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Distance Regular Antipodal Covers of Strongly Regular Graphs

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1. INTRODUCTION

A connected graph G is *distance regular* if, given any two vertices u, v of G and any integers i and j, the number of vertices at distance i from u and j from v depends only on i, j and the distance between u and v. Since u and v may coincide, G must be regular. Some examples are the complete and the complete multipartite graphs, the cycles, the n-cubes, the 1-skeletons of the Platonic solids, Petersen's graph and its line graph. Distance regular graphs were introduced in late 1950's by Biggs [B1] as a generalization of *distance transitive* graphs. These are graphs that have automorphisms sending any ordered pair of vertices at distance r to any other such pair, for all r. All the above examples are distance transitive. Obviously, any distance transitive graph is distance regular, but there are also distance regular graphs which are not distance transitive (see for example [Sh]). We can think about distance regularity as a weakening of the condition of distance transitivity; instead complete symmetry of a graph, there is just numerical regularity. In this sense distance regularity is a combinatorial approximation of the algebraic property of 'being distance transitive'. Thus it is natural to expect an interlacing of the combinatorial and algebraic approaches. Distance regular graphs of diameter two are also known as connected *strongly regular* graphs. These graphs can be treated as extremal graphs and have been studied extensively (for basic properties see Cameron, van Lint [CL], Seidel [Se]). Distance regular graphs also have interesting connections with areas other than algebra: in Combinatorics with Coding theory, Finite geometries, Design theory, Hadamard matrices, in Functional analysis with orthogonal polynomials (see [CS], [BCN] and [God2]). These graphs are a special class of association schemes, a concept which has been studied intensively (see for example BI1).

Distance regular graphs are divided into primitive and imprimitive ones. The latter graphs are either antipodal or bipartite (or both) and they give rise to primitive graphs of half the diameter. Therefore the big project of classifying distance regular graphs goes in two stages:

(a) find all primitive distance regular graphs (see Bannai and Ito [BI1], [BI2])

(b) given a distance regular graph G find all imprimitive graphs, i.e., bipartite distance regular graphs or antipodal distance regular graphs, called distance regular antipodal covers of G, which give rise to G.

(There are also other approaches to classifying distance regular graphs, for example by their valency [BI3] or by multiplicities of their eigenvalues [Zhu].)

The first part of (b) was studied by Hemmeter [He1], [He2]. Also some work has been done on antipodal distance regular covers of complete and complete bipartite graphs (see for example [GH], [Hen]). They give rise to nice combinatorial objects, for example to projective planes, Hadamard matrices and even to more general objects such as square group divisible designs. Inspired by this, we investigate in this report antipodal distance regular covers of strongly regular graphs which are not complete bipartite graphs.

In chapter 2 we give some basic definitions and state few properties of distance regular graphs in general and about strongly regular graphs in particular. In chapter 3 we establish the relation between the parameters of distance regular graphs and their distance regular antipodal covers, and prove few existence conditions for distance regular antipodal covers. Chapter 4 introduces the concept of eigenvalues and equitable partitions of a graph. The eigenvalues and their multiplicities of a distance regular graph can be expressed with its parameters. Integrality conditions on these multiplicities give us very strong existence conditions for a distance regular graph with certain parameters. We prove that the parameters of a distance regular graph determine also the eigenvalues of its distance regular antipodal cover. This result is stated in [BCN] and proved in unpublished manuscript by Brouwer and Gardiner [BG], which we were unable to get. Its enable us to prove new existence conditions for distance regular antipodal covers. In particular, we present a new result by Tilla Schade stating that a distance regular antipodal cover of diameter five can have at most two irrational eigenvalues. The fact that the parameters of the distance regular graph determine parameters, eigenvectors, eigenvalues and their multiplicities of its distance regular antipodal covers is restrictive enough that some constructions of distance regular antipodal covers of complete and complete bipartite graphs were made. But this does not suffice in the case when we are searching for the distance regular antipodal covers of strongly regular graphs which are not complete bipartite. So we are looking for new restrictions for such graphs and study the 'geometry' of distance regular graphs with antipodal covers. Van Bon and Brouwer [BB] proved that manely all classical distance regular graphs of large diameter have no distance regular antipodal covers, but in general nothing is known. In chapter 5 we prove two of their theorems. We use their geometric conditions to show that two important infinite families of strongly regular graphs, namely Steiner graphs and Latin square graphs, cannot have any distance regular antipodal covers (this is joint work with Tilla Schade). In chapter 6 we give an equivalent definition of covers with permutations assigned to the edges of a graph and we prove a new existence condition for antipodal covers in general, which was pointed to us by Chris Godsil. We give an example of its use. In chapter 7 we describe how we searched for feasible parameters of strongly regular graphs with small valency (up to 100) and feasible parameters of their distance regular antipodal covers. We present the list of them which was generated by computer.

2. DISTANCE REGULAR GRAPHS

Let G be a graph. The distance between vertices u and v of a graph G will be the length of a shortest path between u and v, denoted with $dist_G(u,v)$ or just $dist(u,v)$ when this is not ambiguous.

Let u be a vertex of a graph G. Then $S_r(u)$ denotes the set of vertices at distance exactly r from u. We call $S_r(u)$ a *sphere of radius* r centered at u, or the r*-th neighbourhood* of u. In particular we use $S(u)$ for $S_1(u)$ and call it the *neighbourhood* of a vertex u.

Let G be a distance regular graph of diameter d. For vertices u and v at distance r and integers i, j let $p_{ij}(r)$ denote the value $|S_i(u) \cap S_j(v)|$. By the definition of a distance regular graph, this value does not depend on the choice of u and v at a distance r. The numbers $p_{ij}(r)$ are called the *intersection numbers* of G. The valency of G is then $p_{11}(0) = |S(u)|$ and will be denoted by k. We will also give special names to some other intersection numbers. For vertices u and v at distance r we define:

$$k_r = p_{rr}(0) = |S_r(u)|, \quad \text{for } r = 0, 1, \ldots, d,$$

$$a_r = p_{r1}(r) = |S(v) \cap S_r(u)|, \quad \text{for } r = 1, 2, \ldots, d,$$

$$b_r = p_{r+1,1}(r) = |S(v) \cap S_{r+1}(u)|, \quad \text{for } r = 0, 1, \ldots, d-1,$$

$$c_r = p_{r-1,1}(r) = |S(v) \cap S_{r-1}(u)|, \quad \text{for } r = 1, 2, \ldots, d.$$

Set $a_0 = c_0 = b_d = 0$, then $a_r + b_r + c_r = k$ for $r = 0, \ldots, d$ and $b_0 = k$, $c_1 = 1$. All the intersection numbers are determined by the numbers in the *intersection array*

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$$

of G. This can be proved by induction on i using the following recurrence relation:

$$c_{j+1}p_{i,j+1}(r) + a_j p_{ij}(r) + b_{j-1}p_{i,j-1}(r) = c_{i+1}p_{i+1,j}(r) + a_i p_{ij}(r) + b_{i-1}p_{i-1,j}(r)$$

obtained by counting for vertices u and v at distance r the edges with one end in $S_i(u)$ and another in $S_j(v)$ in two different ways. (Note that distance regular graph need not be uniquely determined by its parameters, the smallest such example is the Schrikhande graph, see [BCN].)

The property of the intersection array, that it determines all intersection numbers, suggests viewing a distance regular graph G with diameter d in terms of its *distance partition* $\pi_u = \{\{u\}, S_1(u), \ldots, S_d(u)\}$ corresponding to a vertex u (see Figure 1).

u $\quad S_1(u)$ $\quad S_2(u)$ $\quad S_{r-1}(u)$ $\quad S_r(u)$ $\quad S_{r+1}(u)$ $\quad S_{d-1}(u)$ $\quad S_d(u)$

Figure 1: The distance partition.

The following result gives us the basic properties of the parameters a_i, b_i, c_i and k_i of a distance regular graph (for proof see for example [BCN]):

2.1 LEMMA. *Let G be a distance regular graph with valency k and diameter d. Then the following holds:*

(a) $k_{i-1}b_{i-1} = k_i c_i$, *for* $i = 1, ..., d$.

(b) $1 = c_1 \leq c_2 \leq \ldots \leq c_d$.

(c) $k = b_0 \geq b_1 \geq \ldots \geq b_{d-1} > 0$.

(d) if $i + j \leq d$ *then* $c_i \leq b_j$.

(e) if $i + j \leq d$ *and* $i \leq j$, *then* $k_i \leq k_j$.

(f) the sequence k_i *is unimodal, i.e.,* $k_1 < \ldots < k_h = \ldots = k_l > \ldots > k_d$ *for some* h *and* l *with* $1 \leq h \leq l \leq d$. $\square$

We are now going to have a closer look at a special class of distance regular graphs called strongly regular graphs which were introduced by Bose [Bo] and have been intensively studied since (see e.g. [CL], [Se]). A *strongly regular graph* is a k-regular graph with the property that the number of common neighbours of two vertices u and v is either λ or μ depending on whether u and v are adjacent or not. Some examples are the quadrangle, the pentagon, the direct product of two triangles and the Petersen's graph.

It can be easily seen that the complement of a strongly regular graph with parameters (n, k, λ, μ) is also strongly regular and has parameters $(n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$. Also, by counting in two ways the number of triples of distinct vertices u, v, w with u adjacent to v, u adjacent to w and v not adjacent to w, we find that the parameters of a strongly regular graph satisfy

$$k(k-\lambda-1) = \mu(n-k-1).$$

The connected strongly regular graphs are precisely the distance regular graphs of diameter two. They have intersection array $\{k, k-\lambda-1; 1, \mu\}$, so that $\lambda = a_1$ and $\mu = c_2$. The only disconnected strongly regular graphs are the disjoint unions of a number of isomorphic complete graphs; these are the only strongly regular graphs with $\mu = 0$. As we are not considering disconnected strongly regular graphs nor their complements (these are the complete multipartite graphs), we always assume

$$\max\{0, 2k-n+1\} \leq \lambda \leq k-2 \quad \text{and} \quad \max\{1, 2\mathrm{k}-\mathrm{n}+2\} \leq \mu \leq \mathrm{k}-1$$

Strongly regular graphs which satisfy these inequalities will be called *nontrivial strongly regular graphs*.

For a graph G of diameter d we define the i-th *distance graph* G_i to be the graph with the same vertex set as G, and with two vertices adjacent if and only if they are at distance i in the graph G. We call G *imprimitive* if for some i, $1 \leq i \leq d$, the graph G_i is disconnected or, equivalently, if for some non-empty proper subset I of $\{0, 1, \ldots, d\}$ having distance in I is an equivalence relation on the vertex set of G. A graph which is not imprimitive is *primitive*. Petersen's graph is a primitive graph of diameter two and with intersection array $\{3, 2; 1, 1\}$.

A graph G of diameter d is *antipodal* if the vertices at distance d from a given vertex are all at distance d from each other. Then being at distance d induces an equivalence relation on the vertices of G, and the equivalence classes are called *antipodal classes*. For example the line graph of Petersen's graph is a distance regular antipodal graph of diameter three with intersection array $\{4, 2, 1; 1, 1, 4\}$. The only antipodal graphs of diameter two are complete multipartite graphs $K_{r(1)r(2)\ldots r(s)}$ with $r(1) = \ldots = r(s)$ and they are bipartite only when $s = 2$.

Smith [Sm] proved the following remarkable theorem for distance transitive graphs, but the proof can be easily extended to arbitrary distance regular graphs.

2.2 THEOREM. *An imprimitive distance regular graph with valency greater than two is either bipartite or antipodal (or both).* □

This result can also be expressed as follows: for $k > 2$, if having distance in I is an equivalence relation, then $I = \{0\}$ or $I = \{0, d\}$ or $I = \{0, 2, 4, \ldots\}$ or $I = \{0, 1, \ldots, d\}$.

For a connected bipartite graph G of diameter at least two G_2 has two components. The graphs induced on these components are called *halved* graphs of the graph G and are distance regular (see [BG]). If G is distance transitive both halves are isomorphic, but in general this is not true, an example is Tutte's 12-cage (see [BCN]).

For an antipodal graph G we define the *folded graph* of G to be the graph Q with the antipodal classes (i.e. the components of G_d) as vertices, where two components are adjacent if they contain adjacent vertices. The graph Q is also known as the *antipodal quotient* of G and is distance regular whenever G is (see [Ga] or [Hen]).

Imprimitive distance regular graph G with valency greater than two give us after halving at most once and folding at most once a primitive distance regular graph. For a more precise statement and proof see [BCN] or [Hen].

If the partition of some set consists of the set itself or of the all singletons of the set, we call it trivial partition.

Suppose that G is some graph with a nontrivial partition π of its vertices into cells satisfying the following conditions:

(a) each cell is an independent set,

(b) between any two cells there are either no edges or there is a matching.

Let G/π be the graph with the cells of π as vertices and with two of them adjacent if and only if there is a matching between them. Then we say that G is a *cover* of G/π and we call the cells *fibres*. If G/π is connected, then all cells have the same size which is called the *index* of the cover, and is denoted by r. In this case G is called an r-cover of G/π.

We can give an equivalent definition of a cover H of G using the projection map p from $V(H)$ to $V(G)$. We say that H is a cover of G if there is a map $p : V(H) \to V(G)$ called a *projection* which is a graph morphism and a local isomorphism. Then $\{p^{-1}(u), u \in G\}$ is the set of fibres and $r = |p^{-1}(u)|$ is the index of the covering. If we consider our graphs as simplicial complexes, coverings graphs are covering spaces in the usual topological sense.

If a graph G is a cover of G/π and π consists of its antipodal classes, then G is called an *antipodal cover*. Furthermore, if the graph G is also distance regular, we say that G is a *distance regular antipodal cover*. For example 1-skeleton of the dodecahedron with its pairs of antipodal vertices is a distance regular antipodal 2-cover of Petersen's graph. (Note that if you move each second vertex on emphasized cycle (see Figure 2) to its antipodal vertex, then the emphasized cycle will make two five-stars.) Another example is the line graph of Petersen's graph is a distance regular antipodal 3-cover of K_5.

For the proof of the following lemma we need one more definition. A *geodesic* in a graph G is a path $g_0, \ldots, g_t$, where $dist(g_0, g_t) = t$.

2.3 LEMMA. *A distance regular antipodal graph G of diameter d is a cover of its antipodal quotient with components of G_d as its fibres unless $d = 2$.*

Proof. The complete graphs cannot be covers of any graphs. The statement does not hold for the complete multipartite graphs which are the only distance regular antipodal graphs

Figure 2: Some examples of distance regular antipodal covers.

of diameter two. So we may assume $d > 2$. Let $C(u)$ and $C(v)$ be components of G_d corresponding to adjacent vertices u and v of its antipodal quotient. If there is no other vertex beside u and v in $C(u) \cup C(v)$ we are finished, otherwise we may assume that there exists u' in $C(u)$ different from u. A shortest path between u' and v must be shorter than d (since $u' \notin C(v)$) and longer than $d-2$ (since otherwise there would exist a path of length less than d between u and u'). Thus $dist(u', v) = d - 1$, and v is adjacent to just one vertex in $C(u)$ and at distance $d-1 > 1$ from all other vertices in $S(u)$. By symmetry u has just one neighbour in $C(v)$. Since $b_{d-1} > 0$, the shortest path between v and u' can be extended to geodesic of length d. But this means that u' has also exactly one neighbour in $C(v)$. By symmetry we can now conclude that each vertex from one fibre of u and v has exactly one neighbour in another fibre, what we wanted to prove. □

Note that we used in this proof just the facts that G is antipodal, connected and that $b_{d-1}(u, v) > 0$ for any vertex u and $v \in S_{d-1}(u)$.

3. PARAMETERS OF DISTANCE REGULAR ANTIPODAL COVERS

In order to gain more insight into the structure of the distance regular antipodal covers of distance regular graphs let us first see what we can say about their parameters. The following theorem and corollary are due to Gardiner [Ga], we are presenting our proof of this theorem.

3.1 THEOREM. *Let G be a distance regular graph of diameter d with parameters a_i, b_i, c_i, k_i and H a distance regular antipodal r-cover of G with diameter $D > 2$ and parameters A_i, B_i, C_i, K_i. Then holds:*

(i) *If a vertex $u \in V(H)$ is at distance $i \leq \lfloor \frac{D}{2} \rfloor$ from v, then it is at distance $D-i$ from all other vertices in the fibre of v.*

(ii) *For all $u \in V(H)$ and $i \leq \lfloor \frac{D}{2} \rfloor$*

$$S_{D-i}(u) = \bigcup_{v \in S_i(u)} S_D(v).$$

(iii) *$d = \lfloor \frac{D}{2} \rfloor$ and for $0 \leq i < d$*

$$a_i = A_i = A_{D-i}, \quad b_i = B_i = C_{D-i}, \quad c_i = C_i = B_{D-i}.$$

(iv) *For $D = 2d$ we have $a_d = A_d$, $c_d = B_d + C_d$, $(r-1)C_d = B_d$ and for $D = 2d+1$ we have $c_d = C_d = B_{d+1}$, $(r-1)C_{d+1} = B_d$.*

(v) *$r \leq k$.*

Proof. (i) Let $u \in V(H)$, $S_D(u) = \{u_2, \ldots, u_r\}$ and $v \in S_i(u)$ for $i \leq \lfloor \frac{D}{2} \rfloor$. Then there exists a geodesic P_1: $u = q_1(0), q_1(1), \ldots, q_1(i) = v$ between u and v. Between two fibres there are either no edges or a matching and $i \leq \lfloor \frac{D}{2} \rfloor$, so this geodesic has to be the shortest path between any pair of vertices from the fibres of u and v. For the same reason, there exists, a geodesic P_j: $u_j = q_j(0), q_j(1), \ldots, q_j(i)$, for $2 \leq j \leq r$, with vertices $q_2(s), \ldots, q_r(s)$ from the same fibre as $q_1(s)$, for $s = 0, \ldots, i$. (Intuitively we can say that the paths $P_2, \ldots, P_r$ are parallel with P_1 when we observe them in the antipodal partition.) Because $q_j(0) \in S_D(u)$ and $i \leq \lfloor \frac{D}{2} \rfloor$ we have $|P_1 \cap P_j| \leq 1$. If $|P_1 \cap P_j| = 1$, then $D = 2i$, $v = P_1 \cap P_j$ and $\text{dist}(q_1(i-1), q_j(i-1)) = 2$ which gives us $D = 2$. Contradiction! Thus $P_1 \cap P_j = \emptyset$. For the same reason the paths $P_1, \ldots, P_r$ are all pairwise disjoint. Note that then $S_D(v) = \{q_2(i), \ldots, q_r(i)\}$. Now suppose that $q_j(i) \in S_{D-t}(u)$, for $t < i$. Since $B_{D-t}, \ldots, B_{D-1} > 0$ there would exist a path between some vertices of fibres corresponding to u and v with length less then i. Contradiction! Since a path of length i cannot have one end in $S_D(u)$ and another end in $S_{D-t}(u)$, for $t > i$, hence $q_j(i) \in S_{D-i}(u)$. Finally we conclude that $S_D(v) \subseteq S_{D-i}(u)$.

Statement (ii) is an easy consequence of statement (i) and the fact that one of the paths P_j defined above induces all other pairwise disjoint paths. Statement (iii) follows quickly from (ii). For (iv) we use (ii) and Lemma 2.1 to get in the case $D = 2d$:

$$K_d C_d = K_{d-1} B_{d-1}, \quad K_d B_d = K_{d+1} C_{d+1}, \quad K_{d-1}(r-1) = K_{d+1}, \quad B_{d-1} = C_{d+1},$$

and in the case $D = 2d + 1$:

$$K_{d+1} = (r-1) K_d, \quad K_{d+1} C_{d+1} = K_d B_d.$$

For (v) use (iv), $B_d \le k - 1$ and $C_d, C_{d+1} \ge 1$. □

Statement (ii) gives us an idea how to draw the distance partition of an antipodal cover over the corresponding distance partition of its antipodal quotient and why we say that a distance regular antipodal cover folds to its antipodal quotient (see Figure 3).

Figure 3: The distance partition of a distance regular antipodal cover, with even diameter (left) and odd diameter (right).

3.2 COROLLARY. *H has the following intersection array:*

(i) for $D = 2d$

$$\{b_0, b_1, \ldots, b_{d-1}, (r-1)c_d/r, c_{d-1}, \ldots, c_1; c_1, \ldots, c_{d-1}, \frac{c_d}{r}, b_{d-1}, \ldots, b_1, b_0\}$$

(Since elements of the intersection array are integers, $r|c_d$. By the monotonicity of parameters B_i and C_i we have also $c_{d-1} \le \frac{c_d}{r}$ and $(1 - \frac{1}{r})c_d \le b_{d-1}$),

(ii) for $D = 2d + 1$

$$\{b_0, b_1, \ldots, b_{d-1}, (r-1)t, c_d, c_{d-1}, \ldots, c_1; c_1, \ldots, c_{d-1}, c_d, t, b_{d-1}, \ldots, b_1, b_0\}$$

for some integer t satisfying the conditions $t(r-1) \le \min(b_{d-1}, a_d)$ and $c_d \le t$.

Proof. Use (iii) and (iv) from previous theorem. □

The following corollary can be again found in Gardiner [Ga].

3.3 COROLLARY. *If H is a distance regular antipodal graph, then H has a distance regular antipodal cover only if H is either a cycle, a complete graph or a complete bipartite graph.*

Proof. Let denote by D the diameter of H. Assume further that L is a distance regular antipodal R-cover of H and G the antipodal quotient of H. Let $a_i(X)$, $b_i(X)$, $c_i(X)$ be the intersection numbers of a distance regular graph X. For $D > 2$ the graph H is a cover of G (Lemma 2.3). In this case we have by Lemma 3.1 (iii) that $b_{D-1}(H) = c_1(G) = 1$. Now we use the monotonicity of parameters $b_i(L)$ (Lemma 2.1 (c)) and Corollary 3.2. In the case when the diameter of L is even we get:

$$1 = b_{D-1}(L) = b_{D-1}(H) \geq b_D(L) = (R-1)\frac{c_D(H)}{R} \geq R-1$$

Thus $R = 2$ and $c_D(H) = 2$. Since H is antipodal we have $b_0(H) = c_D(H) = 2$, so the valency of H is two and H is a cycle.

In the case when the diameter of L is odd we get:

$$1 = b_{D-1}(L) = b_{D-1}(H) \geq b_D(L) = (R-1)t(H) \geq R-1$$

Thus $R = 2$ and $t(H) = 1$. Now by monotonicity of $c_i(L)$ we have $1 = c_D(H) = \ldots = c_1(H)$. But then $b_0(H) = c_D(H) = 1$. Contradiction!

It remains to consider the case when $D = 2$. Since H is antipodal it is a complete multipartite graph, i.e. the complement of m copies of K_n, where $m, n \geq 2$. Then we have $b_0(H) = c_2(H) = (m-1)n$ and $b_1(H) = n-1$.

In the case when the diameter of L is four we get by Lemma 3.1 (iv) and (iii) that $a_2(L) = a_2(H) = 0$ and $a_1(L) = a_1(H) = (m-2)n$. But we have also $2a_2(L) \geq a_1(L)$, since for any geodesic path u_0, u_1, u_2, u_3 in the distance regular graph L holds

$$S(u_2) \cap S(u_3) \subseteq \big[S_2(u_0) \cap S(u_2)\big] \cup \big[S_2(u_1) \cap S(u_3)\big]$$

Therefore $a_1(L) = 0$, what give us $m = 2$, i.e. H is a complete bipartite graph.

If the diameter of L is five, we have $n - 1 = b_1(L) \geq b_3(L) = (m-1)n \geq n$. Contradiction! □

From the above theorems we can derive some nonexistence results for distance regular antipodal covers of strongly regular graphs.

3.4 COROLLARY. *Let G be a strongly regular graph with parameters (n, k, λ, μ), i.e. intersection array $\{k, k-\lambda-1; 1, \mu\}$. Then*

(i) if $k > \lfloor \frac{n-1}{2} \rfloor$ then G does not have a distance regular antipodal cover of diameter five,

(ii) if $k > \left\lfloor \frac{r(n-1)}{2r-1} \right\rfloor$ then G does not have a distance regular antipodal r-cover of diameter four.

Proof. (i) Suppose H is a distance regular antipodal cover of G of diameter five. Then H has intersection array $\{k, k-\lambda-1, (r-1)t, \mu, 1; 1, \mu, t, k-\lambda-1, k\}$. Lemma 2.1 yields

$$k - \lambda - 1 \geq \mu = \frac{k(k-\lambda-1)}{n-k-1},$$

i.e. $n - k - 1 \geq k$.

(ii) A distance regular antipodal cover H of G having diameter four would have intersection array $\{k, k-\lambda-1, (r-1)\mu/r, 1; 1, \mu/r, k-\lambda-1, k\}$. Thus $k-\lambda-1 \geq (r-1)\mu/r$, so that $r(n-1) \geq (2r-1)k$. $\square$

4. EIGENVALUES AND EQUITABLE PARTITIONS

Now we will study the relation between the spectrum of a distance regular graph G and its distance regular antipodal r-cover H.

The *adjacency matrix* $A = A(G)$ of a graph G with vertex set $\{1, \ldots, n\}$ is the $n \times n$ matrix with ij-entry equal to 1 if vertex i is adjacent to vertex j and 0 otherwise. Since A is loopless, A has diagonal entries zero, and as A is symmetric matrix all its eigenvalues are real. We will use I to denote an identity matrix, J to denote a square matrix with all entries equal to one and j to denote the column vector with all entries equal to one. Since A is symmetric all its eigenvalues are real. They will be referred to as the eigenvalues of G. By an easy induction argument we get the fundamental property of the adjacency matrix: the number of walks in G from the vertex i to the vertex j with length k is equal to the ij-entry of the matrix A^k. Also some other properties of a graph can be expressed very simple in algebraic way, for example regularity of a graph:

4.1 LEMMA. *Let G be a connected graph. Then G is k-regular if and only if k is its eigenvalue with multiplicity one and eigenvector j.*

Proof. For the adjacency matrix A of the graph G the equation $Ax = \theta x$ is equivalent to $\sum_{j\sim i} x_j = \theta x_i$, so G is k-regular if and only if k is an eigenvalue with eigenvector j. Furthermore for any eigenvector x of the eigenvalue k holds:

$$k|x_i| \leq |\sum_{j\sim i} x_j| \leq \sum_{j\sim i} |x_j| \leq k|x_i|.$$

Therefore $x_i = x_j$ for all adjacent vertices v_i and v_j. Since G is connected x is scalar multiple of vector of all ones and multiplicity of the eigenvalue k is really one. $\square$

Note that from this proof hence also that for any eigenvalue θ of k-regular graph holds $|\theta| \leq k$, with other words k is the spectral radius of $A(G)$.

4.2 THEOREM. *A connected graph G of diameter d on n vertices has at least $d+1$ and at most n distinct eigenvalues.*

Proof. Since the matrices $A^0 = I, A, A^2, ..., A^d$ are all nonzero and linearly independent as elements of the vector space formed by the space of $n \times n$ matrices over the reals, A satisfies no polynomial of degree less than $d+1$. But if G has e distinct eigenvalues, then A satisfies the polynomial of degree e with this eigenvalues as roots. Therefore $d < e$. It is evident that any $n \times n$ matrix $A(G)$ has at most n eigenvalues. $\square$

The rank of $n \times n$ matrix J is one, thus zero is its eigenvalue with multiplicity $n-1$. Since $A(K_n) = J - I$, we have $(A - \theta I)x = (J - (1+\theta)I)x$. Therefore K_n has for its eigenvalues -1 with multiplicity $n-1$ and by Lemma 4.1 also its valency $n-1$ with multiplicity one.

If G is a graph of diameter d, then we define the i-th *distance matrix* A_i to be the adjacency matrix of G_i. We set $A_0 = I$ and $A_r = 0$ for $r > d$ or $r < 0$ and $A = A_1$. Now the uv-entry of $A_i A_j$ is equal to the number of vertices at distance i from u and j from v. This provides us with an equivalent definition of distance regularity:

4.3 THEOREM. *A connected graph G of diameter d is distance regular if and only if there are numbers a_i, b_i and c_i such that*

$$AA_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1} \qquad \textit{for } 0 \le i \le d.$$

If G is a distance regular graph, then $A_i = v_i(A)$ for some polynomial $v_i(x)$ of degree i, for $0 \le i \le d+1$. □

Using this identity for a distance regular graph we find that $A_i A_j$ is a linear combination of distance matrices which are linearly independent. The coefficient at A_r is $p_{ij}(r)$. This means that in order to check if some graph is distance regular it is enough to verify if for any vertex u and $v \in S_i(u)$ the numbers $|S_{i+1}(u) \cap S(v)|$ and $|S_{i-1}(u) \cap S(v)|$ (i.e., the members of the intersection array) are independent of choice of u and v.

The sequence of polynomials $v_i(x)$ is determined with $v_{-1}(x) = 0$, $v_0(x) = 1$, $v_1(x) = x$ and recurrence relation

$$c_{i+1}v_{i+1}(x) = (x - a_i)v_i(x) - b_{i-1}v_{i-1}(x), \qquad i = 0, 1, \dots, d.$$

In this sense distance regular graphs are combinatorial representation of orthogonal polynomials (more about this can be found in [God2]).

Damerell [D] proved the following:

4.4 COROLLARY. *If G is a distance regular graph of diameter d, then it has precisely $d+1$ distinct eigenvalues, namely zeros of $v_{d+1}(x)$.*

Proof. Since $v_{d+1}(A) = A_{d+1} = 0$, the minimal polynomial of A divides a polynomial of degree $d+1$, and so A has at most $d+1$ eigenvalues. But because of Theorem 4.2 we know that G has then exactly $d+1$ eigenvalues. □

The converse does not hold in general, but it is true for strongly regular graphs:

4.5 THEOREM. *A graph G on n vertices is strongly regular if and only if its adjacency matrix A satisfies*

$$A^2 = kI + \lambda A + \mu(J - I - A) \quad \text{and} \quad AJ = kJ$$

for some integers k, λ and μ. Its eigenvalues then are $\theta_1 = k$ with multiplicity one and

$$\theta_{2,3} = \frac{1}{2}\Big[\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\Big],$$

with multiplicities

$$m_2 = \frac{(n-1)\theta_3 + k}{\theta_3 - \theta_2} \quad \text{and} \quad m_3 = n - 1 - m_2.$$

They satisfy $k > \theta_2 > 0 > \theta_3 > -k$.

Proof. The first part of the statement follows from Lemma 4.3. Using the first part of the statement and Corollary 4.4 we get the above formulas for eigenvalues of A. Since k is the spectral radius of $A(G)$ (Lemma 4.1) we can without loss of generality assume $\theta_1 > \theta_2 > \theta_3 > -\theta_1$. Because $\theta_2\theta_3 = \mu - k < 0$ we know also $\theta_2 > 0 > \theta_3$. By Lemma 4.1 we get similarly as in the proof of Corollary 4.15 the multiplicities of these eigenvalues. □

Thus the eigenvalues and their multiplicities can be calculated from the parameters of the strongly regular graph. Conversely the eigenvalues determine all the parameters:

$$k = \theta_1, \qquad \lambda = \theta_1 + \theta_2 + \theta_3 + \theta_2\theta_3, \qquad \mu = \theta_1 + \theta_2\theta_3,$$

so we can use the eigenvalues for a classification of strongly regular graphs. As the multiplicities of the eigenvalues must be integer, the above lemma gives us the following rationality conditions for strongly regular graphs.

4.6 COROLLARY. *If there exists a strongly regular graph with parameters (n, k, λ, μ), then one (or both) of the following holds:*

(a) $k = 2\mu$, $n = 4\mu + 1$ *and* $\lambda = \mu - 1$,

(b) $(\lambda - \mu)^2 + 4(k - \mu) = s^2$, *where s is an integer and s divides $(n-1)(\mu - \lambda) - 2k$.*

Proof. If the expression $(\lambda - \mu)^2 - 4(k - \mu)$ is not a perfect square, then $(n-1)(\mu - \lambda) = 2k$ (i.e. $m_2 = m_3$) what gives us $n = 1 + \frac{2k}{\mu - \lambda}$. Since $n > 1 + k$ we get $0 < \mu - \lambda < 2$ implying $\mu - \lambda = 1$. Therefore $k = 2\mu$ and $n = 4\mu + 1$. □

Strongly regular graphs with parameters $(4\mu + 1, 2\mu, \mu - 1, \mu)$ are called *conference graphs*. They are the only strongly regular graphs with $m_2 = m_3$, and also the only ones which could have irrational eigenvalues. They have the same parameters as their complement.

The next lemma gives us a connection between eigenvectors of G and G_i which has important consequence, as we will see later (Theorem 4.11).

4.7 LEMMA. *Let G be a distance regular graph and θ an eigenvalue with eigenvector x. Then $v_i(\theta)$ is an eigenvalue of G_i with eigenvector x.*

Proof. $Ax = \theta x$ implies $A^r x = \theta^r x$ and $A_i x = v_i(A)x = v_i(\theta)x$. $\square$

We generalized antipodal partitions to covering partitions. Now we generalize these partitions to *equitable partitions*. These are partitions $\pi = \{C_1, \ldots, C_s\}$ of $V(G)$ with the following property: for all i and j the number c_{ij} of neighbours which a vertex in C_i has in the cell C_j is independent of the choice of the vertex C_i. This give rise to a *quotient* graph G/π, which is a directed multigraph with cells as vertices and c_{ij} arcs going from C_i to C_j. So the adjacency matrix $A(G/\pi)$ is a $s \times s$ matrix with ij-entry equal to c_{ij}. The *characteristic matrix* $P = P(\pi)$ of a partition $\pi = \{C_1, \ldots, C_s\}$ of a set of n elements is the $n \times s$ matrix with columns formed by the characteristic vectors of the elements of π (i.e. ij-entry of P is 0 or 1 according to as i is contained in C_j or not).

4.8 LEMMA. *A partition π of $V(G)$ with the characteristic matrix P is equitable if and only if there exists a $s \times s$ matrix B such that $A(G)P = PB$. If π is equitable then $B = A(G/\pi)$.* $\square$

As we will see the quotient graph will often inherit some properties of G. So we actually use quotienting to present this properties in simpler graphs. Here is one important result of this kind due to Haynsworth [Ha]:

4.9 COROLLARY. *Let G be a graph with an equitable partition $\pi = \{C_1, \ldots, C_k\}$, and let θ be an eigenvalue of G/π with eigenvector x, then θ is also an eigenvalue of G (with multiplicity at least as big as its multiplicity in G/π), and x extends to an eigenvector of G which is constant on cells of π. If τ is an eigenvalue of G but not of G/π then the sum of coordinates of any eigenvector of G corresponding to the eigenvalue τ equals zero on each cell C_i.*

Proof. For the first part we use Lemma 4.8 to get that $A(G/\pi)x = \theta x$ implies $APx = \theta Px$. For the second part we note that A is symmetric, and that matrices B and B^t have the same eigenvalues. Then we get similarly as in the first part that $Ay = \theta y$ implies $B^t(P^t y) = \theta(P^t y)$. So if θ is not an eigenvalue of $A(G/\pi)$, then must hold $P^t y = 0$. $\square$

After developing this machinery, it is time to apply it. Let G be a distance regular graph of diameter d and u a vertex of G. Then the distance partition corresponding to a vertex u is an equitable partition and gives rise to a quite simple graph (see Figure 4) which inherits all the eigenvalues of G.

Figure 4: The quotient graph corresponding to the distance partition

The quotient graph G/π_u does not depend on choice of vertex u, since G is distance regular, so we can omit index u. It has just $d+1$ vertices, so by Theorem 4.2 its adjacency matrix

$$A(G/\pi) = \begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix}$$

which is determined by the intersection array of G, has exactly $d+1$ distinct eigenvalues and they are precisely all the eigenvalues of $A(G)$. The vector $v = \big(v_0(\theta), \ldots, v_d(\theta)\big)^T$ is a left eigenvector of this matrix corresponding to the eigenvalue θ. Similarly a vector $u = (u_0(\theta), \ldots, u_d(\theta))^T$ defined with $u_{-1}(x) = 0$, $u_0(x) = 1$, $u_1(x) = x/k$ and recurrence relation

$$xu_i(x) = c_i u_{i-1}(x) + a_i u_i(x) - b_i u_{i+1}(x), \qquad i = 0, 1, \ldots, d$$

is a right eigenvector of this matrix, corresponding to the eigenvalue θ. There is the following relation between coordinates of vectors u and v: $u_i(x)k_i = v_i(x)$. The sequence $\big(u_0(\theta), \ldots, u_d(\theta)\big)$ is fundamental for the study of distance regular graphs and is called *sequence of cosines* corresponding to θ. Using the Sturm's theorem (see for example [Vi]) for the sequence $w_i(x) = b_0 \ldots b_i u_i(x)$ we get:

4.10 THEOREM. *Let $\theta_0 \geq \ldots \geq \theta_d$ be the eigenvalues of a distance regular graph. The sequence of cosines corresponding to the i-th eigenvalue θ_i has precisely i sign changes.* □

If θ is an integer, then $u_i(\theta)$ is also, so it is very easy to find all the integer eigenvalues (and also the integer parts of noninteger eigenvalues).

Let us now return to distance regular antipodal covers.

4.11 THEOREM. *Let G be a distance regular graph and H a distance regular antipodal r-cover of G. Then every eigenvalue θ of G is also an eigenvalue of H with the same multiplicity.*

Proof. From Corollary 4.9 we know that for any eigenvalue of G its multiplicity in G is at most as big as its multiplicity in H. Let us now prove also the opposite inequality. Denote by D the diameter of the graph H and with n the number of vertices of the graph G. Then the graph H_D consists of n copies of K_r, corresponding to the fibres of H. The eigenvalues of a disconnected graph are just the eigenvalues of its components and their multiplicities are sums of the corresponding multiplicities in each component. Therefore H_D has for eigenvalues $r-1$ with multiplicity n and -1 with multiplicity $nr-n$. The eigenvectors corresponding to eigenvalue $r-1$ are constant on fibres and those corresponding to -1 sum to zero on fibres.

Take θ to be an eigenvalue of H which is also an eigenvalue of G. An eigenvector of G corresponding to θ can be extended to an eigenvector of H which is constant on fibres. From Lemma 4.7 we know that the eigenvectors of H are also the eigenvectors of H_D, therefore using again Lemma 4.7 we get $v_D(\theta)=r-1$. So we conclude that all the eigenvectors of H corresponding to θ are constant on fibres and therefore give rise to eigenvectors of G corresponding to θ. □

In the last proof we have seen nice combining of the properties of the antipodal partition of H and the quotient graph of H. Theorem 4.11 can be derived also as a consequence of the following theorem of Biggs [B2]:

4.12 THEOREM. *The multiplicity of an eigenvalue θ of a distance regular graph G with diameter d and n vertices is equal to*

$$\frac{n}{\sum_{i=0}^{d} k_i u_i(\theta)^2}\quad .$$

□

Now we can finally state the main theorem of this chapter which is due to Biggs and Gardiner [BG]:

4.13 THEOREM. *Let H be a distance regular antipodal r-cover with diameter D of the distance regular graph G with diameter d and parameters a_i, b_i, c_i. The $D-d$ eigenvalues of H which are not eigenvalues of G are in the case when $D=2d$ the eigenvalues of the*

$d \times d$ matrix

$$\begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-2} & a_{d-2} & b_{d-2} \\ & & & & c_{d-1} & a_{d-1} \end{pmatrix}$$

(Thus, these eigenvalues do not depend on r and are the roots of $u_d(\theta) = 0$. Their multiplicities are proportional to $r - 1$.) and in the case when $D = 2d + 1$ the eigenvalues of the $(d+1) \times (d+1)$ matrix

$$\begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d - rt \end{pmatrix}$$

(Thus, these eigenvalues depend only on rt and are the roots of $c_d u_{d-1}(\theta) + u_d(\theta)(a_d - rt - \theta) = 0$.) If $\theta_0 \geq \theta_1 \geq \ldots \geq \theta_D$ are the eigenvalues of H and $\xi_0 \geq \xi_1 \geq \ldots \geq \xi_d$ are the eigenvalues of G, then

$$\xi_0 = \theta_0, \quad \xi_1 = \theta_2, \quad \ldots, \quad \xi_d = \theta_{2d}$$

i.e. the eigenvalues of G interlace the 'new' eigenvalues of H.

Proof. We know that $\theta_0 \geq \theta_1 \geq \ldots \geq \theta_D$ are exactly the eigenvalues of the adjacency matrix of the distance partition quotient of H, which is in the 'even diameter' case (i.e.

when $D = 2d$) equal to:

$$
\left(\begin{array}{ccccccccccccc}
0 & b_0 & 0 & 0 & \dots & 0 & & & & & & & \\
c_1 & a_1 & b_1 & 0 & \dots & 0 & & & & & & & \\
0 & c_2 & a_2 & b_2 & \dots & 0 & & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & & & & & 0 & & \\
0 & 0 & \dots & c_{d-2} & a_{d-2} & b_{d-2} & & & & & & & \\
0 & 0 & \dots & 0 & c_{d-1} & a_{d-1} & b_{d-1} & & & & & & \\
 & & & & & \frac{c_d}{r} & k-c_d & (1-\frac{1}{r})c_d & & & & & \\
 & & & & & & b_{d-1} & a_{d-1} & c_{d-1} & 0 & 0 & \dots & 0 \\
 & & & & & & & b_{d-2} & a_{d-2} & c_{d-2} & 0 & \dots & 0 \\
 & & & & & & & 0 & b_{d-3} & a_{d-3} & c_{d-3} & \dots & 0 \\
 & & 0 & & & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & & 0 & 0 & \dots & b_1 & a_1 & c_1 \\
 & & & & & & & 0 & 0 & \dots & 0 & b_0 & 0
\end{array}\right)
$$

Let $\tau_1 \geq \ldots \geq \tau_{D-d}$ be the eigenvalues of H which are not the eigenvalues of G. By Corollary 4.9 a right eigenvector corresponding to τ_i sums to zero on fibres, so by Theorem 3.1 the corresponding right eigenvector of $A(H/\pi)$ has in the 'even diameter' case the following form:

$$u = (u_0, u_1, \ldots, u_{d-1}, 0, \frac{-u_d}{r-1}, \ldots, \frac{-u_0}{r-1})^t.$$

From the symmetry of $A(H/\pi)$ and u it is now easy to conclude that τ_i is an eigenvalue of the first matrix from the statement. But this matrix is a square matrix of order $d = D - d$, so it determines all the 'new' eigenvalues. By Theorem 4.4 these eigenvalues are exactly the roots of $u_d(\theta) = 0$. Similarly we prove also the corresponding parts of the statement for the 'odd diameter' case (i.e. when $D = 2d + 1$).

Let us return to the 'even diameter' case. The first matrix in the statement is the adjacency matrix of the quotient graph of G (corresponding to its distance partition) with deleted the last vertex. Then by Interlacing theorem [God1] the 'new' eigenvalues $\tau_1, \ldots, \tau_{D-d}$ interlace the eigenvalues of G, i.e. $\tau_1 = \theta_1, \tau_2 = \theta_3, \ldots, \tau_i = \theta_{2i-1}, \ldots, \tau_d = \theta_{2d-1}$ or $\xi_0 = \theta_0, \xi_1 = \theta_2, \ldots, \xi_d = \theta_{2d}$.

To prove that 'new' eigenvalues of H interlace the eigenvalues of G also in the 'odd diameter' case, we will use Theorem 4.10 (which could have been used also in the 'even diameter' case). By Corollary 4.9 the eigenvalue ξ_i of G is also an eigenvalue of H. By the some Corollary an eigenvector of G, corresponding to ξ_i, yields the eigenvector of H constant on fibres and which corresponds to the eigenvalue ξ_i. Let π be the distance partition of H. Then by Theorem 3.1 the corresponding right eigenvector of $A(H/\pi)$ has the following form:

$$u = (u_0, u_1, \ldots, u_d, u_d, u_{d-1}, \ldots, u_0)$$

Therefore by Theorem 4.10 we get $\xi_i = \theta_{2i}$. □

The following corollary can be found in van Bon and Brouwer [BB].

4.14 COROLLARY. *In particular, if $d = 2$ and $D = 4$ the two new eigenvalues θ are the two roots of $\theta^2 - a_1\theta - k = 0$ and they occur with multiplicity*

$$m(\theta) = \frac{(r-1)n}{2 + a_1\frac{\theta}{k}} = \frac{n(r-1)}{1 - \theta_5/\theta_4}$$

Consequently, either $a_1 = 0$ or $a_1^2 + 4k$ is a square and these eigenvalues are integral. □

4.15 COROLLARY. *The only Conference graph having a distance regular antipodal cover is the pentagon.*

Proof. Suppose G is a strongly regular graph with parameters $(4\mu+1, 2\mu, \mu-1, \mu)$. For $\mu = 1$ G is the pentagon which has the dodecagon as its distance regular antipodal cover. Thus we can assume $\mu > 1$. Suppose G has a distance regular antipodal cover of diameter four. As $a_1 = \lambda = \mu - 1 > 0$, by Corollary 4.14 we have $a_1^2 + 4k = m^2$ for some integer m. Thus $(\mu-1)^2 + 8\mu = (\mu+3)^2 - 8 = m^2$, and so $(\mu+3+m)(\mu+3-m) = 8$. But $\mu > 1$, so that $\mu + 3 + m = 8$ and $\mu + 3 - m = 1$. Therefore $2\mu = 3$. Contradiction!

Now suppose that G has a distance regular antipodal cover H of diameter five. Then the intersection array of H is

$$\{2\mu, \mu, t(r-1), \mu, 1; 1, \mu, t, \mu, 2\mu\}$$

for some integer t. By Lemma 2.1 we get $t = \mu$ and $r = 2$, which gives us $a_2(H) = a_3(H) = 0$. But as $a_1(H) = \mu - 1 \neq 0$ there must be triangles on edges between $S_2(u)$ and $S_3(u)$, and these triangles have an edge within $S_2(u)$ or within $S_3(u)$. Contradiction! □

Therefore we will be looking just for distance regular antipodal covers of nontrivial strongly regular graphs which are not conference graphs. But then the expression $(\lambda - \mu)^2 - 4(k - \mu)$ is a perfect square and θ_2, θ_3 are rational. Furthermore since θ_2 and θ_3 are algebraic integers, they are integers. Now it has become really clear how we can classify these strongly regular graphs using eigenvalues.

4.16 COROLLARY(Schade). *In particular, if $d = 2$ and $D = 5$ the three new eigenvalues α, β and γ are the three roots of*

$$\theta^3 + (c_2 + rt - k - a_1)\theta^2 + (a_1 k + c_2 - k - kc_2 - a_1 rt)\theta + k(k - c_2 - rt) = 0.$$

They cannot all have the same multiplicity, and at least one of these eigenvalues is an integer.

Proof. If the multiplicities of α, β and γ are not equal, then at least one of these eigenvalues is rational. But since these eigenvalues are roots of a monic polynomial with integer coefficients, any rational eigenvalue has to be an integer. So it suffices to prove that the eigenvalues α, β and γ cannot all have the same multiplicity. Suppose on the contrary that they have the same multiplicity, say m. By Theorem 4.10 we have $m = \frac{n(r-1)}{3}$, where n is the number of vertices of G. Since $\mathrm{tr}(A(G)) = 0 = \mathrm{tr}(A(H))$ the coefficient of θ^2 in the above polynomial is:

$$c_2 + rt - k - a_1 = -(\alpha + \beta + \gamma) = 0,$$

and therefore $rt = k + a_1 - c_2$. From $\mathrm{tr}(A^2(G))r = nkr = \mathrm{tr}(A^2(H))$ follows $m(\alpha^2 + \beta^2 + \gamma^2) = kn(r-1)$. Therefore the coefficient of θ in the above polynomial is equal to

$$a_1 k + c_2 - k - kc_2 - a_1(k + a_1 - c_2)$$

and on the other hand also equal to

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) = -\frac{3}{2}k,$$

This gives us $c_2(k - 1 - a_1) + a_1^2 = \frac{k}{2}$. Using $k = a_1 + b_1 + 1$ we get the equation

$$a_1(a_1 - \frac{1}{2}) + (k - 1 - a_1)(c_2 - \frac{1}{2}) - \frac{1}{2} = 0.$$

Since $c_2 \geq 1$ the left hand side is positive whenever a_1 is positive. Thus $a_1 = 0$ and therefore

$$(k-1)(2c_2 - 1) = 1.$$

This is only possible for $k = 2$, $c_2 = 1$, so $rt = 1$. But then $r = 1$. Contradiction! $\square$

Before we finish this chapter let us mention the *Krein condition* on the intersection numbers of distance regular graphs discovered by Scott [Sc] and the *absolute bound*.

4.17 THEOREM (Krein condition). *Let G be a distance regular graph with n vertices, diameter d and eigenvalues $\theta_0 > \theta_1 > \ldots > \theta_d$ with multiplicities $m_0, \ldots, m_d$. Let the polynomials $v_i(x)$ and the numbers k_i be as above. Then the Krein parameters (also called dual intersection numbers)*

$$q_{ij}(h) = \frac{m_i m_j}{n} \sum_{a=0}^{d} \frac{v_a(\theta_i) v_a(\theta_j) v_a(\theta_h)}{k_a^2}$$

are nonnegative for all $i, j, h \in \{0, \ldots, d\}$. $\square$

4.18 THEOREM (Absolute bound). *Let G be a distance regular graph of diameter d. Then the multiplicities $m_0, \ldots, m_d$ of its eigenvalues satisfy*

$$\sum_{q_{ij}(h) \neq 0} m_h \leq \begin{cases} \frac{1}{2} m_i (m_i + 1) & \text{if } i = j \\ m_i m_j & \text{if } i \neq j \end{cases}$$

where the $q_{ij}(h)$ are the Krein parameters. $\square$

5. GEOMETRY OF DISTANCE REGULAR ANTIPODAL COVERS

Van Bon and Brouwer [BB] found strong necessary conditions for the existence of distance regular antipodal covers of distance regular graphs. Using them they ruled out most of the 'feasible' parameters of distance regular antipodal covers of the classical distance regular graphs. But in general we still do not know much about existence of distance regular antipodal covers. We will give detailed proofs of their two main theorems and show some examples of their use.

For vertices u and v of G we denote by $C(u,v)$ the union of all geodesics between u and v in G.

5.1 THEOREM. *Let G be a distance regular graph of diameter $d \geq 2$ which has a distance regular antipodal r-cover H of diameter $2d$. Then for any two vertices u and v of G at distance d, the subgraph $C(u,v)\backslash\{u,v\}$ is the disjoint union of r subgraphs each having the same number of vertices, and there are no edges joining vertices from different subgraphs.*

Proof. Let $p : V(H) \to V(G)$ be a covering map, u_1 some point in $p^{-1}(u)$ and $p^{-1}(v) = \{v_1, \ldots, v_r\}$. Denote by C_i the union of geodesics in H between u_1 and v_i for $1 \leq i \leq r$, and let be $C = \cup_i (C_i \backslash \{u_1, v_i\})$. Then $p|_C : C \to C(u,v)$ is an isomorphism. The restriction of the map p is injective since the distance between any pair of points in C is less then $2d$ and two vertices of H have the same image under p if and only if they are at distance of the diameter of H. For the same reason, there are no edges between the subgraphs $C_i \backslash \{u_1, u_i\}$, for $i = 1, \ldots, r$. For each geodesic between u and v there exists by Lemma 2.3 the unique geodesic between u_1 and the fibre of v. Thus the restriction of the map p is also surjective. It remains to show that subgraphs $C_i \backslash \{u_1, v_i\}$ have the same size. Since H is a distance regular graph, the numbers $|S_t(u_1) \cap S_{d-t}(v_i)|$, for $t = 1, \ldots, d-1$, do not depend on i. But the number of vertices in $C_i \backslash \{u, v\}$ is equal to the sum of these numbers. □

5.2 THEOREM. *Let G be a distance regular graph of diameter $d \geq 2$ which has a distance regular antipodal r-cover H of diameter $2d+1$. Then for any two vertices u and v of G at distance d and $E = \{v\} \cup \big(S(v) \cap S_d(u)\big)$, the union of $C(u,w)\backslash\{u,w\}$ $(w \in E)$ can be partitioned into r nonempty subgraphs and all edges joining vertices from different subgraphs are in $S(u)$.*

Proof. Again let $p : V(H) \to V(G)$ be a covering map, u_1 some point in $p^{-1}(u)$ and $p^{-1}(v) = \{v_1, \ldots, v_r\}$. By Theorem 3.1 we can assume that $\mathrm{dist}_H(u_1, v_1) = d$ and $\mathrm{dist}_H(u_1, v_i) = d+1$ for $i = 2, \ldots, r$. Each of the vertices $v_2, \ldots, v_r$ has C_{d+1} neighbours in $S_d(u_1)$ and these neighbours are all distinct, since H is an antipodal graph with diameter greater than two. Because p is an adjacency preserving map and $S_d(u_1)$ does not contain any pair of antipodal vertices, these neighbours together with the neighbours of v_1 in $S_d(u_1)$ are mapped by p injectively to the set $E\backslash\{v\}$. But by Corollary 3.2 these sets have the same number $(r-1)C_{d+1}+A_d=a_d$ of elements, so the restriction of p to the first set is a bijection between them. Denote by C_1 the union of the geodesics between u_1 and the vertices from $\{v_1\}\cup(S(v_1)\cap S_d(u_1))$, by C_i the union of the geodesics (in H) between u_1 and v_i for $2\le i\le r$ and finally by C the union of sets $C_i\backslash p^{-1}(E)$ where $i=1,\ldots,r$. Then as in previous proof we obtain that

$$p|_C : C \to \bigcup_{w\in E}\big(C(u,w)\backslash\{u,w\}\big)$$

is an isomorphism, subgraphs $C\cap p(C_i)$ for $i=1,\ldots,r$ are disjoint and all edges joining vertices from different subgraphs are in $S(u)$. $\square$

5.3 COROLLARY. *Let u be some vertex of G with the diameter $d\ge 2$, and i a fixed element from the set $\{1,\ldots,d-1\}$. If for any two adjacent vertices v and w in $S_d(u)$ there is a vertex in $S_{D-i}(u)\cap S_i(v)\cap S_i(w)$, then G does not have any distance regular antipodal covers of diameter $2d+1$.* $\square$

Let us now apply these results. The graph with vertices as q-subsets of an p-set with two q-subsets joined by an edge if and only if they intersect in exactly $q-1$ elements is called *Johnson graph* and is denoted by $J(p,q)$. We assume that q is at least two. (When $q=2$ Johnson graph is the line graph of complete graph with p vertices.) Since graphs $J(p,q)$ and $J(p,p-q)$ are isomorphic we can assume also $q\le\lfloor\frac{p}{2}\rfloor$. It is not difficult to prove that for any two q-subsets A and B holds $\mathrm{dist}_{J(p,q)}(A,B)=q-|A\cap B|$. From this easyly follows that $J(p,q)$ is a distance transitive graph of diameter q and with parameters $b_i=(p-q-i)(q-i)$, $c_i=i^2$ for $i=0,1,\ldots,q$. Let us denote elements of the p-set by $1,2,\ldots,p$. For some elements y and z and a subset X of $\{1,\ldots,p\}$ we define X_{yz} to be the subset $(X\backslash\{y\})\cup z$. Let be $A=\{1,\ldots,q\}$ and $B=\{p-q+1,\ldots,p\}$, so $A\in S_q(B)$. Now for $i_1,i_2\le q$ and $j_1,j_2\ge p-q+1$, where $(i_1,j_1)\ne(i_2,j_2)$, holds $A_{i_1,j_1},A_{i_2,j_2}\in S_{q-1}(B)$. If $i_1=i_2$ or $j_1=j_2$, then A_{i_1,j_1} and A_{i_2,j_2} are adjacent. Otherwise A_{i_1,j_2} is a common neighbour of A_{i_1,j_1}, A_{i_2,j_2} and A. Therefore $S(A)\cap S_{q-1}(B)$ is connected and by Theorem 5.1 $J(p,q)$ has no distance regular antipodal covers of even diameter.

The graph $J(2q,q)$ is antipodal and has therefore by Theorem 3.3 no distance regular antipodal cover, so let us assume that $p\ne 2q$, i.e. $a_q\ne 0$. Thus for $i\le q$ and $j\in\{q+1,\ldots,p-q\}$ we have $A_{ij}\in S_q(B)\cup S(A)$. But $(A_{ij}\cap A)\cup\{1\}\in S_{q-1}(B)\cap S(A)\cap S(A_{ij})$, so by Corollary 5.3 $J(p,q)$ has also no distance regular antipodal covers of odd diameter.

Now we will show that two important infinite families of strongly regular graphs coming from certain finite geometries cannot have distance regular antipodal covers. This is joint work with Tilla Schade.

A *geometric 1-design* is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a set of points and $\mathcal{L}$ is a set of lines, together with an incidence relation such that

(i) every point is on exactly $r \geq 2$ lines,

(ii) every line contains exactly $s \geq 2$ points,

(iii) two distinct points lie on at most one common line.

The *line graph* of a geometric 1-design $\mathcal{D}$ has the lines of $\mathcal{D}$ as vertices, two of them being adjacent whenever they have a point in common; obviously, two distinct lines cannot have more than one point in common. We will look at two special classes of geometric 1-designs which have strongly regular line graphs. For more informations about finite geometries see [BJL].

A 2-$(v, s, 1)$ *design* is a geometric 1-design with the property that any two points are on a unique line, v is the number of all points and s is the number of points on a line (for standard definition see Appendix A). It can be easily shown that the line graph of a non-square 2-$(v, s, 1)$ design $\mathcal{D}$, is strongly regular and has parameters

$$n = \frac{v(v-1)}{s(s-1)}, \qquad k = s\frac{v-s}{s-1}, \qquad a_1 = \frac{v-1}{s-1} - 2 + (s-1)^2 \quad \text{and} \quad c_2 = s^2.$$

As these designs are also called Steiner systems, their line graphs are known as *Steiner graphs* (for $s = 2$ also *triangular graphs*). (If $\mathcal{D}$ is square design, then its line graph is a complete graph.)

5.4 COROLLARY. *Steiner graphs do not admit any distance regular antipodal covers.*

Proof. Suppose G is the line graph of a 2-$(v, s, 1)$ design $\mathcal{D}$ and H is a distance regular antipodal cover of G. Let us first consider the case when the diameter of H is four. Let A be some line of $\mathcal{D}$ and $B \in S_2(A)$, i.e. $A \cap B = \emptyset$. For $a, b \in A$ and $c, d \in B$, where $(a, c) \neq (b, d)$, let C be the line containing the pair $\{a, c\}$ and D the line containing the pair $\{b, d\}$. If $a = b$ or $c = d$, then C and D are adjacent. Otherwise the line E determined by the pair $\{b, c\}$ is a common neighbour of C and D and lies in $S(A) \cap S(B)$. So we have shown that $S(A) \cap S(B)$ is connected and thus by Theorem 5.1 we get a contradiction.

Now consider the case when the diameter of H is five. Suppose that there exist lines B and C in $S_2(A)$ such that $|B \cap C| = 1$. Then the line determined by $B \cap C$ and some point in A is a common neighbour of B and C in $S_1(A)$. By Corollary 5.3 this gives us a contradiction. If there is no such B and C, then $a_2 = 0$ and $v = s^2$. But this means that the distance regular graph G is antipodal (i.e., the complement of $s+1$ copies of K_s), so by Theorem 3.3 it cannot have any distance regular antipodal covers. $\square$

A *transversal design* $TD(s,v)$ is a geometric 1-design with line size s having v points which can be partitioned into s groups of v/s points each such that two distinct points are on a line if and only if they are in distinct groups (for standard definition see Appendix B). For proof that $s \leq v+1$ see Appendix B. The line graph of transversal design is a complete graph if and only if the equality holds. Again, it is not difficult to see that the line graph of a transversal design $TD(s,v)$ is strongly regular for $s \leq v$ and has parameters

$$n = v^2, \qquad k = s(v-1), \qquad a_1 = v - 2 + (s-1)(s-2) \quad and \quad c_2 = s(s-1).$$

The line graphs of these transversal designs are called *Latin square graphs* because a $TD(s,v)$ is equivalent to $s-2$ mutually orthogonal Latin squares of order v (for $s=2$ they are also called *Lattice graphs*).

Figure 5: Example of the line graph of $TD(s,v)$ for $v=3$ and $s=3$.

5.5 COROLLARY. *Latin square graphs do not admit any distance regular antipodal covers, except for $v = 2$ and $s = 2$, which is the quadrangle with the eight-cycle as its distance regular antipodal cover.*

Proof. Suppose that G is the line graph of a $TD(s, v)$ and H a distance regular antipodal cover of G. For $s = 2$ we have $G \cong K_v \times K_v$. Choose a vertex u of G. Then $S_1(u)$ consists of two disjoint copies of K_{v-1} and $S_2(u)$ is isomorphic to $K_{v-1} \times K_{v-1}$. Any two adjacent vertices in $S_2(u)$ lie in the same copy of K_{v-1} and thus there is always a vertex in $S_1(u)$ which is adjacent to all the vertices of this copy. Therefore by Corollary 5.3 the diameter of H has to be even. If the expression $a_1^2 + 4k = (v+2)^2 - 8$ is a square of some integer m, we have

$$(v+2)^2 - m^2 = (v+2+m)(v+2-m) = 8$$

Since the sum of the above factors of eight is even, it is equal to $2 + 4 = 6$. This gives us $v = 1$, which is not possible. So by Corollary 4.14 we get $a_1 = 0$ and thus $v = 2$. Therefore G is the quadrangle which has the eight-cycle as a distance regular antipodal cover.

It remains to consider the case $s \geq 3$. Let A and B be two disjoint lines of the $TD(s, v)$, i.e. $B \in S_2(A)$. Let C and D be any two common neighbours of A and B, and let

$$a = A \cap C, \qquad b = A \cap D, \qquad c = B \cap C, \qquad d = B \cap D.$$

If $a = b$ or $c = d$, then C and D have a common point and thus are adjacent. Otherwise we want to show that C and D lie in the same connected component of $S(A) \cap S(B)$. Suppose that at least one of the pairs $\{a, d\}$ and $\{b, c\}$ is not contained in some group of the $TD(s, v)$. Then that pair determines a line which is a common neighbour of A, B, C and D. Otherwise a and d lie in the same group and b and c lie in the same group. As the number of groups is $s \geq 3$, there is at least one more group which does not contain any of the points a, b, c and d. Let x be the point determined by the intersection of this group and the line A. Then the line E determined by $\{x, c\}$ is a neighbour of A, B and C and the line F determined by $\{x, d\}$ is a neighbour of A, B and D. Obviously E and F are adjacent. It follows that $C(A, B) \backslash \{A, B\}$ consists of only one connected component and so H cannot have diameter four by Theorem 5.1 .

So suppose H has diameter five. For $s = v$ we get $a_2 = 0$, so G is antipodal. Since $s \geq 3$, G is not complete bipartite, and by Theorem 3.3 cannot have any distance regular antipodal covers. If $s < v$ we have $a_2 > 0$, so there are lines $B, C \in S_2(A)$ such that $B \cap C \neq \emptyset$. Then the line D determined by the point $b = B \cap C$ and any point in A not being in the same group as b is a common neighbour of A, B and C. By Corollary 5.3 this contradicts the existence of H. $\square$

The two results above are particularly interesting if we consider the following theorem proved by Neumaier [N], which classifies strongly regular graphs by their smallest eigenvalue.

5.6 THEOREM. *The strongly regular graphs with smallest eigenvalue $-m$, $m \geq 2$ integral, are the following:*

(a) Complete multipartite graphs,

(b) Latin square graphs,

(a) Steiner graphs,

(a) Finitely many other graphs. $\square$

As it can be shown that the smallest eigenvalue of a strongly regular graph cannot be -1 and we have already seen that strongly regular graphs with irrational eigenvalues do not have distance regular antipodal covers (see Corollary 4.15), this yields that only finitely many strongly regular graphs with smallest eigenvalue $-m$ can have distance regular antipodal covers.

6. COVERS AND PERMUTATIONS

For an arbitrary graph G we can construct an r-cover as follows. We associate with each vertex u of G a set $C_u = \{u_1, \ldots, u_r\}$ of r new vertices, and for every edge uv of G we install a matching between the sets C_u and C_v. The sets C_u are then the fibres of the cover. The matching associated with edge uv can be described by the permutation $f(u,v)$ of the set $\{1, \ldots, r\}$ mapping i to j if and only if vertex u_i of C_u is adjacent to vertex v_j in C_v. This gives us a function f from the set of arcs of G to the symmetric group $Sym(r)$ with the property that $f(u,v) = f(v,u)^{-1}$ for all arcs (u,v) of G. We call such a function f a *symmetric arc function of index* r on G and denote the cover it determines by $G(f)$.

Obviously, two different symmetric arc functions may define isomorphic covering graphs. In particular, if we permute the vertices in each set C_u with some permutation τ_u of $Sym(r)$, we get an isomorphic cover $G(g)$, where $g(u,v) = \tau_u^{-1} f(u,v) \tau_v$, for all arcs (u,v) of G. We can choose the permutations τ_u so that g takes the identity value on a spanning forest of G. If a function f has this property, we say that it is *normalized* on this forest.

Let H be a distance regular antipodal r-cover with diameter D of some distance regular graph G. If we number the elements of each antipodal class of G with the elements of $\{1, \ldots, r\}$, this determines a symmetric arc function f so that $H = G(f)$. Suppose that $C = c_1, \ldots, c_h, c_1$ is a cycle of length $h < D$ in G. Without loss of generality we can assume that f is normalized and that it has identity value on at least $h-1$ arcs of the cycle C, say

$$f(c_1, c_2) = \ldots = f(c_{h-1}, c_h) = id \quad \text{and} \quad \mathrm{f}(\mathrm{c_h}, \mathrm{c_1}) = \tau.$$

We want to show that $\tau = id$. The cycle C can be 'lifted' to paths

$$(c_1)_i, (c_2)_i, \ldots, (c_h)_i, (c_1)_{\tau(i)} \quad \text{for} \quad \mathrm{i} \in \{1, \ldots, \mathrm{r}\}$$

in H, where $(c_j)_i$ is the i-th vertex in the fibre $C(c_j)$ of H associated with c_j. Since $dist_H\big((c_1)_i, (c_1)_{\tau(i)}\big) \le h < D$, but $(c_1)_i$ and $(c_1)_{\tau(i)}$ are in the same fibre, this implies $\tau(i) = i$ for each $i \in \{1, \ldots, r\}$. Thus we proved the following statement:

6.1 LEMMA (Godsil). *Let G be distance regular graph and f a symmetric arc function on G determining an antipodal cover of diameter D. Then the product of the values of f on the edges of any cycle of length $h < D$ is the identity.* $\square$

The above argument gives us new existence condition for antipodal covers. Some of the results of the preceding chapter can be proved using this argument. As an example, we prove an extension of a part of corollary 5.5:

6.2 COROLLARY. *Latin square graphs do not admit any antipodal covers of diameter five.*

Proof. First, let $s = 2$, so that $G \cong K_v \times K_v$. We can view G as having vertex set $\{(i,j) : \ i,j \in \{1,\ldots,n\}\ \}$, and vertices (i,j) and (h,m) are adjacent whenever $i = h$ or $j = m$. Then the edge set of G consists of n copies of K_n formed by 'horizontal edges' $\{(a,i),(b,i)\}$ and n copies of K_n formed by 'vertical edges' $\{(i,a),(i,b)\}$. For each of the horizontal copies of K_n we choose a path P_i from $(1,i)$ to (n,i) of length $n-1$ hitting all the vertices (j,i) for $j = 1,\ldots,n$ of this copy. Joining all these paths P_i by edges $(n,i-1),(1,i)$ for $i = 2,\ldots,n$ we get a spanning tree T of G.

Now suppose that f is a symmetric arc function on G determining a cover of diameter five, and that f is normalized on T. By Lemma 5.1 the product of the values of f on any triangle or quadrangle of G is the identity. Thus, taking any triangle or quadrangle in G having all but one edge e in T we find that f has to have identity value on that edge e as well. Iterating this process in each of the horizontal copies of K_n starting with the path P_i, we find that f has identity value on all the horizontal edges. Any vertical edge $\{(a,i-1),(a,i)\}$ lies in a quadrangle with an edge $\{(n,i-1),(1,i)\}$ of T and two horizontal edges of G. Thus, f also has identity value on all these vertical edges; and these edges provide us with paths of length $n-1$ for each of the vertical copies of K_n. Following the same steps as above, f is the identity on all edges of G, so that the cover is trivial.

For $s > 2$, G has $K_n \times K_n$ as a proper subgraph. Using the same argument as above, we find that any symmetric arc function f on G defining a cover of diameter five has to have identity value on this subgraph H. But any edge not in H lies in a triangle with two edges from H. It follows again that f has identity value on all edges of G and that the cover is trivial. $\square$

7. LIST OF FEASIBLE PARAMETER SETS

We generated a list of parameter sets for nontrivial strongly regular graphs which are not conference graphs. Their parameters (n, k, λ, μ) and nontrivial eigenvalues ξ_1 and ξ_2 satisfy the conditions:

(1) $k > \xi_1 > 0$, $-1 > \xi_2$,

(2) $\max\{0, 2k-n+1\} \leq \lambda \leq k-2$,

(3) $\max\{1, 2k-n+2\} \leq \mu \leq k-1$,

(4) $k(k-\lambda-1) = \mu(n-k-1)$.

(5) Krein conditions,

(6) Absolute bounds.

Based on that list of parameters for strongly regular graphs we generated a list of parameter sets for distance regular antipodal r-covers which satisfy the conditions:

in the case $D = 4$

(1) $k \leq \lfloor \frac{2}{3}(n-1) \rfloor$,

(2) $2 \leq r \leq k$,

(3) $r | \mu$,

(4) the sequences b_i and $-c_i$ of the cover are decreasing,

(5) the multiplicities of the new eigenvalues of the cover are integers,

(6) Krein conditions,

(7) Absolute bounds.

(8) If $\mu = 1$ or if $\mu = 2$ and $k < \frac{1}{2}\lambda(\lambda+3)$, then $(\lambda+1)|k$ (for proof of this see [BCN]).

and in the case $D = 5$

(1) $k \leq \lfloor \frac{n-1}{2} \rfloor$,

(2) $2 \leq r \leq k$,

(3) the sequences b_i and $-c_i$ of the cover are decreasing,

(4) the multiplicities of the new eigenvalues of the cover are integers,

(5) Krein conditions,

(6) Absolute bounds.

Our list is sorted by the valency. For each cover we give the number of vertices as a sum of the k_i, the eigenvalues with their multiplicities and the intersection array of it and its quotient. If a parameter set cannot be realised by any (or any known resp.) distance regular antipodal graph then the symbol "//" (or "?" resp.) precedes v. For completeness we included parameter sets which satisfy all the conditions except Krein conditions and absolute bounds.

Here we only list paramewter sets with valency up to 100; a longer list is available from the authors.

Distance regular antipodal covers with diameter four of nontrivial strongly regular graphs

$v = 32 = 1 + 5 + 20 + 5 + 1,$ $\quad 5^1\ 2.2^8\ 1^{10}\ \ -2.2^8\ \ -3^5$

$\{5, 4, 1, 1; 1, 1, 4, 5\}$ is 2-cover of $\{5, 4; 1, 2\}$

$v = 45 = 1 + 6 + 24 + 12 + 2,$ $\quad 6^1\ 3^{12}\ 1^9\ \ -2^{18}\ \ -3^5$

$\{6, 4, 2, 1; 1, 1, 4, 6\}$ is 3-cover of $\{6, 4; 1, 3\}$

// $v = 42 = 1 + 10 + 20 + 10 + 1,$ $\quad 10^1\ 5^6\ 1^{14}\ \ -2^{15}\ \ -4^6$

$\{10, 6, 4, 1; 1, 3, 6, 10\}$ is 2-cover of $\{10, 6; 1, 6\}$

$v = 63 = 1 + 10 + 30 + 20 + 2,$ $\quad 10^1\ 5^{12}\ 1^{14}\ \ -2^{30}\ \ -4^6$

$\{10, 6, 4, 1; 1, 2, 6, 10\}$ is 3-cover of $\{10, 6; 1, 6\}$

// $v = 112 = 1 + 10 + 90 + 10 + 1,$ $\quad 10^1\ 3.2^{28}\ 2^{35}\ \ -3.2^{28}\ \ -4^{20}$

$\{10, 9, 1, 1; 1, 1, 9, 10\}$ is 2-cover of $\{10, 9; 1, 2\}$

$v = 70 = 1 + 16 + 36 + 16 + 1,$ $\quad 16^1\ 8^7\ 2^{20}\ \ -2^{28}\ \ -4^{14}$

$\{16, 9, 4, 1; 1, 4, 9, 16\}$ is 2-cover of $\{16, 9; 1, 8\}$

? $v = 162 = 1 + 20 + 120 + 20 + 1,$ $\quad 20^1\ 5^{36}\ 2^{60}\ \ -4^{45}\ \ -7^{20}$

$\{20, 18, 3, 1; 1, 3, 18, 20\}$ is 2-cover of $\{20, 18; 1, 6\}$

$v = 243 = 1 + 20 + 180 + 40 + 2,$ $\quad 20^1\ 5^{72}\ 2^{60}\ \ -4^{90}\ \ -7^{20}$

$\{20, 18, 4, 1; 1, 2, 18, 20\}$ is 3-cover of $\{20, 18; 1, 6\}$

? $v = 486 = 1 + 20 + 360 + 100 + 5,$ $\quad 20^1\ 5^{180}\ 2^{60}\ \ -4^{225}\ \ -7^{20}$

$\{20, 18, 5, 1; 1, 1, 18, 20\}$ is 6-cover of $\{20, 18; 1, 6\}$

? $v = 486 = 1 + 21 + 420 + 42 + 2,$ $\quad 21^1\ 4.6^{162}\ 3^{105}\ \ -4.6^{162}\ \ -6^{56}$

$\{21, 20, 2, 1; 1, 1, 20, 21\}$ is 3-cover of $\{21, 20; 1, 3\}$

? $v = 200 = 1 + 22 + 154 + 22 + 1,$ $\quad 22^1\ 4.7^{50}\ 2^{77}\ \ -4.7^{50}\ \ -8^{22}$

$\{22,21,3,1;1,3,21,22\}$ is 2-cover of $\{22,21;1,6\}$

? $v=300=1+22+231+44+2,$ $22^1\ 4.7^{100}\ 2^{77}\ -4.7^{100}\ -8^{22}$
$\{22,21,4,1;1,2,21,22\}$ is 3-cover of $\{22,21;1,6\}$

? $v=600=1+22+462+110+5,$ $22^1\ 4.7^{250}\ 2^{77}\ -4.7^{250}\ -8^{22}$
$\{22,21,5,1;1,1,21,22\}$ is 6-cover of $\{22,21;1,6\}$

? $v=352=1+25+300+25+1,$ $25^1\ 5^{88}\ 3^{120}\ -5^{88}\ -7^{55}$
$\{25,24,2,1;1,2,24,25\}$ is 2-cover of $\{25,24;1,4\}$

? $v=704=1+25+600+75+3,$ $25^1\ 5^{264}\ 3^{120}\ -5^{264}\ -7^{55}$
$\{25,24,3,1;1,1,24,25\}$ is 4-cover of $\{25,24;1,4\}$

? $v=704=1+26+650+26+1,$ $26^1\ 5.1^{176}\ 4^{208}\ -5.1^{176}\ -6^{143}$
$\{26,25,1,1;1,1,25,26\}$ is 2-cover of $\{26,25;1,2\}$

? $v=264=1+27+180+54+2,$ $27^1\ 9^{44}\ 3^{55}\ -3^{132}\ -6^{32}$
$\{27,20,6,1;1,3,20,27\}$ is 3-cover of $\{27,20;1,9\}$

$v=128=1+28+70+28+1,$ $28^1\ 14^8\ 4^{28}\ -2^{56}\ -4^{35}$
$\{28,15,6,1;1,6,15,28\}$ is 2-cover of $\{28,15;1,12\}$

// $v=192=1+28+105+56+2,$ $28^1\ 14^{16}\ 4^{28}\ -2^{112}\ -4^{35}$
$\{28,15,8,1;1,4,15,28\}$ is 3-cover of $\{28,15;1,12\}$
absolute bound fails for $1,1$;

// $v=256=1+28+140+84+3,$ $28^1\ 14^{24}\ 4^{28}\ -2^{168}\ -4^{35}$
$\{28,15,9,1;1,3,15,28\}$ is 4-cover of $\{28,15;1,12\}$

? $v=210=1+32+144+32+1,$ $32^1\ 8^{35}\ 2^{84}\ -4^{70}\ -10^{20}$
$\{32,27,6,1;1,6,27,32\}$ is 2-cover of $\{32,27;1,12\}$

? $v=315=1+32+216+64+2,$ $32^1\ 8^{70}\ 2^{84}\ -4^{140}\ -10^{20}$
$\{32,27,8,1;1,4,27,32\}$ is 3-cover of $\{32,27;1,12\}$

? $v = 420 = 1 + 32 + 288 + 96 + 3,$ $\quad 32^1\ 8^{105}\ 2^{84}\ \ -4^{210}\ \ -10^{20}$

$\{32, 27, 9, 1; 1, 3, 27, 32\}$ is 4-cover of $\{32, 27; 1, 12\}$

? $v = 630 = 1 + 32 + 432 + 160 + 5,$ $\quad 32^1\ 8^{175}\ 2^{84}\ \ -4^{350}\ \ -10^{20}$

$\{32, 27, 10, 1; 1, 2, 27, 32\}$ is 6-cover of $\{32, 27; 1, 12\}$

? $v = 420 = 1 + 33 + 352 + 33 + 1,$ $\quad 33^1\ 5.7^{105}\ 3^{154}\ \ -5.7^{105}\ \ -9^{55}$

$\{33, 32, 3, 1; 1, 3, 32, 33\}$ is 2-cover of $\{33, 32; 1, 6\}$

? $v = 630 = 1 + 33 + 528 + 66 + 2,$ $\quad 33^1\ 5.7^{210}\ 3^{154}\ \ -5.7^{210}\ \ -9^{55}$

$\{33, 32, 4, 1; 1, 2, 32, 33\}$ is 3-cover of $\{33, 32; 1, 6\}$

? $v = 1260 = 1 + 33 + 1056 + 165 + 5,$ $\quad 33^1\ 5.7^{525}\ 3^{154}\ \ -5.7^{525}\ \ -9^{55}$

$\{33, 32, 5, 1; 1, 1, 32, 33\}$ is 6-cover of $\{33, 32; 1, 6\}$

// $v = 912 = 1 + 40 + 420 + 440 + 11,$ $\quad 40^1\ 20^{76}\ 7^{19}\ \ -2^{760}\ \ -3^{56}$

$\{40, 21, 22, 1; 1, 2, 21, 40\}$ is 12-cover of $\{40, 21; 1, 24\}$

absolute bound fails for $0, 2$;

? $v = 704 = 1 + 36 + 630 + 36 + 1,$ $\quad 36^1\ 6^{176}\ 4^{231}\ \ -6^{176}\ \ -8^{120}$

$\{36, 35, 2, 1; 1, 2, 35, 36\}$ is 2-cover of $\{36, 35; 1, 4\}$

? $v = 1408 = 1 + 36 + 1260 + 108 + 3,$ $\quad 36^1\ 6^{528}\ 4^{231}\ \ -6^{528}\ \ -8^{120}$

$\{36, 35, 3, 1; 1, 1, 35, 36\}$ is 4-cover of $\{36, 35; 1, 4\}$

? $v = 1408 = 1 + 37 + 1332 + 37 + 1,$ $\quad 37^1\ 6.1^{352}\ 5^{407}\ \ -6.1^{352}\ \ -7^{296}$

$\{37, 36, 1, 1; 1, 1, 36, 37\}$ is 2-cover of $\{37, 36; 1, 2\}$

? $v = 252 = 1 + 45 + 160 + 45 + 1,$ $\quad 45^1\ 15^{21}\ 3^{90}\ \ -3^{105}\ \ -9^{35}$

$\{45, 32, 9, 1; 1, 9, 32, 45\}$ is 2-cover of $\{45, 32; 1, 18\}$

$v = 378 = 1 + 45 + 240 + 90 + 2,$ $\quad 45^1\ 15^{42}\ 3^{90}\ \ -3^{210}\ \ -9^{35}$

$\{45, 32, 12, 1; 1, 6, 32, 45\}$ is 3-cover of $\{45, 32; 1, 18\}$

? $v = 756 = 1 + 45 + 480 + 225 + 5,$ $\quad 45^1\ 15^{105}\ 3^{90}\ \ -3^{525}\ \ -9^{35}$

$\{45, 32, 15, 1; 1, 3, 32, 45\}$ is 6-cover of $\{45, 32; 1, 18\}$

? $v = 392 = 1 + 45 + 300 + 45 + 1,$ $45^1\ 9^{70}\ 3^{150}\ \ -5^{126}\ \ -11^{45}$

$\{45, 40, 6, 1; 1, 6, 40, 45\}$ is 2-cover of $\{45, 40; 1, 12\}$

? $v = 588 = 1 + 45 + 450 + 90 + 2,$ $45^1\ 9^{140}\ 3^{150}\ \ -5^{252}\ \ -11^{45}$

$\{45, 40, 8, 1; 1, 4, 40, 45\}$ is 3-cover of $\{45, 40; 1, 12\}$

? $v = 784 = 1 + 45 + 600 + 135 + 3,$ $45^1\ 9^{210}\ 3^{150}\ \ -5^{378}\ \ -11^{45}$

$\{45, 40, 9, 1; 1, 3, 40, 45\}$ is 4-cover of $\{45, 40; 1, 12\}$

? $v = 1176 = 1 + 45 + 900 + 225 + 5,$ $45^1\ 9^{350}\ 3^{150}\ \ -5^{630}\ \ -11^{45}$

$\{45, 40, 10, 1; 1, 2, 40, 45\}$ is 6-cover of $\{45, 40; 1, 12\}$

? $v = 2352 = 1 + 45 + 1800 + 495 + 11,$ $45^1\ 9^{770}\ 3^{150}\ \ -5^{1386}\ \ -11^{45}$

$\{45, 40, 11, 1; 1, 1, 40, 45\}$ is 12-cover of $\{45, 40; 1, 12\}$

? $v = 798 = 1 + 45 + 660 + 90 + 2,$ $45^1\ 6.7^{266}\ 3^{209}\ \ -6.7^{266}\ \ -12^{56}$

$\{45, 44, 6, 1; 1, 3, 44, 45\}$ is 3-cover of $\{45, 44; 1, 9\}$

? $v = 2394 = 1 + 45 + 1980 + 360 + 8,$ $45^1\ 6.7^{1064}\ 3^{209}\ \ -6.7^{1064}\ \ -12^{56}$

$\{45, 44, 8, 1; 1, 1, 44, 45\}$ is 9-cover of $\{45, 44; 1, 9\}$

? $v = 784 = 1 + 46 + 690 + 46 + 1,$ $46^1\ 6.8^{196}\ 4^{276}\ \ -6.8^{196}\ \ -10^{115}$

$\{46, 45, 3, 1; 1, 3, 45, 46\}$ is 2-cover of $\{46, 45; 1, 6\}$

? $v = 1176 = 1 + 46 + 1035 + 92 + 2,$ $46^1\ 6.8^{392}\ 4^{276}\ \ -6.8^{392}\ \ -10^{115}$

$\{46, 45, 4, 1; 1, 2, 45, 46\}$ is 3-cover of $\{46, 45; 1, 6\}$

? $v = 2352 = 1 + 46 + 2070 + 230 + 5,$ $46^1\ 6.8^{980}\ 4^{276}\ \ -6.8^{980}\ \ -10^{115}$

$\{46, 45, 5, 1; 1, 1, 45, 46\}$ is 6-cover of $\{46, 45; 1, 6\}$

? $v = 1276 = 1 + 49 + 1176 + 49 + 1,$ $49^1\ 7^{319}\ 5^{406}\ \ -7^{319}\ \ -9^{231}$

$\{49, 48, 2, 1; 1, 2, 48, 49\}$ is 2-cover of $\{49, 48; 1, 4\}$

? $v = 2552 = 1 + 49 + 2352 + 147 + 3,$ $49^1\ 7^{957}\ 5^{406}\ \ -7^{957}\ \ -9^{231}$

$\{49, 48, 3, 1; 1, 1, 48, 49\}$ is 4-cover of $\{49, 48; 1, 4\}$

? $v = 2552 = 1 + 50 + 2450 + 50 + 1,$ $50^1\ 7.1^{638}\ 6^{725}\ \ -7.1^{638}\ \ -8^{550}$

$\{50, 49, 1, 1; 1, 1, 49, 50\}$ is 2-cover of $\{50, 49; 1, 2\}$

? $v = 650 = 1 + 54 + 540 + 54 + 1,$ $54^1\ 9^{130}\ 4^{234}\ \ -6^{195}\ \ -11^{90}$

$\{54, 50, 5, 1; 1, 5, 50, 54\}$ is 2-cover of $\{54, 50; 1, 10\}$

? $v = 1625 = 1 + 54 + 1350 + 216 + 4,$ $54^1\ 9^{520}\ 4^{234}\ \ -6^{780}\ \ -11^{90}$

$\{54, 50, 8, 1; 1, 2, 50, 54\}$ is 5-cover of $\{54, 50; 1, 10\}$

? $v = 3250 = 1 + 55 + 2970 + 220 + 4,$ $55^1\ 7.4^{1300}\ 5^{429}\ \ -7.4^{1300}\ \ -10^{220}$

$\{55, 54, 4, 1; 1, 1, 54, 55\}$ is 5-cover of $\{55, 54; 1, 5\}$

? $v = 324 = 1 + 56 + 210 + 56 + 1,$ $56^1\ 14^{36}\ 2^{140}\ \ -4^{126}\ \ -16^{21}$

$\{56, 45, 12, 1; 1, 12, 45, 56\}$ is 2-cover of $\{56, 45; 1, 24\}$

? $v = 486 = 1 + 56 + 315 + 112 + 2,$ $56^1\ 14^{72}\ 2^{140}\ \ -4^{252}\ \ -16^{21}$

$\{56, 45, 16, 1; 1, 8, 45, 56\}$ is 3-cover of $\{56, 45; 1, 24\}$

? $v = 648 = 1 + 56 + 420 + 168 + 3,$ $56^1\ 14^{108}\ 2^{140}\ \ -4^{378}\ \ -16^{21}$

$\{56, 45, 18, 1; 1, 6, 45, 56\}$ is 4-cover of $\{56, 45; 1, 24\}$

? $v = 972 = 1 + 56 + 630 + 280 + 5,$ $56^1\ 14^{180}\ 2^{140}\ \ -4^{630}\ \ -16^{21}$

$\{56, 45, 20, 1; 1, 4, 45, 56\}$ is 6-cover of $\{56, 45; 1, 24\}$

? $v = 1296 = 1 + 56 + 840 + 392 + 7,$ $56^1\ 14^{252}\ 2^{140}\ \ -4^{882}\ \ -16^{21}$

$\{56, 45, 21, 1; 1, 3, 45, 56\}$ is 8-cover of $\{56, 45; 1, 24\}$

? $v = 648 = 1 + 57 + 532 + 57 + 1,$ $57^1\ 7.5^{162}\ 3^{266}\ \ -7.5^{162}\ \ -15^{57}$

$\{57, 56, 6, 1; 1, 6, 56, 57\}$ is 2-cover of $\{57, 56; 1, 12\}$

? $v = 972 = 1 + 57 + 798 + 114 + 2,$ $57^1\ 7.5^{324}\ 3^{266}\ \ -7.5^{324}\ \ -15^{57}$

$\{57, 56, 8, 1; 1, 4, 56, 57\}$ is 3-cover of $\{57, 56; 1, 12\}$

? $v = 1296 = 1 + 57 + 1064 + 171 + 3,$ $57^1\ 7.5^{486}\ 3^{266}\ \ -7.5^{486}\ \ -15^{57}$

$\{57, 56, 9, 1; 1, 3, 56, 57\}$ is 4-cover of $\{57, 56; 1, 12\}$

? $v = 1944 = 1 + 57 + 1596 + 285 + 5,$ $57^1\ 7.5^{810}\ 3^{266}\ \ -7.5^{810}\ \ -15^{57}$

$\{57, 56, 10, 1; 1, 2, 56, 57\}$ is 6-cover of $\{57, 56; 1, 12\}$

? $v = 3888 = 1 + 57 + 3192 + 627 + 11,$ $57^1\ 7.5^{1782}\ 3^{266}\ \ -7.5^{1782}\ \ -15^{57}$

$\{57, 56, 11, 1; 1, 1, 56, 57\}$ is 12-cover of $\{57, 56; 1, 12\}$

? $v = 552 = 1 + 75 + 400 + 75 + 1,$ $75^1\ 15^{69}\ 3^{230}\ \ -5^{207}\ \ -17^{45}$

$\{75, 64, 12, 1; 1, 12, 64, 75\}$ is 2-cover of $\{75, 64; 1, 24\}$

? $v = 828 = 1 + 75 + 600 + 150 + 2,$ $75^1\ 15^{138}\ 3^{230}\ \ -5^{414}\ \ -17^{45}$

$\{75, 64, 16, 1; 1, 8, 64, 75\}$ is 3-cover of $\{75, 64; 1, 24\}$

? $v = 1104 = 1 + 75 + 800 + 225 + 3,$ $75^1\ 15^{207}\ 3^{230}\ \ -5^{621}\ \ -17^{45}$

$\{75, 64, 18, 1; 1, 6, 64, 75\}$ is 4-cover of $\{75, 64; 1, 24\}$

? $v = 1656 = 1 + 75 + 1200 + 375 + 5,$ $75^1\ 15^{345}\ 3^{230}\ \ -5^{1035}\ \ -17^{45}$

$\{75, 64, 20, 1; 1, 4, 64, 75\}$ is 6-cover of $\{75, 64; 1, 24\}$

? $v = 2208 = 1 + 75 + 1600 + 525 + 7,$ $75^1\ 15^{483}\ 3^{230}\ \ -5^{1449}\ \ -17^{45}$

$\{75, 64, 21, 1; 1, 3, 64, 75\}$ is 8-cover of $\{75, 64; 1, 24\}$

? $v = 3312 = 1 + 75 + 2400 + 825 + 11,$ $75^1\ 15^{759}\ 3^{230}\ \ -5^{2277}\ \ -17^{45}$

$\{75, 64, 22, 1; 1, 2, 64, 75\}$ is 12-cover of $\{75, 64; 1, 24\}$

? $v = 1104 = 1 + 76 + 950 + 76 + 1,$ $76^1\ 8.7^{276}\ 4^{437}\ \ -8.7^{276}\ \ -16^{114}$

$\{76, 75, 6, 1; 1, 6, 75, 76\}$ is 2-cover of $\{76, 75; 1, 12\}$

? $v = 1656 = 1 + 76 + 1425 + 152 + 2,$ $76^1\ 8.7^{552}\ 4^{437}\ \ -8.7^{552}\ \ -16^{114}$

$\{76, 75, 8, 1; 1, 4, 75, 76\}$ is 3-cover of $\{76, 75; 1, 12\}$

? $v = 2208 = 1 + 76 + 1900 + 228 + 3,$ $76^1\ 8.7^{828}\ 4^{437}\ \ -8.7^{828}\ \ -16^{114}$

$\{76, 75, 9, 1; 1, 3, 75, 76\}$ is 4-cover of $\{76, 75; 1, 12\}$

? $v = 3312 = 1 + 76 + 2850 + 380 + 5,$ $76^1\ 8.7^{1380}\ 4^{437}\ \ -8.7^{1380}\ \ -16^{114}$

$\{76, 75, 10, 1; 1, 2, 75, 76\}$ is 6-cover of $\{76, 75; 1, 12\}$

? $v = 6624 = 1 + 76 + 5700 + 836 + 11,$ $\quad 76^1\ 8.7^{3036}\ 4^{437}\ -8.7^{3036}\ -16^{114}$

$\{76, 75, 11, 1; 1, 1, 75, 76\}$ is 12-cover of $\{76, 75; 1, 12\}$

? $v = 1080 = 1 + 77 + 924 + 77 + 1,$ $\quad 77^1\ 11^{210}\ 5^{385}\ -7^{330}\ -13^{154}$

$\{77, 72, 6, 1; 1, 6, 72, 77\}$ is 2-cover of $\{77, 72; 1, 12\}$

? $v = 1620 = 1 + 77 + 1386 + 154 + 2,$ $\quad 77^1\ 11^{420}\ 5^{385}\ -7^{660}\ -13^{154}$

$\{77, 72, 8, 1; 1, 4, 72, 77\}$ is 3-cover of $\{77, 72; 1, 12\}$

? $v = 2160 = 1 + 77 + 1848 + 231 + 3,$ $\quad 77^1\ 11^{630}\ 5^{385}\ -7^{990}\ -13^{154}$

$\{77, 72, 9, 1; 1, 3, 72, 77\}$ is 4-cover of $\{77, 72; 1, 12\}$

? $v = 3240 = 1 + 77 + 2772 + 385 + 5,$ $\quad 77^1\ 11^{1050}\ 5^{385}\ -7^{1650}\ -13^{154}$

$\{77, 72, 10, 1; 1, 2, 72, 77\}$ is 6-cover of $\{77, 72; 1, 12\}$

? $v = 2160 = 1 + 78 + 2002 + 78 + 1,$ $\quad 78^1\ 8.8^{540}\ 6^{715}\ -8.8^{540}\ -12^{364}$

$\{78, 77, 3, 1; 1, 3, 77, 78\}$ is 2-cover of $\{78, 77; 1, 6\}$

? $v = 3240 = 1 + 78 + 3003 + 156 + 2,$ $\quad 78^1\ 8.8^{1080}\ 6^{715}\ -8.8^{1080}\ -12^{364}$

$\{78, 77, 4, 1; 1, 2, 77, 78\}$ is 3-cover of $\{78, 77; 1, 6\}$

? $v = 6480 = 1 + 78 + 6006 + 390 + 5,$ $\quad 78^1\ 8.8^{2700}\ 6^{715}\ -8.8^{2700}\ -12^{364}$

$\{78, 77, 5, 1; 1, 1, 77, 78\}$ is 6-cover of $\{78, 77; 1, 6\}$

// $v = 1248 = 1 + 81 + 756 + 405 + 5,$ $\quad 81^1\ 27^{104}\ 3^{168}\ -3^{936}\ -15^{39}$

$\{81, 56, 30, 1; 1, 6, 56, 81\}$ is 6-cover of $\{81, 56; 1, 36\}$

Krein condition fails for $1, 4, 1$;

// $v = 7488 = 1 + 81 + 4536 + 2835 + 35,$ $\quad 81^1\ 27^{728}\ 3^{168}\ -3^{6552}\ -15^{39}$

$\{81, 56, 35, 1; 1, 1, 56, 81\}$ is 36-cover of $\{81, 56; 1, 36\}$

Krein condition fails for $1, 4, 1$;

? $v = 750 = 1 + 81 + 504 + 162 + 2,$ $\quad 81^1\ 27^{50}\ 6^{144}\ -3^{450}\ -9^{105}$

$\{81, 56, 18, 1; 1, 9, 56, 81\}$ is 3-cover of $\{81, 56; 1, 27\}$

? $v = 2250 = 1 + 81 + 1512 + 648 + 8,$ $\quad 81^1\ 27^{200}\ 6^{144}\ -3^{1800}\ -9^{105}$

$\{81,56,24,1;1,3,56,81\}$ is 9-cover of $\{81,56;1,27\}$

? $v=3404=1+81+3240+81+1,$ $81^1\ 9^{851}\ 7^{1035}\ \ -9^{851}\ \ -11^{666}$
$\{81,80,2,1;1,2,80,81\}$ is 2-cover of $\{81,80;1,4\}$

? $v=6808=1+81+6480+243+3,$ $81^1\ 9^{2553}\ 7^{1035}\ \ -9^{2553}\ \ -11^{666}$
$\{81,80,3,1;1,1,80,81\}$ is 4-cover of $\{81,80;1,4\}$

? $v=6808=1+82+6642+82+1,$ $82^1\ 9.1^{1702}\ 8^{1886}\ \ -9.1^{1702}\ \ -10^{1517}$
$\{82,81,1,1;1,1,81,82\}$ is 2-cover of $\{82,81;1,2\}$

? $v=800=1+84+630+84+1,$ $84^1\ 14^{120}\ 4^{315}\ \ -6^{280}\ \ -16^{84}$
$\{84,75,10,1;1,10,75,84\}$ is 2-cover of $\{84,75;1,20\}$

? $v=1600=1+84+1260+252+3,$ $84^1\ 14^{360}\ 4^{315}\ \ -6^{840}\ \ -16^{84}$
$\{84,75,15,1;1,5,75,84\}$ is 4-cover of $\{84,75;1,20\}$

? $v=2000=1+84+1575+336+4,$ $84^1\ 14^{480}\ 4^{315}\ \ -6^{1120}\ \ -16^{84}$
$\{84,75,16,1;1,4,75,84\}$ is 5-cover of $\{84,75;1,20\}$

? $v=4000=1+84+3150+756+9,$ $84^1\ 14^{1080}\ 4^{315}\ \ -6^{2520}\ \ -16^{84}$
$\{84,75,18,1;1,2,75,84\}$ is 10-cover of $\{84,75;1,20\}$

? $v=1600=1+85+1428+85+1,$ $85^1\ 9.2^{400}\ 5^{595}\ \ -9.2^{400}\ \ -15^{204}$
$\{85,84,5,1;1,5,84,85\}$ is 2-cover of $\{85,84;1,10\}$

? $v=4000=1+85+3570+340+4,$ $85^1\ 9.2^{1600}\ 5^{595}\ \ -9.2^{1600}\ \ -15^{204}$
$\{85,84,8,1;1,2,84,85\}$ is 5-cover of $\{85,84;1,10\}$

? $v=8000=1+85+7140+765+9,$ $85^1\ 9.2^{3600}\ 5^{595}\ \ -9.2^{3600}\ \ -15^{204}$
$\{85,84,9,1;1,1,84,85\}$ is 10-cover of $\{85,84;1,10\}$

? $v=644=1+96+450+96+1,$ $96^1\ 24^{46}\ 4^{252}\ \ -4^{276}\ \ -16^{69}$
$\{96,75,16,1;1,16,75,96\}$ is 2-cover of $\{96,75;1,32\}$

? $v=1288=1+96+900+288+3,$ $96^1\ 24^{138}\ 4^{252}\ \ -4^{828}\ \ -16^{69}$

$\{96,75,24,1;1,8,75,96\}$ is 4-cover of $\{96,75;1,32\}$

? $v = 2576 = 1 + 96 + 1800 + 672 + 7,$ $\quad 96^1\ 24^{322}\ 4^{252}\ \ -4^{1932}\ \ -16^{69}$
$\{96,75,28,1;1,4,75,96\}$ is 8-cover of $\{96,75;1,32\}$

// $v = 1170 = 1 + 96 + 588 + 480 + 5,$ $\quad 96^1\ 48^{39}\ 6^{104}\ \ -2^{936}\ \ -8^{90}$
$\{96,49,40,1;1,8,49,96\}$ is 6-cover of $\{96,49;1,48\}$
absolute bound fails for $1,1$;
Krein condition fails for $1,4,1$;

// $v = 3120 = 1 + 96 + 1568 + 1440 + 15,$ $\quad 96^1\ 48^{117}\ 6^{104}\ \ -2^{2808}\ \ -8^{90}$
$\{96,49,45,1;1,3,49,96\}$ is 16-cover of $\{96,49;1,48\}$
Krein condition fails for $1,4,1$;

? $v = 1650 = 1 + 96 + 1456 + 96 + 1,$ $\quad 96^1\ 12^{330}\ 6^{572}\ \ -8^{495}\ \ -14^{252}$
$\{96,91,6,1;1,6,91,96\}$ is 2-cover of $\{96,91;1,12\}$

? $v = 2475 = 1 + 96 + 2184 + 192 + 2,$ $\quad 96^1\ 12^{660}\ 6^{572}\ \ -8^{990}\ \ -14^{252}$
$\{96,91,8,1;1,4,91,96\}$ is 3-cover of $\{96,91;1,12\}$

? $v = 3300 = 1 + 96 + 2912 + 288 + 3,$ $\quad 96^1\ 12^{990}\ 6^{572}\ \ -8^{1485}\ \ -14^{252}$
$\{96,91,9,1;1,3,91,96\}$ is 4-cover of $\{96,91;1,12\}$

? $v = 4950 = 1 + 96 + 4368 + 480 + 5,$ $\quad 96^1\ 12^{1650}\ 6^{572}\ \ -8^{2475}\ \ -14^{252}$
$\{96,91,10,1;1,2,91,96\}$ is 6-cover of $\{96,91;1,12\}$

? $v = 3300 = 1 + 97 + 3104 + 97 + 1,$ $\quad 97^1\ 9.8^{825}\ 7^{1067}\ \ -9.8^{825}\ \ -13^{582}$
$\{97,96,3,1;1,3,96,97\}$ is 2-cover of $\{97,96;1,6\}$

? $v = 4950 = 1 + 97 + 4656 + 194 + 2,$ $\quad 97^1\ 9.8^{1650}\ 7^{1067}\ \ -9.8^{1650}\ \ -13^{582}$
$\{97,96,4,1;1,2,96,97\}$ is 3-cover of $\{97,96;1,6\}$

? $v = 9900 = 1 + 97 + 9312 + 485 + 5,$ $\quad 97^1\ 9.8^{4125}\ 7^{1067}\ \ -9.8^{4125}\ \ -13^{582}$
$\{97,96,5,1;1,1,96,97\}$ is 6-cover of $\{97,96;1,6\}$

? $v = 3534 = 1 + 99 + 3234 + 198 + 2,$ $\quad 99^1\ 9.9^{1178}\ 6^{836}\ \ -9.9^{1178}\ \ -15^{341}$

$\{99, 98, 6, 1; 1, 3, 98, 99\}$ is 3-cover of $\{99, 98; 1, 9\}$

? $v = 10602 = 1 + 99 + 9702 + 792 + 8,$ 99^1 9.9^{4712} 6^{836} -9.9^{4712} -15^{341}
$\{99, 98, 8, 1; 1, 1, 98, 99\}$ is 9-cover of $\{99, 98; 1, 9\}$

? $v = 5152 = 1 + 100 + 4950 + 100 + 1,$ 100^1 10^{1288} 8^{1540} -10^{1288} -12^{1035}
$\{100, 99, 2, 1; 1, 2, 99, 100\}$ is 2-cover of $\{100, 99; 1, 4\}$

? $v = 10304 = 1 + 100 + 9900 + 300 + 3,$ 100^1 10^{3864} 8^{1540} -10^{3864} -12^{1035}
$\{100, 99, 3, 1; 1, 1, 99, 100\}$ is 4-cover of $\{100, 99; 1, 4\}$

Distance regular antipodal but not bipartite covers with diameter five of non-trivial strongly regular graphs

$v = 20 = 1 + 3 + 6 + 6 + 3 + 1,$ 3^1 2.2^3 1^5 0^4 -2^4 -2.2^3
$\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$ is 2-cover of $\{3, 2; 1, 1\}$

// $v = 224 = 1 + 10 + 45 + 135 + 30 + 3,$ 10^1 5.3^{30} 2^{35} 0^{108} -4^{20} -5.3^{30}
$\{10, 9, 6, 2, 1; 1, 2, 2, 9, 10\}$ is 4-cover of $\{10, 9; 1, 2\}$
Krein condition fails for $1, 5, 1$;

? $v = 324 = 1 + 21 + 140 + 140 + 21 + 1,$ 21^1 9^{21} 3^{105} 0^{120} -6^{56} -9^{21}
$\{21, 20, 9, 3, 1; 1, 3, 9, 20, 21\}$ is 2-cover of $\{21, 20; 1, 3\}$

// $v = 486 = 1 + 21 + 140 + 280 + 42 + 2,$ 21^1 9^{42} 3^{105} 0^{240} -6^{56} -9^{42}
$\{21, 20, 12, 3, 1; 1, 3, 6, 20, 21\}$ is 3-cover of $\{21, 20; 1, 3\}$
Krein condition fails for $1, 5, 1$;

// $v = 972 = 1 + 21 + 140 + 700 + 105 + 5,$ 21^1 9^{105} 3^{105} 0^{600} -6^{56} -9^{105}
$\{21, 20, 15, 3, 1; 1, 3, 3, 20, 21\}$ is 6-cover of $\{21, 20; 1, 3\}$
Krein condition fails for $1, 5, 1$;

? $v = 729 = 1 + 22 + 220 + 440 + 44 + 2,$ 22^1 7^{132} 4^{132} -2^{330} -5^{110} -11^{24}
$\{22, 20, 18, 2, 1; 1, 2, 9, 20, 22\}$ is 3-cover of $\{22, 20; 1, 2\}$

$v = 252 = 1 + 25 + 100 + 100 + 25 + 1,$ 25^1 15^9 7^{35} 1^{75} -3^{90} -5^{42}

$\{25, 16, 9, 4, 1; 1, 4, 9, 16, 25\}$ is 2-cover of $\{25, 16; 1, 4\}$

// $v = 378 = 1 + 25 + 100 + 200 + 50 + 2,$ $25^1\ 15^{18}\ 7^{35}\ 1^{150}\ -3^{90}\ -5^{84}$

$\{25, 16, 12, 4, 1; 1, 4, 6, 16, 25\}$ is 3-cover of $\{25, 16; 1, 4\}$

Krein condition fails for $1, 5, 1$;

absolute bound fails for $1, 1$;

// $v = 875 = 1 + 30 + 144 + 576 + 120 + 4,$ $30^1\ 15^{56}\ 5^{84}\ 0^{560}\ -5^{90}\ -10^{84}$

$\{30, 24, 20, 5, 1; 1, 5, 5, 24, 30\}$ is 5-cover of $\{30, 24; 1, 5\}$

Krein condition fails for $1, 4, 1$;

// $v = 693 = 1 + 30 + 200 + 400 + 60 + 2,$ $30^1\ 15^{44}\ 9^{55}\ 0^{308}\ -3^{175}\ -6^{110}$

$\{30, 20, 18, 3, 1; 1, 3, 9, 20, 30\}$ is 3-cover of $\{30, 20; 1, 3\}$

Krein condition fails for $1, 5, 1$;

? $v = 704 = 1 + 36 + 315 + 315 + 36 + 1,$ $36^1\ 13.3^{36}\ 4^{231}\ 0^{280}\ -8^{120}\ -13.3^{36}$

$\{36, 35, 16, 4, 1; 1, 4, 16, 35, 36\}$ is 2-cover of $\{36, 35; 1, 4\}$

// $v = 1408 = 1 + 36 + 315 + 945 + 108 + 3,$ $36^1\ 13.3^{108}\ 4^{231}\ 0^{840}\ -8^{120}\ -13.3^{108}$

$\{36, 35, 24, 4, 1; 1, 4, 8, 35, 36\}$ is 4-cover of $\{36, 35; 1, 4\}$

Krein condition fails for $1, 5, 1$;

// $v = 2816 = 1 + 36 + 315 + 2205 + 252 + 7,$ $36^1\ 13.3^{252}\ 4^{231}\ 0^{1960}\ -8^{120}\ -13.3^{252}$

$\{36, 35, 28, 4, 1; 1, 4, 4, 35, 36\}$ is 8-cover of $\{36, 35; 1, 4\}$

Krein condition fails for $1, 5, 1$;

// $v = 432 = 1 + 40 + 175 + 175 + 40 + 1,$ $40^1\ 20^{12}\ 4^{140}\ 0^{189}\ -8^{75}\ -16^{15}$

$\{40, 35, 16, 8, 1; 1, 8, 16, 35, 40\}$ is 2-cover of $\{40, 35; 1, 8\}$

Krein condition fails for $1, 4, 1$;

absolute bound fails for $1, 1$;

// $v = 864 = 1 + 40 + 175 + 525 + 120 + 3,$ $40^1\ 20^{36}\ 4^{140}\ 0^{567}\ -8^{75}\ -16^{45}$

$\{40, 35, 24, 8, 1; 1, 8, 8, 35, 40\}$ is 4-cover of $\{40, 35; 1, 8\}$

Krein condition fails for $1, 4, 1$;

absolute bound fails for $1, 1$;

$v = 512 = 1 + 45 + 210 + 210 + 45 + 1,$ $\quad 45^1\ 27^{10}\ 13^{45}\ 3^{120}\ \ -3^{210}\ \ -5^{126}$
$\{45, 28, 15, 6, 1; 1, 6, 15, 28, 45\}$ is 2-cover of $\{45, 28; 1, 6\}$

// $v = 768 = 1 + 45 + 210 + 420 + 90 + 2,$ $\quad 45^1\ 27^{20}\ 13^{45}\ 3^{240}\ \ -3^{210}\ \ -5^{252}$
$\{45, 28, 20, 6, 1; 1, 6, 10, 28, 45\}$ is 3-cover of $\{45, 28; 1, 6\}$
Krein condition fails for $1, 5, 1$;
absolute bound fails for $1, 1$;

// $v = 1280 = 1 + 45 + 210 + 840 + 180 + 4,$ $\quad 45^1\ 27^{40}\ 13^{45}\ 3^{480}\ \ -3^{210}\ \ -5^{504}$
$\{45, 28, 24, 6, 1; 1, 6, 6, 28, 45\}$ is 5-cover of $\{45, 28; 1, 6\}$
Krein condition fails for $1, 5, 1$;
absolute bound fails for $1, 1$;

? $v = 2352 = 1 + 50 + 1125 + 1125 + 50 + 1,$ $\quad 50^1\ 14^{175}\ 8^{500}\ 0^{756}\ \ -6^{675}\ \ -10^{245}$
$\{50, 45, 24, 2, 1; 1, 2, 24, 45, 50\}$ is 2-cover of $\{50, 45; 1, 2\}$

? $v = 3528 = 1 + 50 + 1125 + 2250 + 100 + 2,$ $\quad 50^1\ 14^{350}\ 8^{500}\ 0^{1512}\ \ -6^{675}\ \ -10^{490}$
$\{50, 45, 32, 2, 1; 1, 2, 16, 45, 50\}$ is 3-cover of $\{50, 45; 1, 2\}$

? $v = 4704 = 1 + 50 + 1125 + 3375 + 150 + 3,$ $\quad 50^1\ 14^{525}\ 8^{500}\ 0^{2268}\ \ -6^{675}\ \ -10^{735}$
$\{50, 45, 36, 2, 1; 1, 2, 12, 45, 50\}$ is 4-cover of $\{50, 45; 1, 2\}$

? $v = 7056 = 1 + 50 + 1125 + 5625 + 250 + 5,$ $\quad 50^1\ 14^{875}\ 8^{500}\ 0^{3780}\ \ -6^{675}\ \ -10^{1225}$
$\{50, 45, 40, 2, 1; 1, 2, 8, 45, 50\}$ is 6-cover of $\{50, 45; 1, 2\}$

? $v = 9408 = 1 + 50 + 1125 + 7875 + 350 + 7,$ $\quad 50^1\ 14^{1225}\ 8^{500}\ 0^{5292}\ \ -6^{675}\ \ -10^{1715}$
$\{50, 45, 42, 2, 1; 1, 2, 6, 45, 50\}$ is 8-cover of $\{50, 45; 1, 2\}$

? $v = 14112 = 1 + 50 + 1125 + 12375 + 550 + 11,$ $\quad 50^1\ 14^{1925}\ 8^{500}\ 0^{8316}\ \ -6^{675}\ \ -10^{2695}$
$\{50, 45, 44, 2, 1; 1, 2, 4, 45, 50\}$ is 12-cover of $\{50, 45; 1, 2\}$

? $v = 18816 = 1 + 50 + 1125 + 16875 + 750 + 15,$ $\quad 50^1\ 14^{2625}\ 8^{500}\ 0^{11340}\ \ -6^{675}\ \ -10^{3675}$
$\{50, 45, 45, 2, 1; 1, 2, 3, 45, 50\}$ is 16-cover of $\{50, 45; 1, 2\}$

// $v = 972 = 1 + 51 + 272 + 544 + 102 + 2,$ $\quad 51^1\ 27^{34}\ 15^{51}\ 0^{512}\ \ -3^{272}\ \ -9^{102}$
$\{51, 32, 30, 6, 1; 1, 6, 15, 32, 51\}$ is 3-cover of $\{51, 32; 1, 6\}$

Krein condition fails for $1, 5, 1$;
absolute bound fails for $1, 1$;

? $v = 1300 = 1 + 55 + 594 + 594 + 55 + 1$, $\quad 55^1\ 18.0^{55}\ 5^{429}\ 0^{540}\ -10^{220}\ -18.0^{55}$
$\{55, 54, 25, 5, 1; 1, 5, 25, 54, 55\}$ is 2-cover of $\{55, 54; 1, 5\}$

// $v = 3250 = 1 + 55 + 594 + 2376 + 220 + 4$, $\quad 55^1\ 18.0^{220}\ 5^{429}\ 0^{2160}\ -10^{220}\ -18.0^{220}$
$\{55, 54, 40, 5, 1; 1, 5, 10, 54, 55\}$ is 5-cover of $\{55, 54; 1, 5\}$
Krein condition fails for $1, 5, 1$;

// $v = 6500 = 1 + 55 + 594 + 5346 + 495 + 9$, $\quad 55^1\ 18.0^{495}\ 5^{429}\ 0^{4860}\ -10^{220}\ -18.0^{495}$
$\{55, 54, 45, 5, 1; 1, 5, 5, 54, 55\}$ is 10-cover of $\{55, 54; 1, 5\}$
Krein condition fails for $1, 5, 1$;

// $v = 4725 = 1 + 64 + 880 + 3520 + 256 + 4$, $\quad 64^1\ 20^{400}\ 10^{350}\ -1^{3072}\ -6^{594}\ -16^{308}$
$\{64, 55, 52, 4, 1; 1, 4, 13, 55, 64\}$ is 5-cover of $\{64, 55; 1, 4\}$
Krein condition fails for $1, 5, 1$;

? $v = 2160 = 1 + 78 + 1001 + 1001 + 78 + 1$, $\quad 78^1\ 23.2^{78}\ 6^{715}\ 0^{924}\ -12^{364}\ -23.2^{78}$
$\{78, 77, 36, 6, 1; 1, 6, 36, 77, 78\}$ is 2-cover of $\{78, 77; 1, 6\}$

// $v = 3240 = 1 + 78 + 1001 + 2002 + 156 + 2$, $\quad 78^1\ 23.2^{156}\ 6^{715}\ 0^{1848}\ -12^{364}\ -23.2^{156}$
$\{78, 77, 48, 6, 1; 1, 6, 24, 77, 78\}$ is 3-cover of $\{78, 77; 1, 6\}$
Krein condition fails for $1, 5, 1$;

// $v = 4320 = 1 + 78 + 1001 + 3003 + 234 + 3$, $\quad 78^1\ 23.2^{234}\ 6^{715}\ 0^{2772}\ -12^{364}\ -23.2^{234}$
$\{78, 77, 54, 6, 1; 1, 6, 18, 77, 78\}$ is 4-cover of $\{78, 77; 1, 6\}$
Krein condition fails for $1, 5, 1$;

// $v = 6480 = 1 + 78 + 1001 + 5005 + 390 + 5$, $\quad 78^1\ 23.2^{390}\ 6^{715}\ 0^{4620}\ -12^{364}\ -23.2^{390}$
$\{78, 77, 60, 6, 1; 1, 6, 12, 77, 78\}$ is 6-cover of $\{78, 77; 1, 6\}$
Krein condition fails for $1, 5, 1$;

// $v = 8640 = 1 + 78 + 1001 + 7007 + 546 + 7$, $\quad 78^1\ 23.2^{546}\ 6^{715}\ 0^{6468}\ -12^{364}\ -23.2^{546}$
$\{78, 77, 63, 6, 1; 1, 6, 9, 77, 78\}$ is 8-cover of $\{78, 77; 1, 6\}$
Krein condition fails for $1, 5, 1$;

// $v = 9720 = 1+78+1001+8008+624+8$, $\quad 78^1\ 23.2^{624}\ 6^{715}\ 0^{7392}\ -12^{364}\ -23.2^{624}$

$\{78,77,64,6,1;1,6,8,77,78\}$ is 9-cover of $\{78,77;1,6\}$

Krein condition fails for $1,5,1$;

// $v = 12960 = 1+78+1001+11011+858+11$, $\quad 78^1\ 23.2^{858}\ 6^{715}\ 0^{10164}\ -12^{364}\ -23.2^{858}$

$\{78,77,66,6,1;1,6,6,77,78\}$ is 12-cover of $\{78,77;1,6\}$

Krein condition fails for $1,5,1$;

? $v = 22780 = 1+78+5616+16848+234+3$, $\quad 78^1\ 14^{4355}\ 11^{2210}\ -3^{10452}\ -7^{3484}\ -13^{2278}$

$\{78,72,63,1,1;1,1,21,72,78\}$ is 4-cover of $\{78,72;1,1\}$

? $v = 39865 = 1+78+5616+33696+468+6$, $\quad 78^1\ 14^{8710}\ 11^{2210}\ -3^{20904}\ -7^{3484}\ -13^{4556}$

$\{78,72,72,1,1;1,1,12,72,78\}$ is 7-cover of $\{78,72;1,1\}$

// $v = 2187 = 1+78+650+1300+156+2$, $\quad 78^1\ 39^{54}\ 24^{78}\ 3^{702}\ -3^{650}\ -6^{702}$

$\{78,50,42,6,1;1,6,21,50,78\}$ is 3-cover of $\{78,50;1,6\}$

Krein condition fails for $1,5,1$;

absolute bound fails for $1,1$;

? $v = 6808 = 1+82+3321+3321+82+1$, $\quad 82^1\ 10^{1517}\ 8^{1886}\ -8^{1886}\ -10^{1517}\ -82^1$

$\{82,81,80,2,1;1,2,80,81,82\}$ is 2-cover of $\{82,81;1,2\}$

// $v = 810 = 1+96+308+308+96+1$, $\quad 96^1\ 54^8\ 6^{264}\ 0^{385}\ -12^{140}\ -36^{12}$

$\{96,77,36,24,1;1,24,36,77,96\}$ is 2-cover of $\{96,77;1,24\}$

Krein condition fails for $1,4,1$;

absolute bound fails for $1,1$;

// $v = 1215 = 1+96+308+616+192+2$, $\quad 96^1\ 54^{16}\ 6^{264}\ 0^{770}\ -12^{140}\ -36^{24}$

$\{96,77,48,24,1;1,24,24,77,96\}$ is 3-cover of $\{96,77;1,24\}$

Krein condition fails for $1,4,1$;

absolute bound fails for $1,1$;

? $v = 5152 = 1+100+2475+2475+100+1$, $\quad 100^1\ 12^{1035}\ 8^{1540}\ -8^{1540}\ -12^{1035}\ -100^1$

$\{100,99,96,4,1;1,4,96,99,100\}$ is 2-cover of $\{100,99;1,4\}$

// $v = 4752 = 1+100+2275+2275+100+1$, $\quad 100^1\ 20^{429}\ 12^{945}\ -4^{1925}\ -8^{1430}\ -40^{22}$

$\{100,91,64,4,1;1,4,64,91,100\}$ is 2-cover of $\{100,91;1,4\}$

absolute bound fails for $5,5$;

Distance regular antipodal and bipartite covers with diameter five of nontrivial strongly regular graphs

$v=20=1+3+6+6+3+1,\quad 3^1\ 2^4\ 1^5\ -1^5\ -2^4\ -3^1$

$\{3,2,2,1,1;1,1,2,2,3\}$ is 2-cover of $\{3,2;1,1\}$

$v=32=1+5+10+10+5+1,\quad 5^1\ 3^5\ 1^{10}\ -1^{10}\ -3^5\ -5^1$

$\{5,4,3,2,1;1,2,3,4,5\}$ is 2-cover of $\{5,4;1,2\}$

$v=100=1+7+42+42+7+1,\quad 7^1\ 3^{21}\ 2^{28}\ -2^{28}\ -3^{21}\ -7^1$

$\{7,6,6,1,1;1,1,6,6,7\}$ is 2-cover of $\{7,6;1,1\}$

$v=112=1+10+45+45+10+1,\quad 10^1\ 4^{20}\ 2^{35}\ -2^{35}\ -4^{20}\ -10^1$

$\{10,9,8,2,1;1,2,8,9,10\}$ is 2-cover of $\{10,9;1,2\}$

// $v=112=1+10+45+45+10+1,\quad 10^1\ 5.3^{10}\ 2^{35}\ 0^{36}\ -4^{20}\ -5.3^{10}$

$\{10,9,4,2,1;1,2,4,9,10\}$ is 2-cover of $\{10,9;1,2\}$

$v=154=1+16+60+60+16+1,\quad 16^1\ 6^{21}\ 2^{55}\ -2^{55}\ -6^{21}\ -16^1$

$\{16,15,12,4,1;1,4,12,15,16\}$ is 2-cover of $\{16,15;1,4\}$

? $v=324=1+21+140+140+21+1,\quad 21^1\ 6^{56}\ 3^{105}\ -3^{105}\ -6^{56}\ -21^1$

$\{21,20,18,3,1;1,3,18,20,21\}$ is 2-cover of $\{21,20;1,3\}$

$v=200=1+22+77+77+22+1,\quad 22^1\ 8^{22}\ 2^{77}\ -2^{77}\ -8^{22}\ -22^1$

$\{22,21,16,6,1;1,6,16,21,22\}$ is 2-cover of $\{22,21;1,6\}$

? $v=352=1+25+150+150+25+1,\quad 25^1\ 7^{55}\ 3^{120}\ -3^{120}\ -7^{55}\ -25^1$

$\{25,24,21,4,1;1,4,21,24,25\}$ is 2-cover of $\{25,24;1,4\}$

? $v=704=1+26+325+325+26+1,\quad 26^1\ 6^{143}\ 4^{208}\ -4^{208}\ -6^{143}\ -26^1$

$\{26,25,24,2,1;1,2,24,25,26\}$ is 2-cover of $\{26,25;1,2\}$

? $v = 420 = 1 + 33 + 176 + 176 + 33 + 1,$ $\quad 33^1\ 9^{55}\ 3^{154}\ \ -3^{154}\ \ -9^{55}\ \ -33^1$

$\{33, 32, 27, 6, 1; 1, 6, 27, 32, 33\}$ is 2-cover of $\{33, 32; 1, 6\}$

? $v = 704 = 1 + 36 + 315 + 315 + 36 + 1,$ $\quad 36^1\ 8^{120}\ 4^{231}\ \ -4^{231}\ \ -8^{120}\ \ -36^1$

$\{36, 35, 32, 4, 1; 1, 4, 32, 35, 36\}$ is 2-cover of $\{36, 35; 1, 4\}$

? $v = 1408 = 1 + 37 + 666 + 666 + 37 + 1,$ $\quad 37^1\ 7^{296}\ 5^{407}\ \ -5^{407}\ \ -7^{296}\ \ -37^1$

$\{37, 36, 35, 2, 1; 1, 2, 35, 36, 37\}$ is 2-cover of $\{37, 36; 1, 2\}$

? $v = 532 = 1 + 45 + 220 + 220 + 45 + 1,$ $\quad 45^1\ 12^{56}\ 3^{209}\ \ -3^{209}\ \ -12^{56}\ \ -45^1$

$\{45, 44, 36, 9, 1; 1, 9, 36, 44, 45\}$ is 2-cover of $\{45, 44; 1, 9\}$

? $v = 784 = 1 + 46 + 345 + 345 + 46 + 1,$ $\quad 46^1\ 10^{115}\ 4^{276}\ \ -4^{276}\ \ -10^{115}\ \ -46^1$

$\{46, 45, 40, 6, 1; 1, 6, 40, 45, 46\}$ is 2-cover of $\{46, 45; 1, 6\}$

? $v = 1276 = 1 + 49 + 588 + 588 + 49 + 1,$ $\quad 49^1\ 9^{231}\ 5^{406}\ \ -5^{406}\ \ -9^{231}\ \ -49^1$

$\{49, 48, 45, 4, 1; 1, 4, 45, 48, 49\}$ is 2-cover of $\{49, 48; 1, 4\}$

? $v = 2552 = 1 + 50 + 1225 + 1225 + 50 + 1,$ $\quad 50^1\ 8^{550}\ 6^{725}\ \ -6^{725}\ \ -8^{550}\ \ -50^1$

$\{50, 49, 48, 2, 1; 1, 2, 48, 49, 50\}$ is 2-cover of $\{50, 49; 1, 2\}$

? $v = 1300 = 1 + 55 + 594 + 594 + 55 + 1,$ $\quad 55^1\ 10^{220}\ 5^{429}\ \ -5^{429}\ \ -10^{220}\ \ -55^1$

$\{55, 54, 50, 5, 1; 1, 5, 50, 54, 55\}$ is 2-cover of $\{55, 54; 1, 5\}$

? $v = 648 = 1 + 57 + 266 + 266 + 57 + 1,$ $\quad 57^1\ 15^{57}\ 3^{266}\ \ -3^{266}\ \ -15^{57}\ \ -57^1$

$\{57, 56, 45, 12, 1; 1, 12, 45, 56, 57\}$ is 2-cover of $\{57, 56; 1, 12\}$

? $v = 6500 = 1 + 57 + 3192 + 3192 + 57 + 1,$ $\quad 57^1\ 8^{1520}\ 7^{1729}\ \ -7^{1729}\ \ -8^{1520}\ \ -57^1$

$\{57, 56, 56, 1, 1; 1, 1, 56, 56, 57\}$ is 2-cover of $\{57, 56; 1, 1\}$

? $v = 2146 = 1 + 64 + 1008 + 1008 + 64 + 1,$ $\quad 64^1\ 10^{406}\ 6^{666}\ \ -6^{666}\ \ -10^{406}\ \ -64^1$

$\{64, 63, 60, 4, 1; 1, 4, 60, 63, 64\}$ is 2-cover of $\{64, 63; 1, 4\}$

? $v = 1104 = 1 + 76 + 475 + 475 + 76 + 1,$ $\quad 76^1\ 16^{114}\ 4^{437}\ \ -4^{437}\ \ -16^{114}\ \ -76^1$

$\{76, 75, 64, 12, 1; 1, 12, 64, 75, 76\}$ is 2-cover of $\{76, 75; 1, 12\}$

? $v = 2160 = 1 + 78 + 1001 + 1001 + 78 + 1,\qquad 78^1\ 12^{364}\ 6^{715}\ -6^{715}\ -12^{364}\ -78^1$
$\{78,77,72,6,1;1,6,72,77,78\}$ is 2-cover of $\{78,77;1,6\}$

? $v = 3404 = 1 + 81 + 1620 + 1620 + 81 + 1,\qquad 81^1\ 11^{666}\ 7^{1035}\ -7^{1035}\ -11^{666}\ -81^1$
$\{81,80,77,4,1;1,4,77,80,81\}$ is 2-cover of $\{81,80;1,4\}$

? $v = 1600 = 1 + 85 + 714 + 714 + 85 + 1,\qquad 85^1\ 15^{204}\ 5^{595}\ -5^{595}\ -15^{204}\ -85^1$
$\{85,84,75,10,1;1,10,75,84,85\}$ is 2-cover of $\{85,84;1,10\}$

? $v = 1334 = 1 + 96 + 570 + 570 + 96 + 1,\qquad 96^1\ 20^{115}\ 4^{551}\ -4^{551}\ -20^{115}\ -96^1$
$\{96,95,80,16,1;1,16,80,95,96\}$ is 2-cover of $\{96,95;1,16\}$

? $v = 3300 = 1 + 97 + 1552 + 1552 + 97 + 1,\qquad 97^1\ 13^{582}\ 7^{1067}\ -7^{1067}\ -13^{582}\ -97^1$
$\{97,96,91,6,1;1,6,91,96,97\}$ is 2-cover of $\{97,96;1,6\}$

? $v = 2356 = 1 + 99 + 1078 + 1078 + 99 + 1,\qquad 99^1\ 15^{341}\ 6^{836}\ -6^{836}\ -15^{341}\ -99^1$
$\{99,98,90,9,1;1,9,90,98,99\}$ is 2-cover of $\{99,98;1,9\}$

The only four parameter sets in our list which are known not to exist are:

$$\{10,6,4,1;1,3,6,10\},$$

$$\{10,9,1,1;1,1,9,10\},$$

$$\{10,9,4,2,1;1,2,4,9,10\},$$

$$\{28,15,8,1;1,4,15,28\}.$$

The first one should be a distance regular antipodal cover of complement of the triangular graph $T(7)$. Using Theorem 4.1. it can be proven that such graph has no distance regular antipodal cover of diameter four (see [BB]). The fourth one should be a distance regular antipodal cover of the halved 8-cube. Again using Theorem 4.1. it can be shown that the 8-cube is the only distance regular antipodal cover of it (see [BB]). The second and third one should be a distance regular antipodal covers of the Gewirtz graph. But the only distance regular antipodal cover of this graph is its bipartite double with intersection array $\{10,9,8,2,1;1,2,8,9,10\}$ (for the proof of this see [BCN]).

APPENDIX A

A t-(v, s, λ_t) *design* is a collection of s-subsets (called *blocks*) of a set of v elements called *points*, such that each t-set of points lies in exactly λ_t blocks. We assume that $t < s$ to exclude degenerate cases. If $\lambda_t = 1$, then a t-design is called *Steiner system*.

In a t-design let λ_i denotes the number of blocks containing a given set of i points, with $0 \leq i \leq t$. Let S be some i-set. Then S is contained in λ_i blocks and each of them contains $\binom{s-i}{t-i}$ distinct t-sets with S as subset. On the other hand the set S can be enlarged to t-set in $\binom{v-i}{t-i}$ ways and each of these t-set is contained in λ_t blocks. So we conclude:

$$\lambda_i \binom{s-i}{t-i} = \lambda_t \binom{v-i}{t-i}$$

Therefore λ_i is independent of S. This actually means that a t-design is also i-design, for $0 \leq i \leq t$. The number of blocks in a design is equal to λ_0 and is denoted by b. Every point in a 1-design lies in λ_1 blocks and this number is often denoted by r. When $t \geq 2$ we get from upper identity for $i = 0$, $t = 1$ and $i = 1$, $t = 2$ the following:

$$bs = rv \quad \text{and} \quad r(s-1) = \lambda_2(v-1)$$

or

$$r = \lambda_2 \frac{v-1}{s-1} \quad \text{and} \quad b = \lambda_2 \frac{v(v-1)}{s(s-1)}$$

For any design with more then one block $b \geq v$, which is known as Fisher's inequality [R]. Designs with $b = v$ are called *square* (or *symmetric*) and have property that any two blocks meet in exactly λ_t points. A design is said to be *quasi symmetric* if the cardinality of the intersection of two distinct blocks takes only two values. Any 2-$(v, s, 1)$ design is an example of such design.

APPENDIX B

An *orthogonal array* $OA(s,v)$ is an $v^2 \times s$ array of v symbols such that in any pair of distinct columns of the array each ordered pair of symbols occurs exactly once. A geometric representation of $OA(s,v)$ can be given in the following way. Take a set P of sv symbols (elements) called *points* and partition it into s subsets $G_1, \ldots, G_s$ (called *groups*) of size v. Now choose from the set of points P a family L of s subsets (called *lines*) which has the property that every pair of distinct points in P occurs in precisely one group or one line but not both. In other words we require that

(a) each line intersects each group in exactly one point and each other line in at most one point.

(b) for any two points from distinct groups there is a line which contains them (by (a) this line is unique).

It is easy to see that the number of lines has to be v^2. This geometric representation is known as *transversal design* $TD(s,v)$.

Figure 6: A transversal design

If we denote the elements in each group by the numbers $1, \ldots, v$, we can present each line l as s-tuple, where i-th entry is equal to the label of $l \cap G_i$. This gives us an orthogonal array. On the other hand if for each column of an orthogonal array we identify positions with the same entries, new columns can be treated as groups and rows of $OA(s,v)$ as lines of a transversal design $TD(s,v)$.

Now choose a line l of $TD(s,v)$. Then all lines through the point $l \cap G_1$ partition each G_i for $i \geq 2$. There is exactly v such lines, since G_i has v elements. So the line l intersects in each group with $v-1$ lines. Of course these lines are pairwise distinct for different groups and their number is less or equal to the number of all lines in $TD(s,v)$. This gives us $s(v-1)+1 \leq v^2$, and thus $s \leq v+1$.

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