THE MERCEDES KNOT PROBLEM

Aleksandar Jurišić

IMFM, Univerza v Ljubljani

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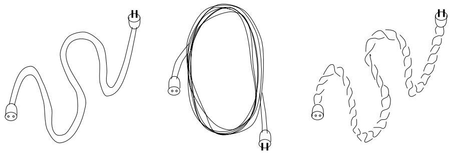
Have you ever coiled a long extension cord in order to store it.

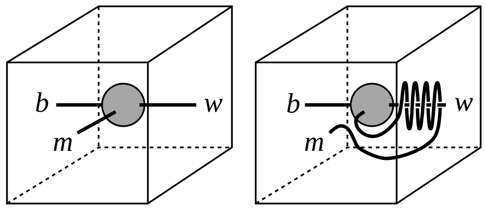
After you unwind it, you may notice that it is twisted many times.

Can you explain this ‘phenomena’?



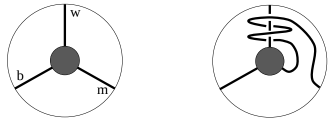
DEMONSTRATION!!!

The elastic bands can be moved in any way but cannot be torn and their ends are fixed.

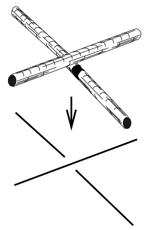


The Mercedes knot problem:

Is it possible to move elastic bands so that m becomes wrapped several times around w without moving the ball?



This Figure explains our choice of name for the problem.



OUR PLAN

- translation, partial solution
- invariants
  - **(a)** in 3-dim space
  - **(b)** in hollow ball
- complete solution
- application

MODELING

$S_i = \{x \in \mathrm{R}^3 : ||x|| = i\}$ i=1,2

H = the space between S_1 and S_2
(i.e., the hollow ball $S^2 \times [1, 2]$)

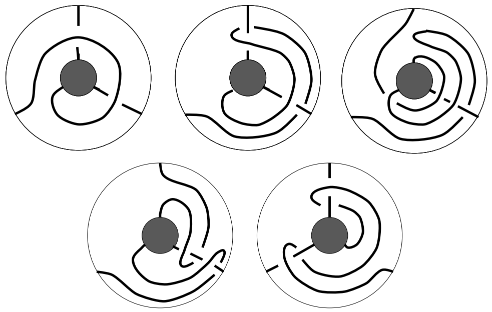
b, m, w are the intervals in which H intersects the positive coordinate axes x, y, z.

QUESTION 1 (Jože Vrabec, 1985/86).
Is there an ambient isotopy of H fixing S_1 and S_2 which wraps m twice around w?

SOLUTION:

a topological proof: John Milnor
(he used the nontrivial element of $\pi_1(SO(3))$ to build the desired ambient isotopy)

a graphical proof:



Unfortunately neither solution can be used to answer the following question:

QUESTION 2.
Is there an ambient isotopy of H fixing S_1 and S_2 which wraps m once around w?

HISTORY

Question 1:

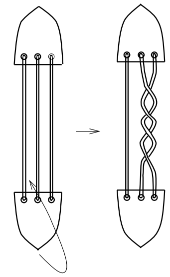
- Dirac's strings (1928)
(English physicist P. A. M. Dirac)

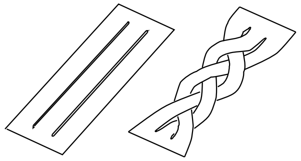
- game Tangloids (early thirties)
(Piet Hein, Niels Bohr's Institute)

Question 2:

- M. H. A. Newman (1942) used Artin's braid theory

- E. Fadel (1962) used fundamental groups of the configuration spaces





A quote of Hilbert:
" The art of doing mathematics consists in finding that special case which contains all the germs of generality."

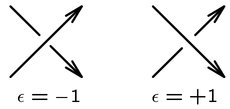
INVARIANTS.

- follow twisting of elastic bands

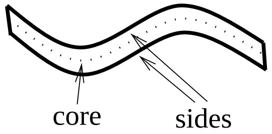
- replace intervals b, m, w with narrow bands.

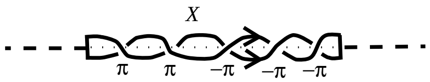
First we study a single band in:

- the three dimensional ball B^3 (Lemma 1)
- the hollow ball H (Lemma 2).



(Right-hand-rule). Define the <u>*linking number*</u> to be the sum of signs over all crossings of one closed curve with the other divided by 2.





Twisting number of the band X

$\text{Tw}(X) := \frac{1}{2}(\#\text{ of twists for }\pi - \#\text{ of twists for }-\pi)$

In the 19th century when knot theory was in its infancy Gauss computed inductance (linking numbers) in a system of linked circular wires.

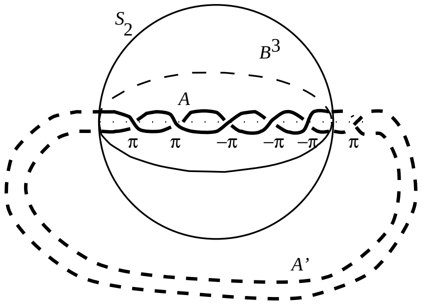
The linking number plays an important role in the study of the effects of enzymes on a circular DNA (i.e., the one which can be represented by a closed band in $\mathbb{R}^3$ whose boundary has two components).

Their packing in the cell of a nucleus is very complex. De Witt Summers writes: "If one scales up the cell nucleus to the size of basketball, the DNA inside scales up to the thickness of a thin fishing line, with 200 km of that line inside the nuclear basketball."

IN THREE DIMENSIONAL BALL:

LEMMA 1.

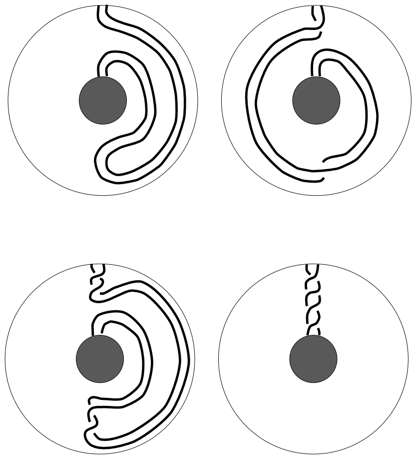
Tw(X) *cannot be altered by an ambient isotopy of* B^3.



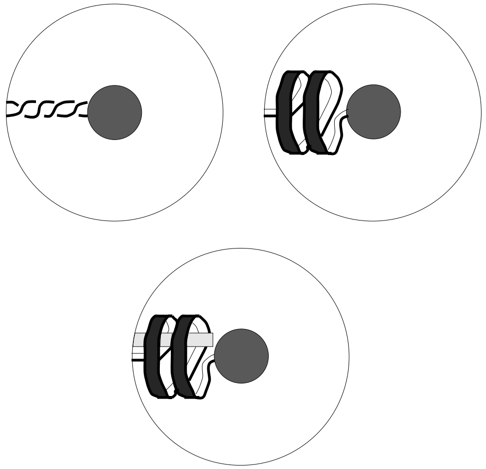
QED

IN HOLLOW BALL:

There exists an ambient isotopy of H that alters the number of twists of A by 2:



IN HOLLOW BALL:



DEMONSTRATION!!!

We now come to the heart of our problem. We are in situation where we have to use a little bit of topology to prove the nonexistence part. However, due to the following two facts (see the book by Burde and Zieschang)

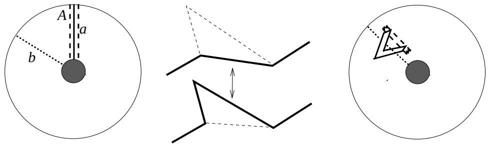
(1) Since our starting position can be drawn with polygonal lines, any position can be drawn with polygonal lines after a small deformation.

(2) If there is an ambient isotopy between two polygonal positions, then there exists a piecewise linear ambient isotopy between these positions.

LEMMA 2.

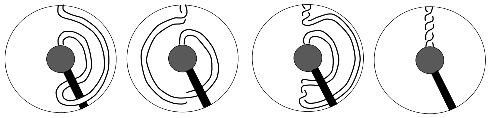
An ambient isotopy of H fixing S_1 and S_2 can alter Tw(A) *only by a multiple of 2.*

Sketch of the proof:

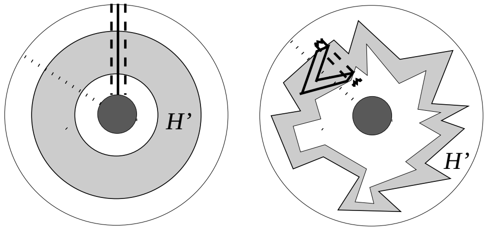


(a) (b) Delta move (c)

We can assume that the ambient isotopy of H is piecewise linear, so it can be realized by a finite number of Δ-moves, and thus the band crosses b finite number of times.



Two plane sections:



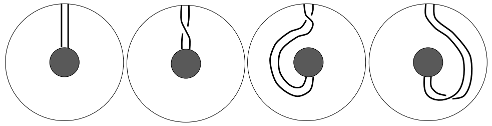
Therefore $2|$Tw(A). QED

QUATERNIONS

Let i, j and k be the rotations of S_1 around the coordinate axes x, y and z respectively through π radians.

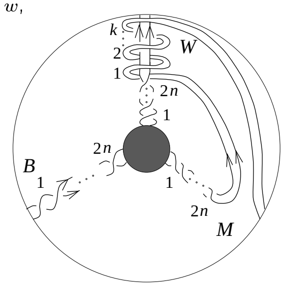
Let $G \le SO(3)$ generated by i, j, k modulo the equivalence relation of an ambient isotopy in H with A keeping S_1 and S_2 fixed.

Then $i^4 = j^4 = k^4 = 1$, $i^2 = j^2 = k^2$, and $ij = k$, $jk = i$, $ki = j$, and if we denote i^2 with minus one (-1) then also $ji = -k$, $kj = -i$, $ik = -j$, so we have exactly the quaternion group.


k ik $j^{-1} = j^3 = -j$

SOLUTION. Let B, M, W be untwisted bands in H along intervals b, m, w respectively.

Suppose that there exists an ambient isotopy of H fixing S_1 and S_2 that wraps m k-times around w,

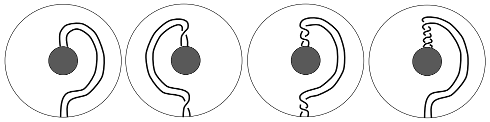


By Lemma 2 Tw$(B) = 2n$, for $n \in \mathbb{N}$.
By Lemma 1 Tw$(W) = -2n$ & Tw$(M) = -2n$.
Then Tw(M) becomes $-2n+2k$.
By Lemma 1 Tw$(M) = 2n$.
Hence $k = 2n$. QED

APPLICATION.

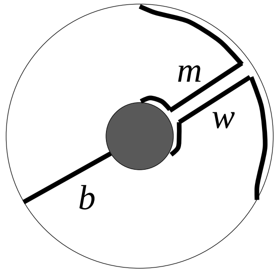
D. A. Adams built a device which can supply electric power to a rotating device through flexible wires

(i.e., one that prevents the wires from twisting up and breaking).



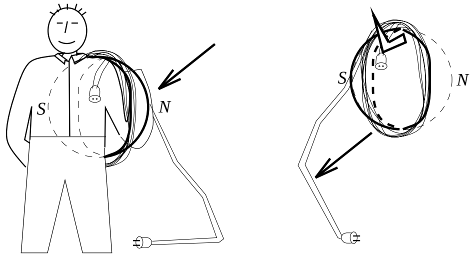
Apparatus for Providing Energy Communication Between a moving and a Stationary Terminal, U.S. Patent 3,586,413, June 22, 1971. Copies available from the U.S. Patent Office

Let us now give a more natural solution of Question 1.



DEMONSTRATION!!!

Let us explain the ‘phenomena’ mentioned at the beginning of the talk.



(a) (b)

DEMONSTRATION!!!

The same principle is used in a *periscope*, i.e., an instrument by which an observer obtains an otherwise obstructed view, to be able to look around while not moving the head.