Theorem: We can construct a spanning tree T with $cr(T) := O(\sqrt{n})$ in time $O(n^{3/2})$.

This can be improved...

4. Closing

- results from the nineties
- reweighting technique (Erdös)
- spanning tree of n points with $cr = O(\sqrt{n})$
- range searching in 2D
- homotopic paths in $\mathbb{R}^2 \setminus S$
- randomized techniques for algorithms
- different divide and conquer in the plane
- Hopcroft's problem
- most things generalize to higher dimension (exponents change)
- some problems are still open