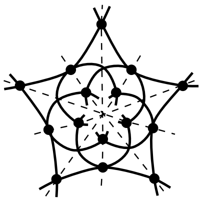
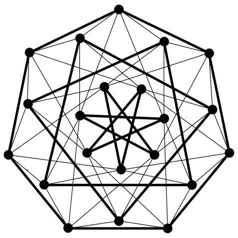
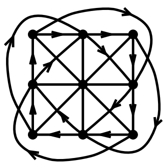
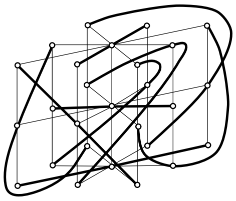
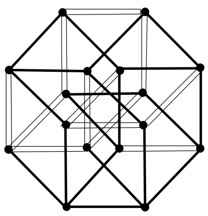
ANTIPODAL COVERS

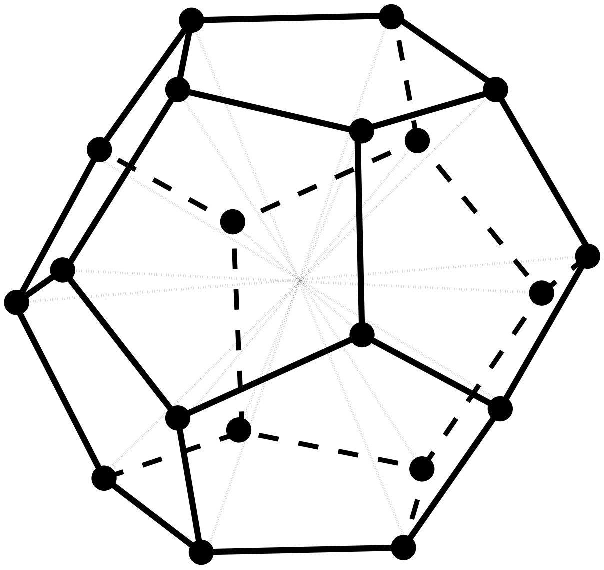




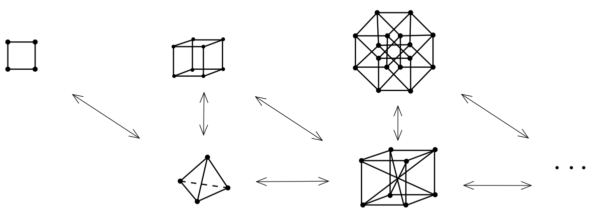








A. JURIŠIĆ



Cover:

1) The Petersen graph is hidden inside the dodecahedron. Where?
   For more on distance-regular graphs with $b_i = 1$ for small i see Theorem 7.1.1, which is a joint work with Araya and Hiraki.

Small figures (from top to bottom):

2) The generalized quadrangle $GQ(2,2)$ with a spread deleted (dashed). Its point graph is a unique distance-regular cover of K_5, cf. Section 3.1, Table 3.1 and Brouwer's Theorem 3.3.1.
3) The distance-regular cover of K_8 (with one antipodal class deleted). Each of the three missing vertices is connected to one of the three 'bold' heptagons. This cover is called the *Klein graph* and it is a unique distance-regular graph which is locally a heptagon, cf. Section 3.2 (Lemma 3.2.2) and Mathon's construction on page 27. Distance-regular graphs which are locally strongly regular are studied in Chapter 4, see Theorem 4.5.7, which is a joint work with Koolen.
4) The cyclic cover of K_9 is described by orienting some lines of the affine plane $AG(2,3)$, cf. Theorem 3.3.2 by Brouwer and Wilbrink, Section 3.4 and Godsil's Theorem 3.5.4. The other distance-regular cover of K_9 can be obtained by 'switching' any directed line with three directed loops, cf. Section 3.5 (Theorem 3.5.3).
5) A unique spread of the symplectic generalized quadrangle $W(3)$ has been find by studying the generalized quadrangle $GQ(2,4)$, cf. Payne's Construction 3.5.1, Section 3.6 and Section 3.7.
6) A unique double-cover of the 4-cube Q_4 with no quadrangles is described by a 2-colouring of Q_4, cf. Proposition 6.1.5 by Cohen and Tits and Chapter 6, where antipodal covers (of strongly regular graphs), which are not necessarilly distance-regular, are studied.
7) Merging (skew arrows) in n-cubes which cover folded n-cubes is realized by connecting vertices at maximum distance, cf. Chapter 5, which contains a characterization of certain distance-regular antipodal covers with regular near polygons (Theorem 5.3.3).

ANTIPODAL COVERS

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ANTIPODAL COVERS

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Abstract

We study antipodal distance-regular graphs. We start with an investigation of cyclic covers and spreads of generalized quadrangles and find a switching, which uses some known infinite families of antipodal distance-regular graphs of diameter three to produce new ones. Then we examine antipodal distance-regular graphs of diameter four and five. P. Terwilliger has shown, using the theory of subconstituent algebras, that in a Q-polynomial antipodal distance-regular graph the neighbourhood of any vertex is a strongly regular graph. We use representations of graphs to extend this result and to derive from that new nonexistence conditions for covers. This study relates to the above switching and to extended generalized quadrangles.

In an imprimitive association scheme there always exists a merging (i.e., a grouping of the relations) which gives a new nontrivial association scheme. We determine when merging in an antipodal distance-regular graph produces a distance-regular graph. This leads to our main result, a characterization of certain antipodal distance-regular graphs with regular near polygons containing a spread. In case of diameter three we get Brouwer's characterization of certain distance-regular graphs with generalized quadrangles containing a spread.

Finally, antipodal covers of strongly regular graphs which are not necessarily distance-regular are studied. In most cases, the structure of short cycles provides a tool to determine the existence of an antipodal cover. A relationship between antipodal covers of a graph and its line graph is investigated. Antipodal covers of complete bipartite graphs and their line graphs (lattice graphs) are characterized in terms of weak resolvable transversal designs which are, in the case of maximal covering index, equivalent to affine planes with a parallel class deleted.

We conclude by mentioning two results which indicate the importance of antipodal distance-regular graphs.

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My work has benefited from conversations, correspondence, and electronic communications with Makoto Araya, Andries Brouwer, Akira Hiraki, Jack Koolen, Robert Liebler, and Paul Terwilliger.

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Finally, I wish to thank to my father Dragoš Jurišić, who is unfortunately not among us any more, for his continuous enthusiasm to discuss with me about my research and my graduate studies, and to my wife Alenka Trojar-Jurišić for her unwavering support.

To my parents

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1

INTRODUCTION

First, equitable partitions are used to define distance-regular graphs and antipodal covers. Then we explain why distance-regular antipodal covers are interesting and finally we give a summary of our main results.

As it is quite natural to introduce new subjects through examples, we start with two remarkable combinatorial objects, the Petersen graph and the dodecahedron,

Figure 1.1: The Petersen graph and the dodecahedron.

and explain how they are related through *equitable partitions*. These are partitions $\pi = \{C_1, \ldots, C_s\}$ of the vertex set of a graph G, such that for all i and j the number c_{ij} of neighbours, which a vertex in C_i has in the cell C_j, is independent of the choice of the vertex in C_i. In other words each cell C_i induces a regular graph of valency c_{ii}, and between any two cells C_i and C_j there is a biregular graph, with vertices of the cells C_i and C_j having valencies c_{ij} and c_{ji} respectively. The antipodal pairs of vertices of the dodecahedron, determine one such partition. Another example is the *distance partition*, i.e., partition of the vertices corresponding to their distances from a particular vertex, of the dodecahedron, the Petersen graph, or its line graph.

Equitable partitions give rise to *quotient* graphs G/π, which are directed multigraphs with cells as vertices and c_{ij} arcs going from C_i to C_j. From the distance partitions of the dodecahedron, the Petersen graph and its line graph, and the partition of dodecahedron into antipodal pairs of vertices we get the following quotients, see Figure 1.2 and Figure 1.3.

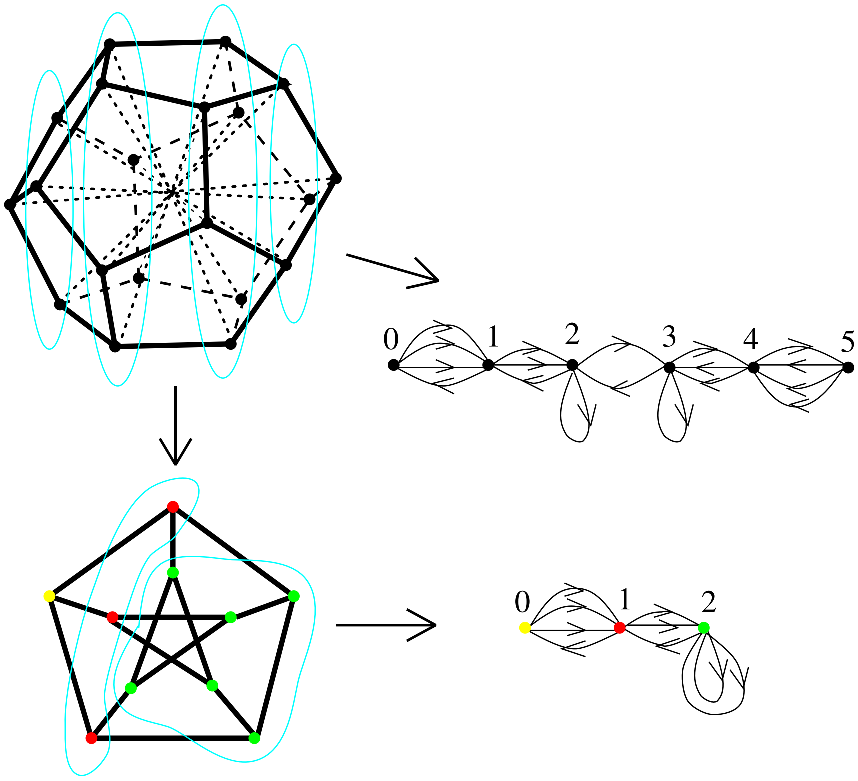

Figure 1.2: The quotients corresponding to the distance and antipodal partitions of the dodecahedron, and the distance partition of the Petersen graph.

One of the most important properties of the equitable partitions is that the eigenvalues of the quotient graph G/π are also the eigenvalues of the original graph G, thus all the eigenvalues of the Petersen graph are also eigenvalues of the dodecahedron.

A graph of diameter d has at least $d+1$ eigenvalues (cf. Theorem 2.2.2), and in many cases this lower bound is tight. If a distance partition is equitable for each vertex u of a connected graph G and the parameters c_{ij} of these equitable partitions do not depend on choice of u, then G is called *distance-regular graph*. Suppose that a graph G is distance-regular and that C_i is the set of vertices at distance i from u, then $c_{ij} = 0$ for $|i-j| > 1$, and the parameters c_{ii}, $c_{i,i+1}$ and $c_{i,i-1}$ are denoted by a_i, b_i and c_i respectively. Since for a distance-regular graph the quotient graph corresponding to the distance partition inherits all the eigenvalues of the original graph, the above lower bound on the number of eigenvalues is tight in this case. Distance-regular graphs of diameter two are called *strongly regular graphs*. Some examples of distance-regular graphs are complete graphs, complete multipartite graphs, cycles, n-cubes, 1-skeletons of the Platonic solids, the Petersen graph and its line graph. The *intersection arrays* $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ of the last two graphs and of the dodecahedron are respectively $\{3, 2; 1, 1\}$, $\{4, 2, 1; 1, 1, 4\}$, and $\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$.

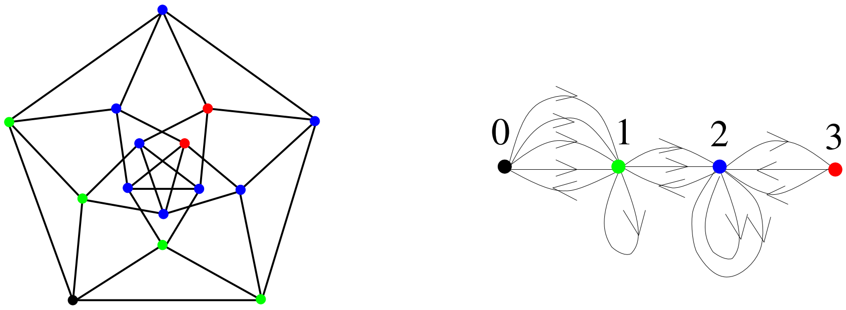

Figure 1.3: The quotient corresponding to the distance partition of the line graph of the Petersen graph. For example, the '2' in the intersection array means that each green vertex (i.e., a vertex at distance one from a chosen vertex) has exactly two blue neighbours (i.e., two neighbours at distance two from the chosen vertex).

There are also many infinite families of distance-regular graphs, for example Johnson and Hamming graphs, which offer, through the study of distance-regular graphs, a unifying approach to design and coding theories. For a detailed treatment of distance-regular graphs see Biggs [13], Bannai and Ito [8], Brouwer, Cohen and Neumaier [27] and Godsil [64].

A *distance transitive* graph, i.e., a graph where any two vertices can be mapped by an automorphism to any other two vertices at the same distance, is obviously distance-regular. Although all the above examples are distance transitive there are also distance-regular graphs which are not distance transitive, see for example Shrikhande [123] or Figure 3.3. We can consider distance-regularity as a weakening of the condition of distance transitivity; instead of complete symmetry of a graph, there is just a numerical regularity. In this sense distance-regularity is a combinatorial approximation of the algebraic property of 'being distance transitive'. This correctly forecasts an interlacing of the combinatorial and the algebraic approach.

Distance-regular graphs also have important connections with areas other than algebra: in combinatorics with finite geometries, coding theory, design theory, and in functional analysis with orthogonal polynomials. These graphs are a special class of association schemes, one of the most important unifying concepts in algebraic combinatorics.

The above relation between the Petersen graph and the dodecahedron is the central theme of this thesis, so let us explore some additional properties of the partition corresponding to the diagonals of the dodecahedron. First observe that in this particular case $c_{ii} = 0$ for each i, and c_{ij} either is zero or one when $i \neq j$. In other words, the cells must be independent sets and there is either a 1-factor or nothing between any two cells. A graph with such a partition is called a *cover* of its quotient G/π and the cells are called *fibres*. If G/π is connected then all the fibres have the same size called *covering index* and usually denoted by r. This terminology comes from topology, since the geometric realization of these covers are covering spaces in the usual topological sense.

A graph G of diameter d is *antipodal* if the vertices at distance d from a given vertex are all at distance d from each other. Then 'being at distance d or zero' induces an equivalence relation on the vertices of G, and the equivalence classes are called *antipodal classes*. Antipodal distance-regular graphs of diameter at least three are covers with antipodal classes as fibres (diameter two case is excluded because of complete multipartite graphs, e.g., the octahedron). A cover of index r, in which the fibres are antipodal classes, is called *antipodal r-cover* of its quotient. For example, the cube is the unique distance-regular antipodal double-cover of the tetrahedron, i.e., K_4, the line graph of the the Petersen graph is the unique distance-regular antipodal triple-cover of K_5, the icosahedron is a distance-regular antipodal double-cover of K_6, and the dodecahedron is a distance-regular antipodal double-cover of the Petersen graph.

In the line graph of the Petersen graph, each edge lies in a unique maximal clique, which is a triangle in this case. A graph with this property is a *collinearity graph* (also called the *point graph*) of the partial linear space, formed by the vertices of the graph as points and the maximal cliques of it as lines, see Figure 1.4.

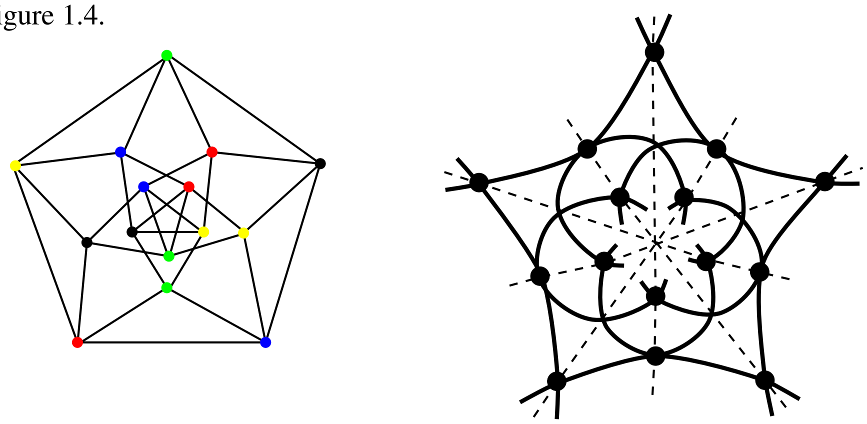

Figure 1.4: The line graph of the Petersen graph together with its antipodal classes is a collinearity graph of a near polygon (called generalized quadrangle $GQ(2,2)$, since the incidence structure does not contain triangles, there are 2+1 points on each line and 2+1 lines through each point).

The line graph of the Petersen graph, together with its antipodal classes actually determines an incidence structure called *near polygon*, since additionally for each vertex u and a maximal clique C with $d(u, C)$ less then the diameter of the graph, there is a unique vertex of C nearest to u. This property is modeled after a property of ordinary polygons.

In general, antipodal distance-regular graphs of small diameter give rise to various important combinatorial structures such as projective planes, Hadamard matrices, and even to more general objects such as symmetric group divisible designs and resolvable transversal designs, see Gardiner [59], Drake [57], Delorme [53], Shad [121], Shawe-Taylor [122]. Distance-regular graphs serve as an alternative approach to these interesting combinatorial objects and allow the use of graph eigenvalues, graph representations and the theory of association schemes. Inspired by this, we investigate (distance-regular) antipodal covers of small diameter.

Our goal is to gain insight into the structure of antipodal distance-regular graphs, to construct new antipodal distance-regular graphs (an infinite family or some sporadic example) or to prove that for certain intersection arrays there are no such graphs, as well as to characterize certain antipodal distance-regular graphs, or to find some new technique which could be used in the study of distance-regular graphs in general.

New constructions usually lead to constructions of some other combinatorial objects (mentioned above), while sporadic examples show the true nature of this field. For the latest advances in the construction business of distance-regular graphs see de Caen, Mathon and Moorhouse [38].

Strong characterizations, which, with some luck, eventually lead to new constructions or provide a tool to prove uniqueness of a certain object, are very rare. In diameter three case, there are only a few characterizations of this kind. Two should be mentioned: distance-regular double-covers are characterized by (regular) two-graphs, and Brouwer's characterization by near polygons of diameter two, i.e., *generalized quadrangles*, containing a set of lines which partition the point set, called a *spread*. The second characterization is especially strong in conjunction with results on geometric graphs, see Cameron, Goethals and Seidel [46]. In diameter four case, strong characterizations have been known only for covers of complete bipartite graphs, for larger diameter no characterizations are known.

A graph G is said to be *locally* $\mathcal{C}$, where $\mathcal{C}$ is a graph or a class of graphs, when for each vertex u of G the neighbours of u induce a graph isomorphic to (respectively a member of) $\mathcal{C}$. For example, the icosahedron is locally a pentagon, and the point graphs of generalized quadrangles is locally a union of cliques. Local graphs of a distance-regular graph are regular, and in some special cases provide an alternative tool to study distance-regular graphs.

After having introduced some basic results about distance-regular graphs in Chapter 2, we start an investigation of antipodal distance-regular graphs with diameter three case in Chapter 3. We find a switching, an operation which uses some known infinite families of antipodal distance-regular graphs of diameter three, to produce new ones. A similar idea (switching on a regulus) has been used to construct non-Desarguesian planes from Desarguesian ones. Additionally, we prove uniqueness of some distance-regular covers of small complete graphs (more precisely K_i, $i < 9$), and geometric distance-regular cover of K_{10}.

In Chapter 4, we examine antipodal distance-regular graphs of diameter four and five. Their intersection numbers, Krein and absolute bounds are determined. We parametrize Q-polynomial antipodal distance-regular graphs of diameter four with two parameters. Among these graphs we investigate those which are locally generalized quadrangles $(q + 1, q - 1)$ (there are two known examples: $q = 2, 3$). This study relates to the above switching and to extended generalized quadrangles. Terwilliger showed, using the theory of Krein modules, that a Q-polynomial antipodal distance-regular graph is locally strongly regular. Together with J. Koolen we use graph representations to extend this result and to derive from that a new nonexistence conditions for covers. For example, from the set of feasible intersection arrays of antipodal distance-regular graphs one quarter of those which are Q-polynomial are ruled out.

In Chapter 5, we study mergings (i.e., groupings of the relations) in an imprimitive association scheme, which give new nontrivial association schemes. Merging was, for example, used to construct some new strongly regular graphs, see Brouwer and Van Lint [31]. We determine when merging the first and the last classes in an antipodal distance-regular graph (i.e., joining all the pairs of antipodal vertices with edges) produce a distance-regular graph. Conversely, given a distance-regular graph with the same intersection array as the merged graph and a certain clique partition, an antipodal distance-regular graph is constructed. This leads to the main result of this thesis, a characterization of certain antipodal distance-regular graphs (of arbitrary diameter) with regular near polygons containing a spread. In the case of diameter three, we get Brouwer's characterization of certain distance-regular graphs, with generalized quadrangles containing a spread. For example, certain distance-regular antipodal double-covers of strongly regular graphs are equivalent to certain triangle-free strongly regular graphs. As there are only a few such graphs known, this means that it will be extremely difficult to find such covers. Known examples of distance-regular graphs in which mergings works are: the D-cube as the double-cover of the folded D-cube and the folded $(D+1)$-cube as the merged graph; the bipartite double of the coset graph of the binary Golay code merges to the coset graph of the extended binary Golay code; the coset graph of the shortened ternary Golay code is a distance-regular antipodal three-fold cover with diameter four of the point graph of the truncated ternary Golay code and the Berlekamp- Van Lint-Seidel graph (the coset graph of ternary Golay code) as the merged graph. We find new infinite families of feasible parameters of distance-regular antipodal covers with diameter four.

Finally, in Chapter 6, antipodal covers of strongly regular graphs, which are not necessarily distance-regular, are studied. In most cases the structure of short cycles provides a tool to determine the existence of an antipodal cover. A relationship between antipodal covers of a graph and its line graph is investigated. Antipodal covers of complete bipartite graphs and their line graphs (the lattice graphs) are characterized in terms of weak resolvable transversal designs, which are, in the case of maximal covering index, equivalent to affine planes with a parallel class deleted.

At the end we mention two results which indicate the importance of antipodal distance-regular graphs. The first one is the result of collaboration with Araya and Hiraki. Let G be a distance-regular graph of diameter d and valency $k > 2$. If $b_t = 1$ and $2t \leq d$, then G is an antipodal double-cover. Consequently, if $m > 2$ is the multiplicity of an eigenvalue of the adjacency matrix of G and if G is not an antipodal double-cover, then $d \leq 2m - 3$. This result is an improvement of Godsil's diameter bound, which is very important for the classification of distance-regular graphs with an eigenvalue of small multiplicity (as opposed to a dual classification of distance-regular graphs with small valency). The second result is joint work with Godsil. We show that distance-regular graphs, that contain maximal independent geodesic paths of short length, are antipodal. A new infinite family of feasible parameters of antipodal distance-regular graphs of diameter four is found. As an auxiliary result, we use equitable partitions to show that the determinant of a Töplitz matrix can be written as a product of two determinants of approximately half the size of the original one.

In tables "!" means that a graph is uniquely determined by its parameters, "//" means that no graph realizes this particular parameter set, and "?" means that nothing is known.

2

PRELIMINARIES

In this Chapter we introduce some basic result about distance-regular graphs, eigenvalues of graphs, association schemes, antipodal graphs and generalized quadrangles, which are not our own and which are needed later in the thesis for reference purposes.

1. Distance-regular graphs

Let G be a graph. The *distance* between vertices u and v of a graph G will be the length of a shortest path between u and v, denoted by $\text{dist}_G(u, v)$ or just by $\text{dist}(u, v)$ when this is not ambiguous. Let u be a vertex of a graph G. Then $S_r(u)$ denotes the set of vertices at distance exactly r from u. We call $S_r(u)$ a *sphere* of radius r centered at u, or the r*-th neighbourhood* of u. In particular, we use $S(u)$ for $S_1(u)$ and call it the *neighbourhood* of a vertex u. Let $B_r(u) := \{u\} \cup S_1(u) \cup \cdots \cup S_r(u)$ be called a *ball* of radius r centered at u.

Let G be a distance-regular graph of diameter d. For vertices u and v at distance r and integers i, j let $p_{ij}(r)$ denote the value $|S_i(u) \cap S_j(v)|$. The numbers $p_{ij}(r)$ are called the *intersection numbers* of G. The valency of G is then $p_{11}(0) = |S(u)|$ and is usually denoted by k. Some other intersection numbers also have special names:

$$k_r = p_{rr}(0) = |S_r(u)|, \ \text{ for } r = 0, 1, \ldots, d,$$

$$a_r = p_{r1}(r) = |S(v) \cap S_r(u)|, \ \text{ for } r = 1, 2, \ldots, d,$$

$$b_r = p_{r+1,1}(r) = |S(v) \cap S_{r+1}(u)|, \ \text{ for } r = 0, 1, \ldots, d-1,$$

$$c_r = p_{r-1,1}(r) = |S(v) \cap S_{r-1}(u)|, \ \text{ for } r = 1, 2, \ldots, d,$$

and $\lambda = a_1$, $\mu = c_2$. Set $a_0 = c_0 = b_d = 0$, then $a_r + b_r + c_r = k$ for $r = 0, \ldots, d$ and $b_0 = k$, $c_1 = 1$.

All the intersection numbers are determined by the numbers in the *intersection array*

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$$

of G. This can be proved by induction on i using the following recurrence relation:

$$c_{j+1}p_{i,j+1}(r) + a_j p_{ij}(r) + b_{j-1}p_{i,j-1}(r) = c_{i+1}p_{i+1,j}(r) + a_i p_{ij}(r) + b_{i-1}p_{i-1,j}(r)$$

obtained by counting for vertices u and v at distance r the edges with one end in $S_i(u)$ and another in $S_j(v)$ in two different ways. Therefore the intersection numbers do not depend on the choice of u and v at distance r. Note that a distance-regular graph need not be uniquely determined by its parameters, the smallest such example is the Shrikhande graph, see [27, p. 104] or Figure 3.3., which has the same parameters as the 4×4-grid graph.

Distance-regular graphs of diameter two are called *strongly regular graphs*. They were introduced by Bose [19] and have been intensively studied since, see e.g. Seidel [118], Cameron and Van Lint [48], and Brouwer and Van Lint [31].

The following result gives us the basic properties of the parameters a_i, b_i, c_i and k_i of a distance-regular graph, see Brouwer et al. [27, pp. 127, 133, 167]:

2.1.1 LEMMA. *Let G be a distance-regular graph with valency k and diameter d. Then the following holds:*
(a) $k_{i-1}b_{i-1} = k_i c_i$, *for* $i = 1, ..., d$.
(b) $1 = c_1 \le c_2 \le \cdots \le c_d$.
(c) $k = b_0 \ge b_1 \ge \cdots \ge b_{d-1} > 0$.
(d) if $i + j \le d$ *then* $c_i \le b_j$.
(e) if $i + j \le d$ *and* $i \le j$, *then* $k_i \le k_j$.
(f) the sequence k_i *is unimodal, i.e.,* $k_1 < \cdots < k_h = \cdots = k_l > \cdots > k_d$ *for some* h *and* l *with* $1 \le h \le l \le d$. □

For a graph G of diameter d we define the i-th *distance graph* G_i to be the graph with the same vertex set as G, and with two vertices adjacent if and only if they are at distance i in the graph G. We call G *imprimitive* if for some i, $1 \le i \le d$, the graph G_i is disconnected. A graph which is not imprimitive is called *primitive*. Smith [126] has proved the following theorem for distance transitive graphs, but the proof can be easily extended to arbitrary distance-regular graphs.

2.1.2 THEOREM (Smith [126]). *An imprimitive distance-regular graph with valency greater than two is either bipartite or antipodal (or both).* □

If G is a connected bipartite graph of diameter at least two, then G_2 has two components. The graphs induced on these components are called *halved* graphs of the graph G and they are distance-regular if G is, see Biggs and Gardiner [16] or Brouwer et al. [27]. If G is distance transitive, the two halves are isomorphic, but in general this is not necessarily so. An example is Tutte's 12-cage, see Brouwer et al. [27].

For an antipodal graph G we define the *folded graph* of G, also called the *antipodal quotient* of G, to be the graph Q with the antipodal classes as vertices, where two components are adjacent if they contain adjacent vertices. The graph Q is distance-regular whenever G is distance-regular, see Gardiner [60] or Hensel [83]. The only antipodal graphs of diameter two are complete multipartite graphs $K_{t(m)}$, i.e., the complement of t cliques of size m, and they are bipartite only when $t = 2$.

An imprimitive distance-regular graph G with valency greater than two, gives us, after halving at most once and folding at most once, a primitive distance-regular graph. For a more precise statement and proof see Biggs and Gardiner [16], Brouwer et al. [27] or Hensel [83].

2. Eigenvalues and graph representations

The *adjacency matrix* $A = A(G)$ of a graph G with vertex set $\{1, \ldots, n\}$ is the $n \times n$ matrix with the ij-entry equal to 1 if the vertex i is adjacent to the vertex j and equal to 0 otherwise. Since G is loopless, diagonal entries are zero, and since A is a symmetric matrix, all the eigenvalues of A are real. They will be referred to as the eigenvalues of G. In vector/matrix equations we will use I to denote the identity matrix, J to denote the square matrix with all entries equal to one.

By an easy induction argument we get the fundamental property of the adjacency matrix: the number of walks in G from the vertex i to the vertex j with length k is equal to the ij-entry of the matrix A^k. Some other properties of a graph can also be expressed very elegantly in algebraic way, for example, regularity of a graph:

2.2.1 LEMMA. *Let G be a connected graph. Then G is k-regular if and only if k is its eigenvalue with multiplicity one and eigenvector* $(1, 1, \ldots, 1)^T$. □

The valency k is the spectral radius of $A(G)$. The following two results can be found, for example, in Godsil [64, Lemma 2.5.2 and Lemma 11.2.2],

2.2.2 THEOREM. *A connected graph G of diameter d on n vertices has at least $d+1$ and at most n distinct eigenvalues.* □

The rank of $n \times n$ matrix J is one, thus zero is an eigenvalue with multiplicity $n-1$. Since $A(K_n) = J - I$, we have $(A - \theta I)x = (J - (1+\theta)I)x$. Therefore K_n has for its eigenvalues -1 with multiplicity $n-1$ and by Lemma 2.2.1 also its valency $n-1$ with multiplicity one. The corresponding eigenvectors are $(1, 1, \ldots, 1)^T$ and the $n-1$ vectors obtained from the regular n-simplex in $(n-1)$-dimensional space with the center of mass at the origin. The n projections of its vertices to the i-th coordinate define one such vector.

If G is a graph of diameter d, then we define the i-th *distance matrix* A_i to be the adjacency matrix of G_i. We set $A_0 = I$ and $A_r = 0$ for $r > d$ or $r < 0$ and $A = A_1$. Now the uv-entry of $A_i A_j$ is equal to the number of vertices at distance i from u and j from v. This provides an equivalent definition of distance regularity:

2.2.3 THEOREM. *A connected graph G of diameter d is distance-regular if and only if there are numbers a_i, b_i and c_i such that*

$$AA_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1} \quad \textit{for} \quad 0 \le i \le d.$$

If G is a distance-regular graph, then $A_i = v_i(A)$ for some polynomial $v_i(x)$ of degree i, for $0 \le i \le d+1$. □

Using this identity for a distance-regular graph we find that $A_i A_j$ is a linear combination of distance matrices which are linearly independent. The coefficient at A_r is $p_{ij}(r)$. We have already pointed out once, this means that, in order to check if some graph is distance-regular it is enough to verify if for any vertices u and $v \in S_i(u)$ the numbers $|S_{i+1}(u) \cap S(v)|$ and $|S_{i-1}(u) \cap S(v)|$, i.e., the members of intersection array, are independent of the choice of u and v.

The sequence of polynomials $v_i(x)$ is determined with $v_{-1}(x) = 0$, $v_0(x) = 1$, $v_1(x) = x$ and with the recurrence relation

$$c_{i+1}v_{i+1}(x) = (x - a_i)v_i(x) - b_{i-1}v_{i-1}(x), \qquad i = 0, 1, \ldots, d.$$

In this sense distance-regular graphs are combinatorial representation of orthogonal polynomials, see [64] for further details. Damerell [52] has proved the following:

2.2.4 COROLLARY. *If G is a distance-regular graph of diameter d, then it has precisely $d + 1$ distinct eigenvalues, namely zeros of $v_{d+1}(x)$.* □

The converse does not hold in general, but it is true for diameter two strongly regular graphs. The next lemma gives us a connection between eigenvectors of G and G_i which has an important consequence, cf. Theorem 2.4.4).

2.2.5 LEMMA. *Let G be a distance-regular graph and θ an eigenvalue with eigenvector x. Then $v_i(\theta)$ is an eigenvalue of G_i with eigenvector x.* □

The *characteristic matrix* $P = P(\pi)$ of a partition $\pi = \{C_1, \ldots, C_s\}$ of a set of n elements is the $n \times s$ matrix with columns formed by the characteristic vectors of the elements of π (i.e., the ij-entry of P is 0 or 1 according to i being contained in C_j or not).

2.2.6 LEMMA. *A partition π of $V(G)$ with the characteristic matrix P is equitable if and only if there exists a $s \times s$ matrix B such that $A(G)P = PB$. If π is equitable then $B = A(G/\pi)$.* □

Here is one important result of this kind due to Haynsworth [80]:

2.2.7 COROLLARY. *Let G be a graph with an equitable partition $\pi = \{C_1, \ldots, C_k\}$, and let θ be an eigenvalue of G/π with eigenvector x, then θ is also an eigenvalue of G (with multiplicity at least as large as its multiplicity in G/π), and x extends to an eigenvector of G which is constant on cells of π. If τ is an eigenvalue of G but not of G/π then the sum of coordinates of any eigenvector of G corresponding to the eigenvalue τ equals zero on each cell C_i.*
□

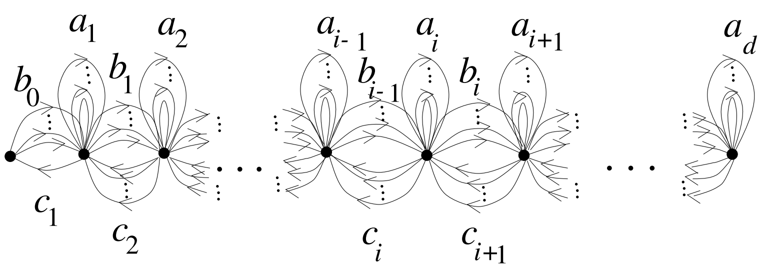

Figure 2.1: The quotient graph corresponding to the distance partition.

Having developed this machinery, it is time to apply it. Let G be a distance-regular graph of diameter d and u a vertex of G. Then the distance partition corresponding to a vertex u is an equitable partition and gives rise to a quite simple graph (see Figure 2.1) which inherits all the eigenvalues of G. The quotient graph G/π_u does not depend on the choice of a vertex u, since G is distance-regular, so we can omit index u. It has just $d+1$ vertices, so by Theorem 2.2.2 its adjacency matrix

$$A(G/\pi) = \begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix}$$

which is determined by intersection array of G, has exactly $d+1$ distinct eigenvalues and they are precisely all the eigenvalues of $A(G)$. The vector $v = \big(v_0(\theta), \dots, v_d(\theta)\big)^T$ is a left eigenvector of this matrix corresponding to the eigenvalue θ. Similarly a vector $w = (w_0(\theta), \dots, w_d(\theta))^T$ defined by $w_{-1}(x) = 0$, $w_0(x) = 1$, $w_1(x) = x/k$ and by the recurrence relation

$$xw_i(x) = c_i w_{i-1}(x) + a_i w_i(x) + b_i w_{i+1}(x), \qquad i = 0, 1, \dots, d$$

is a right eigenvector of this matrix, corresponding to the eigenvalue θ. There is the following relation between coordinates of vectors w and v: $w_i(x)k_i = v_i(x)$. The sequence $\big(w_0(\theta), \dots, w_d(\theta)\big)$ is fundamental for the study of distance-regular graphs and is called the *standard sequence* (or also a *sequence of cosines*) corresponding to θ. Using Sturm's theorem (see Brouwer et al. [27, p. 129] or [140]) the following result is obtained.

2.2.8 THEOREM. *Let $\theta_0 > \cdots > \theta_d$ be the eigenvalues of a distance-regular graph. The sequence of cosines corresponding to the i-th eigenvalue θ_i has precisely i sign changes.* □

Now, we will obtain the sequence of cosines by graph representations. Let G be a distance-regular graph with vertex set $\{1, \dots, n\}$, and let θ be an eigenvalue of $A = A(G)$ with multiplicity m. Let U_θ be an $n \times m$ matrix with columns forming an orthonormal basis for the eigenspace associated with θ. Let $u_\theta(i)$ be the i-th row of U_θ. Then this defines a mapping from $V(G)$ into $\mathbb{R}^m$, called a *graph representation* corresponding to θ. We have $AU_\theta = \theta U_\theta$ and therefore,

$$\sum_{j\sim i} u_\theta(j) = \theta u_\theta(i).$$

We mention two results which demonstrate that this concept is natural. All this and more on representations can be found in Godsil [64, Ch. 13] and Brouwer et al. [27, Ch. 3].

2.2.9 LEMMA. *Let G be a distance-regular graph and let θ be an eigenvalue of G. If i and j are two vertices of G then $(u_\theta(i), u_\theta(j))$ is determined by the distance between i and j in G.* □

This result implies that the vectors $u_\theta(i)$ all have the same length. Now it is not difficult to derive from Theorem 2.2.3 that for $s = \mathrm{dist}(i, j)$ we have:

$$w_s(\theta) = \frac{(u_\theta(i), u_\theta(j))}{(u_\theta(i), u_\theta(i))}.$$

2.2.10 THEOREM. *Let G be a distance-regular graph of diameter d and valency $k > 2$. Let θ be an eigenvalue of G. Then u_θ is not injective if and only if*
(a) $\theta = k$, or
(b) $\theta = -k$ and G is bipartite, or
(c) G is antipodal and θ is not an eigenvalue of its antipodal quotient. □

This result implies that u_θ is locally injective when G is not complete multipartite graph and θ is not k or $-k$.

3. Association schemes

The set of d graphs $G_1, \dots, G_d$ with the vertex set X, for which we have:
(a) for any pair of vertices x and y exactly one graph G_i contains the edge xy,
(b) for any pair of vertices x, y and any integers i, j the number

$$|\{z \in X \mid xz \in G_i \text{ and } yz \in G_j\}| \quad (=: p_{ij}(h))$$

depends only on i, j and the distance graph G_h which contains the edge xy (and not on the edge xy itself), is called a d-class *(symmetric) association scheme on the set* X with *intersection numbers* $p_{ij}(h)$. Note that each graph G_i could be substituted with the corresponding relation $\mathcal{R}_i$. A d-class association scheme on X is essentially a colouring of the edges of the complete graph $K_{|X|}$ with d colours, such that the number of triangles with a given colouring on a given edge depends only on the colouring and not on the edge.

If we denote the adjacency matrices of graphs G_i by A_i and the identity matrix of order $|X|$ by A_0, the condition (a) translates to $\sum_{i=0}^{d} A = J$ (all ones matrix) and the condition (b) to $A_iA_j = \sum_{h=0}^{d} p_{ij}^{h} A_h$. The algebra $\mathcal{A}$ generated by matrices $A_0, \dots, A_d$ is called the *Bose-Mesner algebra* of the association scheme. Since the 01-matrices of $\mathcal{A}$ which are idempotents for the Schur product (i.e., the entry-wise product, also known as the Hadamard product), and with minimal number of ones are uniquely determined, we us usually denote an association scheme just by $(X, \mathcal{A})$. It can be shown that $\mathcal{A}$ is an association scheme if and only if the matrices $A_0, \dots, A_d$ span a $(d+1)$-dimensional commutative subalgebra of symmetric complex matrices $\mathbb{C}(X)$, where X is the index set. Furthermore, it can be shown that this algebra admits a basis of $d+1$ pairwise orthogonal symmetric idempotents E_i, $i = 1, \dots, d$, with $E_0 = |X|^{-1}J$ and $\sum_{i=0}^{d} E_i = I$. These E_i's are in fact projections onto $d+1$ pairwise orthogonal eigenspaces common to all elements of $\mathcal{A}$.

If the graph G is distance-regular then its distance graphs $G_1, \dots, G_d$ form an association scheme and for $i = 1, \dots, d$ the matrix A_i is a polynomial of degree i in A_1, Such an association scheme is said to be *P-polynomial*. Equivalently, $(\mathcal{A}, X)$ is P-polynomial if for all integers i, j, h $(0 \le i, j, h \le d)$, $p_{ij}(h) = 0$ (resp. $p_{ij}(h) \neq 0$) whenever one of i, j, h is greater than (resp. equal to) the sum of the other two. Similarly, an association scheme is *Q-polynomial* if there are polynomials q_i of degree i such that $E_i = q_i(E_1)$, i.e., if for all integers i, j, h $(0 \le i, j, h \le d)$, $q_{ij}(h) = 0$ (resp. $q_{ij}(h) \neq 0$) whenever one of i, j, h is greater than (resp. equal to) the sum of the other two.

The definition of primitivity is extended from distance-regular graphs to association schemes. An association scheme is *primitive* if all the graphs G_i are connected, and *imprimitive* otherwise. The condition (b) implies that the graphs G_i are regular. Let us denote their valency by k_i. We will use some well known identities for the intersection numbers $p_{ij}(h)$:

2.3.1 LEMMA. *(a)* $p_{i0}(j) = \delta_{ij}$, *(b)* $p_{ij}(h) = p_{ji}(h)$, *(c)* $\sum_{t=0}^{d} p_{st}(i) = k_s$, *(d)* $p_{ij}(h)k_h = p_{ih}(j)k_j = p_{jh}(i)k_i$. □

Before we finish this section let us mention the *Krein condition* on intersection numbers of association schemes discovered by Scott [117] and the *absolute bound* discovered by Neumaier [106].

2.3.2 THEOREM (Krein condition). *Let G be a distance-regular graph with n vertices, diameter d and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$ with multiplicities $m_0, \ldots, m_d$. Let the polynomials $v_i(x)$ and the numbers k_i be as above. Then the Krein parameters (also called the dual intersection numbers)*

$$q_{ij}(h) = \frac{m_i m_j}{n} \sum_{a=0}^{d} \frac{v_a(\theta_i) v_a(\theta_j) v_a(\theta_h)}{k_a^2}$$

are nonnegative for all $i, j, h \in \{0, \ldots, d\}$. □

2.3.3 THEOREM (Absolute bound). *Let G be a distance-regular graph of diameter d. Then the multiplicities $m_0, \ldots, m_d$ of its eigenvalues satisfy*

$$\sum_{q_{ij}(h) \neq 0} m_h \leq \begin{cases} \frac{1}{2} m_i(m_i + 1) & \textit{if } i = j \\ m_i m_j & \textit{if } i \neq j \end{cases}$$

where the $q_{ij}(h)$ are the Krein parameters. □

4. Antipodal distance-regular graphs

Let G be a graph with a partition π of its vertices into cells satisfying the following conditions:

(a) each cell is an independent set,

(b) between any two cells there are either no edges or there is a matching.

Let G/π be the graph with the cells of π as vertices and with two of them adjacent if and only if there is a matching between them. Then we say that G is a *cover* of G/π and we call the cells and the matchings the *fibres of vertices* and the *fibres of edges* respectively. If G/π is connected, then all cells have the same size which is called the *index* of the cover, and is denoted by r. In this case G is called an r-cover of G/π. In this thesis we will always require that $r > 1$.

We can give an equivalent definition of a cover H of G using the projection map p from $V(H)$ to $V(G)$. We say that H is a cover of G if there is a map $p : V(H) \to V(G)$ called a *projection* which is a graph morphism, i.e., preserves adjacency, and a local isomorphism, i.e., for each vertex u of H the map p restricted to $\{u\} \cup S(u)$ is bijective. Then $\{p^{-1}(u), u \in G\}$ is the set of fibres and $r = |p^{-1}(u)|$ is the index of the covering. If we consider our graphs as simplicial complexes, coverings graphs are covering spaces in the usual topological sense.

If a graph G is a cover of G/π and π consists of its antipodal classes, then G is called an *antipodal cover*. Furthermore, if the graph G is also distance-regular, we say that G is a *distance-regular antipodal cover*.

2.4.1 LEMMA. *A distance-regular antipodal graph G of diameter d is a cover of its antipodal quotient with components of G_d as its fibres unless $d = 2$.*

To prove the above result we need only the facts that G is antipodal, connected and that $b_{d-1}(u, v) > 0$ for any vertex u and $v \in S_{d-1}(u)$.

In order to gain more insight into the structure of the distance-regular antipodal covers of distance-regular graphs let us first prove the following extension of a result due to Gardiner [60]. The part (i) is new, and (ii) modified, however the proofs of (i)⇒(ii) and (i)&(ii)⇒(iii) are motivated by his proof.

For each vertex u of a cover H we denote the fibre which contains u by $F(u)$. A *geodesic* in a graph G is a path $g_0, \ldots, g_t$, where dist$(g_0, g_t) = t$.

2.4.2 THEOREM. *Let G be a distance-regular graph of diameter d with parameters b_i, c_i and H its r-cover of diameter $D > 2$. Then the following statements are equivalent:*

(i) The graph H is antipodal with its fibres as the antipodal classes (hence an antipodal cover of G) and each geodesic of length at least $\lfloor (D+1)/2 \rfloor$ in H can be extended to a geodesic of length D.

(ii) For any $u \in V(H)$ and $i \in \{0, 1, \ldots, \lfloor D/2 \rfloor\}$ we have

$$S_{D-i}(u) = \cup\{F(v) \backslash \{v\} \ : \ v \in S_i(u)\}.$$

(iii) The graph H is distance-regular with $D \in \{2d, 2d+1\}$ and intersection array

$\left\{b_0, \ldots, b_{d-1}, \frac{(r-1)c_d}{r}, c_{d-1}, \ldots, c_1; c_1, \ldots, c_{d-1}, \frac{c_d}{r}, b_{d-1}, \ldots, b_0\right\}$

for D even, and

$\left\{b_0, \ldots, b_{d-1}, (r-1)t, c_d, \ldots, c_1; c_1, \ldots, c_d, t, b_{d-1}, \ldots, b_0\right\}$

for D odd and some integer t.

Proof. Let H be an antipodal cover. If two paths both have length less than D and they go through the same fibres in the same order, then we will say that they are *parallel*. Note that two parallel paths have the same length and that one of them and a vertex from the other one uniquely determine the other path. By antipodality, a path of length less than D contains at most one vertex from each fibre, therefore two parallel paths are either disjoint or equal corresponding to their intersection being empty or not. Finally, the parallelism is an equivalence relation, each parallel class corresponds bijectively to a path in the antipodal quotient of H, and each parallel class contains r elements. The last, for example, implies that for two distinct fibres any vertex from them lies in a shortest path between them (cf. [27, Lemma 11.1.4]).

(i)$\Rightarrow$(ii): Let u and v be any two vertices of H which are at distance $i \leq \lfloor D/2 \rfloor$. Since $F(v)$ is an antipodal class, the distance from u to any vertex of $F(v)$ is at least i. Let P be a path of length i between u and v. Then P is a shortest path between $F(u)$ and $F(v)$. Note that the set of all ends of paths from the parallel class of P equals $F(u) \cup F(v)$, and consider the distance partition corresponding to u. Let P' be a parallel path of P which has one end in $S_D(u)$ and the other end in $S_j(u)$ for some $j \geq D - i$. The required property of geodesics implies the existence of a path of length $D - j$ between $V(P') \cap S_j(u)$ and $S_D(u)$. But this is also a path between $F(v)$ and $F(u)$, so $D - j \geq i$. Therefore $j = D - i$ and $S_D(v) \subseteq S_{D-i}(u)$. Now let w be any vertex in $S_{D-i}(u)$. Then the extension of a geodesic from u to w to $S_D(u)$ is a shortest path between $F(u)$ and $F(w)$ and a path from its parallel class starting at u has to end in $S_i(u)$. Hence $\cup\{S_D(v) : v \in S_i(u)\} \subseteq S_{D-i}(u)$.

(i)$\Leftarrow$(ii): $i = 0$ implies that the graph H is antipodal with its fibres as the antipodal classes and therefore $S_{D-i}(u) = \cup\{S_D(v) \; : \; v \in S_i(u)\}$. The rest is now straightforward.

(i)&(ii)$\Rightarrow$(iii): A geodesic P of length d corresponds to a parallel class of geodesics of length $d < D$. These are the shortest paths between two fibres since P is a geodesic. Therefore by (ii) $D \geq 2d$. If $D \geq 2d + 2$ then by (ii) there exists a geodesic in H of length $d + 1$ which is the shortest path between two fibres and therefore $\mathrm{diam}(G) \geq d + 1$. Contradiction! The remainder of this part of the proof is only sketched. Suppose $D = 2d$ and let $\{u_1, \ldots, u_r\}$ be an antipodal class of H. Then the balls $B_{d-1}(u_i)$ of radius $d - 1$ centered at u_i (i.e., $\{u_i\} \cup S_1(u_i) \cup \ldots \cup S_{d-1}(u_i)$) for $i = 1, \ldots, r$ are disjoint and there are no edges between any two of them. Their induced graphs are parallel in the above sense and therefore isomorphic to their projection. This implies the desired parameters of H. The case when D is odd can be treated similarly.

(i)$\Leftarrow$(iii) It suffices to prove that, for a vertex $u \in V(H)$, any two distinct vertices v and w in $S_D(u)$ are at distance at least D. Suppose that $B_i(v) \cap B_i(w) = \emptyset$ and that there are no edges between $B_{i-1}(v)$ and $B_{i-1}(w)$ for some

$i \in \{1, \ldots, \lfloor D/2 \rfloor\}$. This is certainly satisfied for $i = 1$, since $a_D(H) = 0$ and $b_{D-1}(H) = 1$. If $i = (D-1)/2$ our work is done, otherwise $a_{D-i}(H) = a_i(H)$ implies that there is no edges between $S_i(v)$ and $S_i(w)$. If $i = D/2$ our work is done again, otherwise by $c_{D-i-1}(H) = b_{i+1}(H)$ the sets $S_{i+1}(v)$ and $S_{i+1}(w)$ are disjoint. So the induction assumption is satisfied for $i + 1$. □

Remarks: Statement (ii) gives us an idea how to draw the distance partition of an antipodal cover over the corresponding distance partition of its antipodal quotient and why we say that a distance-regular antipodal cover folds to its antipodal quotient (see Figures 3.1, 4.1 and 4.3).

(i) In $D = 2d$ case the integrality of entries in the intersection array implies $r|c_d$. By the monotonicity of parameters B_i and C_i there is also $c_{d-1} \leq \frac{c_d}{r}$ and $(1 - \frac{1}{r})c_d \leq b_{d-1}$.

(ii) In $D = 2d + 1$ case the integer t satisfies the conditions $t(r-1) \leq \min(b_{d-1}, a_d)$ and $c_d \leq t$.

The following corollary can again be found in Gardiner [60].

2.4.3 COROLLARY. *If H is a distance-regular antipodal graph, then H has a distance-regular antipodal cover only if H is either a cycle, a complete graph or a complete bipartite graph.* □

In the reminder of this section we determine also the eigenvalues of antipodal distance-regular graphs and their multiplicities.

2.4.4 THEOREM. *Let G be a distance-regular graph and H a distance-regular antipodal r-cover of G. Then every eigenvalue θ of G is also an eigenvalue of H with the same multiplicity.* □

The above result can be proved by combining the properties of the antipodal partition of H and the quotient graph of H, but it can also be derived as a consequence of the following theorem of Biggs [14]:

2.4.5 THEOREM. *The multiplicity of an eigenvalue θ of a distance-regular graph G with diameter d and n vertices is equal to*

$$\frac{n}{\sum_{i=0}^{d} k_i w_i(\theta)^2} \ .$$

□

Now we can finally state a result due to Biggs and Gardiner [16]:

2.4.6 THEOREM. *Let H be a distance-regular antipodal r-cover with diameter D of the distance-regular graph G with diameter d and parameters a_i, b_i, c_i. The $D-d$ eigenvalues of H which are not eigenvalues of G are, in the case when $D = 2d$, the eigenvalues of the $d \times d$ matrix*

$$\begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-2} & a_{d-2} & b_{d-2} \\ & & & & c_{d-1} & a_{d-1} \end{pmatrix}$$

and, in the case when $D = 2d+1$, the eigenvalues of the $(d+1) \times (d+1)$ matrix

$$\begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d - rt \end{pmatrix}$$

If $\theta_0 \geq \theta_1 \geq \cdots \geq \theta_D$ are the eigenvalues of H and $\xi_0 \geq \xi_1 \geq \cdots \geq \xi_d$ are the eigenvalues of G, then

$$\xi_0 = \theta_0, \;\; \xi_1 = \theta_2, \;\; \cdots, \;\; \xi_d = \theta_{2d}$$

i.e., the eigenvalues of G interlace the 'new' eigenvalues of H. □

Thus, in the even case the new eigenvalues do not depend on r and are the roots of $w_d(\theta) = 0$. Their multiplicities are proportional to $r-1$. In the odd diameter case the new eigenvalues depend only on rt and are the roots of $c_d w_{d-1}(\theta) + w_d(\theta)(a_d - rt - \theta) = 0$.

5. Generalized quadrangles

A *quadratic form* $Q(x_0, x_1, \ldots, x_n)$ over $GF(q)$ is a homogeneous polynomial of degree 2, i.e., for $x = (x_0, x_1, \ldots, x_n)$ and an $(n+1) \times (n+1)$ matrix C over $GF(q)$:

$$Q(x) = \sum_{i,j=0}^{n} c_{ij} x_i x_j = xCx^T.$$

A *quadric* in $PG(n,q)$ is the set of *isotropic* points

$$Q = \{\langle x \rangle : Q(x) = 0\},$$

where $\langle x \rangle$ is the one-dimensional subspace of $GF(q)^n$ generated by $x \in GF(q)^n$. Two quadratic forms $Q_1(x)$ and $Q_2(x)$ are *projectively equivalent* if there is an invertible matrix A and a nonzero λ such that

$$Q_2(x) = \lambda Q_1(xA).$$

The *rank* of a quadratic form is the smallest number of indeterminates that occur in a projectively equivalent quadratic form. A quadratic form $Q(x_0, \ldots, x_n)$ (or the quadric Q in $PG(n,q)$ determined by it) is *nondegenerate* if its rank is $n+1$. For q odd a subspace U is degenerate whenever $U \cap U^\perp \neq \emptyset$, i.e., whenever its orthogonal complement $U^\perp$ is degenerate, where $\perp$ denotes the inner product on the vector space $V(n+1,q)$ defined by

$$(x,y) := Q(x+y) - Q(x) - Q(y).$$

A *flat* of a projective space $PG(n,q)$ defined over $(n+1)$-dimensional space V consists of 1-dimensional subspaces of V that are contained in some subspace of V, and it is said to be *isotropic* when all its points are isotropic. In Theorem 3.7.1 the dimension of maximal isotropic flats is determined.

A *generalized quadrangle* an incidence structure of points and lines with $s+1$ points on each line, $t+1$ lines through each point, and for a point p not on a line ℓ there is exactly one point of ℓ collinear with p. Let us now give a brief description of *classical* generalized quadrangles, which are all associated with classical groups and are due to J. Tits, see [113].

An *orthogonal* generalized quadrangle $Q(d,q)$ is determined by isotropic points and lines of a nondegenerate quadratic form in $PG(d,q)$, for $d \in \{3,4,5\}$. An orthogonal generalized quadrangle $Q(4,q)$ has parameters (q,q). Its dual is called *symplectic* (or *null*) generalized quadrangle $W(q)$ (since it can be defined on points of $PG(3,q)$, together with the self-polar lines of a null polarity), and it is isomorphic to $Q(4,q)$ for even q.

Let H be a nondegenerate hermitian variety (e.g., $V(x_0^{q+1}+\cdots+x_d^{q+1})$) in $PG(d,q^2)$. Then its points and lines form a generalized quadrangle called a *unitary* (or *Hermitean*) generalized quadrangle $\mathcal{U}(d,q^2)$. A unitary generalized quadrangle $\mathcal{U}(3,q^2)$ has parameters (q^2,q) and is isomorphic to a dual of orthogonal generalized quadrangle $Q(5,q)$.

Finally, we describe one construction found by Ahrens and Szekeres [1] and independently by M. Hall, Jr [77]. Let $\mathcal{O}$ be a hyperoval of the projective plane $PG(2,q)$, $q=2^h$, i.e., a set of $(q+2)$ points meeting every line in zero or two points, and let $PG(2,q)=H$ be imbedded as a plane in $PG(3,q)=P$. Define a generalized quadrangle $T_2^*(\mathcal{O})$ with parameters $(q-1,q+1)$ by taking for points just the points of $P\backslash H$, and for lines just the lines of P which are not contained in H and meet $\mathcal{O}$ (necessarily in a unique point).

For a systematic combinatorial treatment of generalized quadrangles see the book by Payne and Thas [113]. The order of each known generalized quadrangle or its dual is one of the following: $(s,1)$ for $s\geq 1$; (q,q), (q,q^2), (q^2,q^3), $(q-1,q+1)$, for q a prime power.

3

DISTANCE-REGULAR COVERS OF THE COMPLETE GRAPH

In this chapter we investigate distance-regular covers of the complete graphs K_n for small n. Our main goal is to use known infinite families of these graphs to produce new ones. Before we start with the first section a short introduction and the summary of sections are to be given. The introduction explains why we study this particular case of distance-regular covers.

In 1982 Biggs [15] set up a classification scheme for distance-regular graphs of diameter three. They fall into three classes: antipodal, bipartite, and primitive. The antipodal distance-regular graphs of diameter three share many properties with certain graphs derived from finite geometries and in some cases they are even equivalent to them. Group divisible designs (see Godsil and Hensel [68, Theorems 5.2 and 5.3]), projective planes and generalized quadrangles with a spread are such examples. The last two families correspond to distance-regular graphs with parameters $\{k, k-2, 1; 1, 1, k\}$, see Biggs [15], Godsil and Hensel [68, Construction 4.2] or Brouwer et al. [27, p. 387], and $\{st, s(t-1), 1; 1, t-1, st\}$, see Brouwer [24] [27, p. 385]. Since the antipodal distance-regular graphs of diameter one and two are complete graphs and complete multipartite graphs (e.g., the octahedron), the antipodal distance-regular graphs of diameter three are the first nontrivial case of antipodal distance-regular graphs. They cover complete graphs. Note that Theorem 2.4.2 implies that a distance-regular cover of a complete graph is always antipodal with its fibres as antipodal classes, i.e., an antipodal cover of a complete graph. Godsil and Hensel [68] refined the classification scheme for the antipodal distance-regular graphs of diameter three and made a survey of the known constructions of them. For a complete survey of constructions of antipodal distance-regular graphs of diameter three and isomorphisms between them, see Brouwer, Godsil and Wilbrink [29] (cf. Gardiner [61] and [62]).

In Section 3.1 we mention two characterizations, which correspond to the extreme values of the covering index, and Mathon's construction. Using the feasibility conditions from Godsil and Hensel [68] a list of small feasible parameters was made by Godsil and Hensel. In the next section we study small cases of locally cyclic graphs.

In Section 3.3 we introduce Brouwer's characterization and give an elementary proof of the uniqueness of the distance-regular three-fold covers of K_9. There are two of them and they correspond to the two nonisomorphic spreads of a generalized quadrangle $GQ(2,4)$. Noticing that one of the two covers is cyclic we investigate, in Section 3.4, the cyclic covers of complete graphs. A generalized quadrangle $GQ(2,4)$ with a spread can be constructed from a generalized quadrangle $GQ(3,3)$ with a regular point. By exploring this relation we finally find, in Section 3.5, an operation, called switching, which yields new infinite families of distance-regular covers.

In Section 3.6 we use the cyclic spread of the generalized quadrangle $GQ(2,4)$ to show that there is a unique spread of the symplectic generalized quadrangle $GQ(3,3)$. In the following section we present a geometric proof (due to Brouwer and Wilbrink) of the same result. In the last section we discuss some open problems.

1. Some constructions

Since distance-regular covers of complete graphs (i.e., antipodal distance-regular graphs of diameter three) motivate the study of antipodal distance-regular graphs with larger diameter and give considerable insight into the structure of these graphs, we provide some fundamentals, the most important constructions or characterizations and their connections.

Let G be a distance-regular antipodal r-cover of the complete graph K_n. By Theorem 2.4.2, its intersection array is

$$\{n-1,(r-1)c_2,1;1,c_2,n-1\},$$

so it depends only on the *parameter set* (n,r,c_2), and the distance partition of a distance-regular cover of a complete graph corresponding to an antipodal class has the following shape (see Figure 3.1):

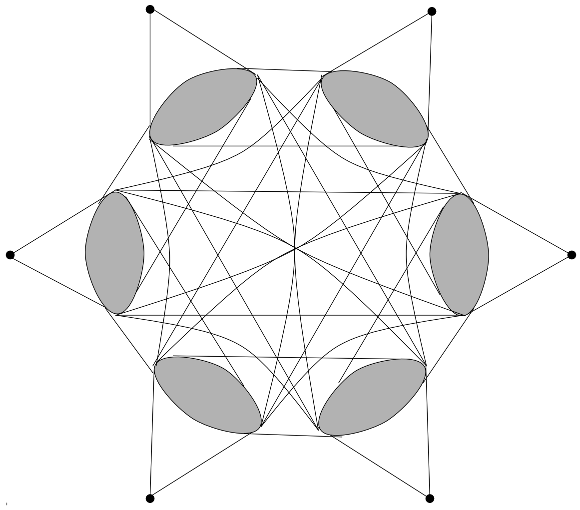

Figure 3.1: A distance-regular graph of diameter three and index six (the distance partition corresponding to an antipodal class).

Note that if we deleted the distinguished antipodal class, this would remind us of a figure of a complete multipartite graph, but in a complete multipartite graph shaded sets would be antipodal classes, while here each shaded area contains one vertex from each antipodal class.

The distance-regular double-covers $(n, 2, c)$ are equivalent to certain designs, called *(regular) two-graphs* and they can be derived from the strongly regular graphs with parameters $\{n-2-c, c/2; 1, (n-2-c)/2\}$, see Biggs [15, Thm. 5]. They have been surveyed by Taylor [129] and Seidel and Taylor [120]. The case when the covering index is equal to the valency, was investigated by Gardiner [60]. He has shown that in this case the distance-regular covers of complete graphs are equivalent to the *Moore graphs of diameter two* (i.e., k-regular graphs of diameter two and girth five, and they can exist only for $k \in \{2, 3, 7, 57\}$).

We present explicitly only Mathon's construction [103] of a distance-regular r-cover of K_{q+1}, where $q = rc+1$ is a prime power and either c is even or $q-1$ is a power of two, mainly because it covers so many parameter sets. The following is a neat version of the construction, due to Neumaier [15]. Let K be a subgroup of the multiplicative group of $GF(q)$ of index r. Let $\mathcal{R}$ be the equivalence relation for the elements of $GF(q)^2\{0\}$ defined by $(v_1, v_2)\mathcal{R}(u_1, u_2)$ if and only if there exists $h \in K$ such that $(v_1 h, v_2 h) = (u_1, u_2)$. Then the graph G with the equivalence classes vK, $v \in GF(q)^2\backslash\{0\}$ of $\mathcal{R}$ as vertices, and $(v_1, v_2)K \sim (u_1, u_2)K$ if and only if $v_1 u_2 - v_2 u_1 \in K$, is an antipodal distance-regular graph of diameter three, with $r(q+1) = (q^2-1)/c$ vertices, index r and $c_2 = c$. G is vertex transitive, and when r is prime and the characteristic of $GF(q)$ is a primitive element modulo r, also distance transitive, see Biggs [15] and Brouwer et al. [27, p. 386]. The smallest example is the line graph of the Petersen graph as the distance-regular three-fold cover of K_5. For $r = 2$ the double-cover comes from the *Paley graph* $\{2s, s; 1, s\}$, and for $c = 1$ Bondy [15] has shown that a graph with this parameters implies the existence of a projective plane of order q. This suggests that the general problem of finding all the covers of complete graphs will be hard to solve.

Let us now summarize the feasibility conditions from Godsil and Hensel [68]:

(F1) n, r, c_2 *are integers with* $0 \leq a_1 \leq n-3$, *i.e.,* $1 \leq (r-1)c_2 \leq n-2$,
(F2) *if* n *is even then* c_2 *is even,*
(F3) *the multiplicities of the nontrivial eigenvalues are integers.*

Krein conditions for a distance-regular cover with parameters (n, r, c_2) and $r > 2$ imply only one restriction: $\theta^3 \geq n-1$. Using the feasibility conditions (F1), (F2), (F3), the Krein and the absolute bounds, the group divisible design condition [68, Thm. 5.4], and [68, Lemma 3.5], Godsil and Hensel have obtained a list of small *feasible parameter sets*. The only distance-regular covers of complete graphs which are also bipartite are the graphs $K_{n,n}$ minus a perfect matching with parameters $\{n-1, n-2, 1; 1, n-2, n-1\}$ (e.g., the 3-cube is a double-cover of the tetrahedron), and are omitted from the following table:

n	r	a_1	c_2	a cover G of K_n	$\#of$G
5	3	1	1	L(Petersen)	1
6	2	2	2	Icosahedron	1
7	6	0	1	S_2(Hoffman-Singleton)	1
8	3	2	2	Klein graph	1
9	3	1	3	$GQ(2,4)\backslash$ spread	2
9	7	1	1	equivalent to the unique $PG(2,8)$	1
10	2	4	4	Johnson graph $J(6,3)$	1
10	4	2	2	$GQ(3,3)\backslash$ unique spread, Thm. 3.3.1	≥ 1

Table 3.1: List of small distance-regular covers of complete graphs.

The i*-th subconstituent graph* corresponding to a vertex u is the graph induced by the set $S_i(u)$. The first three parameter sets uniquely determine the line graph of the Petersen graph [27, p. 2], the icosahedron [27, Prop. 1.1.4] and the second subconstituent graph of the Hoffman-Singleton graph (i.e., the Moore graph of valency seven). Next, there is only one feasible intersection array of distance-regular covers of K_8: $\{7, 4, 1; 1, 2, 7\}$. Biggs [15], cf. Brouwer et al.

[28, p. 386], mentioned that this parameter set is realized by the Klein graph, i.e., the dual of the famous Klein map on a surface of genus 3. Further, in Brouwer et al. [27, p. 386], it is mentioned that there is only one such graph, so it must be the one coming from Mathon's construction. We give our own proof of this uniqueness in the following section as the part of the treatment of locally cyclic graphs, since it gives us an idea (see Figure 3.4(a) how to study larger locally cyclic distance-regular graphs with $\mu = 2$.

2. Locally cyclic graphs

Let G be a graph which is locally a cycle of length k. Then this is a k-regular graph with $\lambda = 2$, and it defines an imbedding in a surface. Let g' be the *crosscap number* $c(G)$ if the surface is nonorientable, and twice the genus $g(G)$ otherwise. Euler's formula, which relates the number of vertices n, the number of edges e and the number of faces f of this embedding, gives us together with $kn = 2e = 3f$ the following relation:

$$n - e + f = n - \frac{kn}{2} + \frac{kn}{3} = 2 - g', \quad \text{i.e.,} \qquad n(6-k) = 6(2-g').$$

If $k < 6$ then $g' \geq 0$ is equivalent to $n \leq 12/(6-k)$. Therefore $k = 3$ implies $n = 4$, $g' = 0$; $k = 4$ implies $n = 6$, $g' = 0$; and $k = 5$ implies $n = 12$, $g' = 0$. As $g' = 0$ in all three cases we must have the 1-skeleton of a Platonic solid with triangles as faces, i.e., the graph G is the tetrahedron, the octahedron and the icosahedron for $k = 3$, $k = 4$ and $k = 5$ respectively, cf. [27, Prop. 1.1.4 and Prop. 1.1.5].

For $k = 6$ an infinite example of G is a tiling of a plane by equilateral triangles. The above relation implies that in the finite case the orientable genus equals one. This suggests quotienting of the plane to a torus, and we obtain this way a two parameter family of graphs which are locally a hexagon, see Figure 3.2 and Figure 3.3(a). For general solution see Thomassen [138]. We will show that there exist only two examples when we restrict to distance-regular graphs. Before we do that, let us make a few general remarks about locally cyclic graphs.

Remarks: Let G be a distance-regular graph of diameter d, which is locally a k-cycle. Then:

(i) μ is a proper divisor of $(k-3)k$, since the number of edges between the first and the second neighbourhood of a vertex is $(k-\lambda-1)k = \mu k_2$, μ is less then k, and because of $\lambda = 2$ also greater than one.

(ii) $d \leq (k + c_d)/4$, by [27, Corollary 5.2.2], since $\mu \geq 2$ implies an existence of a quadrangle.

(iii) Let u be a vertex of G and a k-cycle $u_1, \dots, u_k$ be its local graph. Then $S(u_i) \cap S_2(u)$ induces a path P_i with $k-3$ vertices. Since $\lambda = 2$, the paths P_i and P_{i+1} have exactly one vertex in common, and thus $k_2 \geq 2k - 7$. Therefore $\mu \leq k(k-3)/(2k-7)$. The paths $P_1, \dots, P_k$ guarantee an existence of at least $k(k-4)/2$ edges in $S_2(u)$ and thus $a_2 \geq k(k-4)/k_2$.

Let q be a prime power congruent 1 (modulo 4). The *Paley graph* $P(q)$ has the elements of the finite field $GF(q)$ for vertices and two adjacent if their difference is a non-zero square. Note that -1 is a square in F, so this graph is undirected. Furthermore, it is a strongly regular graph of valency $k = (q-1)/2$, $\lambda = (q-5)/4$ and $\mu = (q-1)/4$. Seidel [119] showed that this graphs are uniquely determined with their parameters for $q \leq 17$.

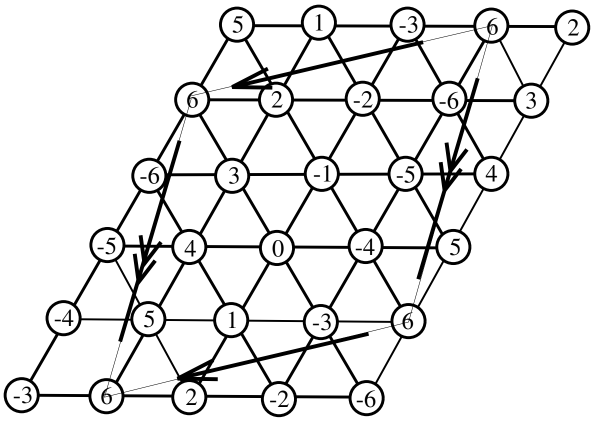

Figure 3.2: The Paley graph $P(13)$. In order to get an embedding we identify the sides with the same type of arrows.

3.2.1 LEMMA. *The Shrikhande graph and the Paley graph $P(13)$ are the only distance-regular graphs which are locally a hexagon (the first one has $\mu = 2$ and the second one has $\mu = 3$).*

Proof. Let u be a vertex of a distance-regular graph which is locally a hexagon and let the cycle $u_1, \dots, u_6$ be the local graph of u, see the darker shaded hexagon on Figure 3.3 (a). Then the above Remark (i) implies that μ is either three or two. As $\lambda = 2$ the edge $u_i u_{i+1}$ lies in two triangles: $u u_i u_{i+1}$ and $v_i u_i u_{i+1}$ for some vertex v_i of G, $i = 1, \dots, 6$ (where indices are taken modulo six, with representatives $1, \dots, 6$). If $\mu = 3$ then $k_2 = \mu + \lambda + 1$ implies, by [27, Thm. 1.5.5] and the fact that there is no double-cover of K_5, that diameter is two and the parameters correspond to the Paley graph $P(13)$. In the neighbourhood of zero we find a hexagon $1, 4, 3, -1, -4, -3$.

It remains to consider the case $\mu = 2$. Then the vertices $v_1, \ldots, v_6$ are distinct. Furthermore, $k_2 = 9$, so there exist exactly three more vertices w_1, w_2 and w_3 of $S_2(u)$, and we can label them so that w_i is a common neighbour of u_i and u_{i+3} for $i = 1, 2, 3$. Since the local graphs of u_i and u_{i+3} are hexagons, w_i is adjacent to v_i, v_{i-1} v_{i+3} and v_{i+2} for $i = 1, 2, 3$. So far we already have all the edges in the lighter shaded hexagon on Figure 3.3 (a).

Now, v_i has already one common neighbour with u_{i+2}, therefore v_{i+2} has to be the other common neighbour for $i = 1, \ldots, 6$, the local graphs of w_i and v_i are hexagons as well, and we really obtained the Shrikhande graph, see Figure 3.3. $\square$

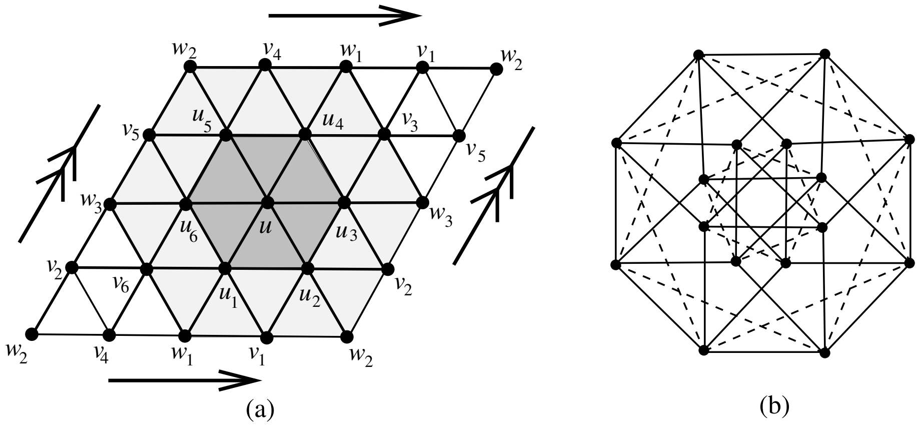

Figure 3.3: The Shrikhande graph drawn on two ways: (a) on a torus, (b) with imbedded four-cube.

The Shrikhande graph is not distance transitive, since some μ-graphs, i.e., the graphs induced by common neighbours of two vertices at distance two, are K_2 (e.g., for a pair u_i, u_{i+2}) and some are $2{\cdot}K_1$ (e.g., for a pair u_i, u_{i+3}). A similar approach works also when we study graphs which are locally a heptagon.

3.2.2 LEMMA. *The Klein graph is the unique distance-regular graph which is locally a heptagon. In particular, it is a unique distance-regular cover of* K_8.

Proof. Suppose that G is a distance-regular graph which is locally a heptagon. Then, by the above Remark (i), μ is either two or four. If $\mu = 4$ then Remark (iii) implies $a_2 = 3$, however, this is not possible, since G has an odd valency and would have an odd number of vertices (1+7+7) as well. Therefore $\mu = 2$.

Let u be a vertex of G with a heptagon $u_1, \ldots, u_7$ as its local graph, and its second neighbourhood $v_1, \ldots, v_7, w_1, \ldots, w_7$. Since $\lambda = 2$, we can assume that v_i is adjacent to both ends of the edge u_{i+3}, u_{i+4}, for $i = 1, \ldots, 7$, where indices are taken modulo seven, with representatives $1, \ldots, 7$. Vertices

u_i and u_{i+3} are at distance 2 and must have beside u exactly one more common neighbour, which can be without loss of generality w_{i-2}, for $i = 1, \dots, 7$, see Figure 3.4(a).

Since the neighbourhood of u_i induces a heptagon, and $v_{i+3}, u_{i+6}, u, u_{i+1}, v_{i+4}$ is a path in this neighbourhood, the vertices w_{i+2} and w_{i+5} are adjacent, for $i = 1, \dots, 7$, (see the smallest bold heptagon on Figure 3.4.(b)). Further, since $S_1(u_{i+6}) \cap S_1(w_{i+5}) = \{w_{i+1}, u_i\}$, vertices v_{i+4} and w_{i+5} are adjacent, for $i = 1, \dots, 7$. Now, there is already a path of length six in the neighbourhood of u_i, so also vertices v_{i+3} and w_{i+2} are adjacent, for $i = 1, \dots 7$.

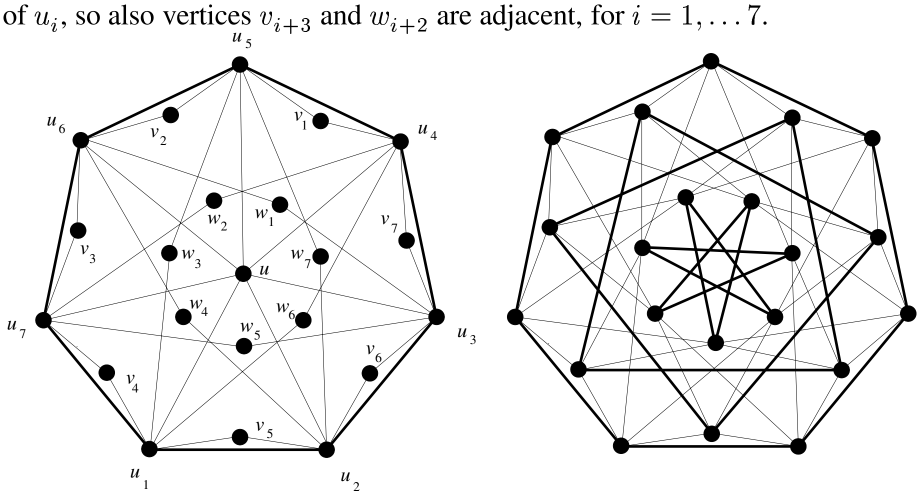

Figure 3.4: (a) (b) The distance-regular cover of K_8 minus a fibre

The neighbourhood of w_{i+3} contains paths w_i, u_{i+5}, v_{i+2} and $v_{i+4}, u_{i+1}, w_{i+5}$, therefore vertices v_{i+2} and v_{i+4} are adjacent, for $i = 1, \dots 7$ (this gives us the third bold cycle on Figure 3.4(b)).

Finally, all the vertices $v_1, \dots, v_7, w_1, \dots, z_7$ have already a path of length six in their neighbourhood, so G has only two more vertices v and w and their first neighbourhoods are $v_1, \dots, v_7$ and $w_1, \dots, w_7$ respectively.

If G is a distance-regular cover of K_8, then $(8, 2, 2)$ is the only feasible parameter set and a local graph is a union of cycles. Since $\mu = 2$, there are no four-cycles in the local graph and G must be locally a heptagon. $\square$

The graph G is antipodal with antipodal classes $\{u, v, w\}$ and $\{u_i, v_i, w_i\}$ for $i = 1, \dots, 7$ (see Figure 3.4, where each vertex from the missing fibre $\{u, v, w\}$ should be joined to all the vertices of its own bold heptagon). This graph is not distance transitive, for the same reason as the Shrikhande graph.

There is no feasible parameter set for a distance-regular cover of K_9 or K_{11} with $\lambda = 2$, however inspired by the Shrikhande graph we searched also for distance-regular graphs which are locally an octagon or a decagon, and obtained as an immediate consequence of the above Remark (ii) and [27, Ch. 14] the following result. There is no distance-regular graph which is locally an octagon or decagon. Furthermore, there is no distance-regular graph which is locally a nine-gon. The second result has been proved by first using Remark (ii) and Brouwer et al. [27, Ch. 14] to show that the graph must be a distance-regular cover of K_{10}, and then start with setting similar as on Figure 3.4(a). The important step was to show that antipodal classes must lie on axis of symmetry of the regular nine-gon.

We conclude this section by mentioning that the distance-regular covers coming from Mathon's construction are locally a k-cycle when k is a prime (e.g., (12,2,2), (14,2,2), (18,2,2)).

3. Brouwer's characterization

Let $(X, \mathcal{A})$, be a 3-class association scheme. Brouwer [24] derived from $\mathcal{A}$ a new association scheme $(X, \mathcal{A}')$ with two classes by *merging* its classes: $A'_1 := A_1 + A_2$ and $A'_2 := A_3$. Then $\mathcal{A}'$ is an association scheme if and only if $A'^2_1, A'_1A'_2, A'_2 \in \mathcal{A}'$, i.e.,

$$p_{11}(1) + 2p_{12}(1) + p_{22}(1) = p_{11}(2) + 2p_{12}(2) + p_{22}(2),$$

$$p_{13}(1) + p_{23}(1) = p_{13}(2) + p_{23}(2) \qquad \text{and} \qquad p_{33}(1) = p_{33}(2).$$

By Lemma 2.3.1(c), the sum of entries in the j-th row of the matrix P_i, defined by $(P_i)_{s,t} = p_{s,t}(i)$, equals k_j. Now, the left-hand side of the first two equations equals to the sum of the entries of the first two rows of P_1 minus one, and the the right-hand side of the first two equations equals to the sum of entries of the first two rows of P_2 minus one. Therefore the first two of the above conditions are equivalent. similarly, by considering the third rows of matrices P_1 and P_2, we derive that also the last two conditions are equivalent. Hence we need to check only one of the above conditions.

If A_i is the i-th distance matrix of an antipodal distance-regular graph H of diameter three, then A'_1 determines a nontrivial strongly regular graph M only when we merge A_1 and A_3 and $p_{22}(1) = p_{22}(3)$, i.e., $a_2b_1 = (a_3 + a_2 - a_1)c_3$ (by Lemma 2.3.1). The last equality is equivalent to $n - 1 = (c_2 + 1)(r - 1)$ and it implies that for $s := r - 1$ and $t := c_2 + 1$ intersection array of H is $\{st, s(t-1), 1; 1, t-1, st\}$ and of M is $\{s(t+1), st; 1, t+1\}$. (Instead of using Lemma 2.3.1 we can obtain $p_{22}(3) = kc_2$ from Figure 3.1 of an antipodal distance-regular graph of diameter three, so that by $p_{22}(1) = (r-1)p_{21}(2) =$

$(r-1)(k-1-c_2)$ we immediately get $n-1 = (c_2+1)(r-1)$.) Graphs with the same parameter set as H were characterized by Brouwer [24] [27, Prop. 12.5.2]:

3.3.1 THEOREM (Brouwer). *Let $(X, \mathcal{L})$ be a generalized quadrangle of order (s,t), (where $t > 1$) with a spread $\mathcal{S}$. Then the point graph H of $(X, \mathcal{L}\backslash\mathcal{S})$ is distance-regular, of diameter three and with intersection array $\{st, s(t-1), 1; 1, t-1, st\}$, i.e., an antipodal $(s+1)$-cover of the complete graph K_{st+1}. More generally, given a strongly regular graph M with intersection array $\{s(t+1), st; 1, t+1\}$ such that a partition $\mathcal{S}$ of its vertices into $(s+1)$-cliques exists, we obtain a distance-regular graph of diameter three with intersection array as given above, by deleting the edges with both ends in the same member of $\mathcal{S}$. Conversely, any graph H with these parameters arises this way.* □

Note that the above discussion is actually a proof of the converse. In this case "*unmerging*" (or *splitting*) is deleting the edges of $(s+1)$-cliques in M whose vertex sets partition $V(M)$.

The generalized quadrangles which are known to have spreads have orders $(q,1)$, $(1,q)$, (q,q), (q,q^2), $(q-1,q+1)$ for all q and $(q+1,q-1)$ for even q.

Now we return to our list of feasible parameters. There are two possible parameter sets for distance-regular covers of K_9. A cover with parameters $(9,7,1)$ is equivalent to the unique projective plane $PG(2,8)$, see Brouwer et al. [27, p. 40] or Koolen [96, Thm. 7.33.]. So it remains to consider covers with parameters $(9,3,3)$. In 1982 Biggs [15] pointed out that this was still an open question. Cameron, Goethals and Seidel [46, Thm. 7.9] have proved that any strongly regular graph with the same parameters as the point graph of $GQ(q,q^2)$ must be a point graph of a generalized quadrangle. Furthermore, it is known that there is a unique generalized quadrangle $GQ(2,4)$, see Payne and Thas [113, p. 123] or Brouwer et al. [27, Thm. 1.15.2]. Therefore Brouwer's Theorem 3.3.1 implies that covers $(9,3,3)$ are equivalent to spreads of the generalized quadrangle $GQ(2,4)$. Brouwer and Wilbrink [34] (cf. Cameron, Hughes and Pasini [47], in the solution of Lemma 3.18) have used orthogonal geometries (see Batten [11], Cameron [39], Higman [84], Hirschfeld [85], [86], Hirschfeld and Thas [87], Van Lint and Wilson [101]) to prove the following:

3.3.2 THEOREM (Brouwer and Wilbrink). *There are, up to isomorphism, two distinct spreads in the generalized quadrangle $GQ(2,4)$.*

Additionally, they have explained the relation between the two spreads of $GQ(2,4)$ in the dual by describing an operation, which applies to a planar ovoid in the unitary generalized quadrangle $GQ(q^2,q)$, and produces a non-planar ovoid. An *ovoid* is a set of points, no two collinear, which intersects all the lines (i.e., the dual of a spread).

Since this is the first time that we have two covers for the same parameter set, we have decided to investigate this situation more carefully. The driving idea has been to see if the above operation can be 'modified' or generalized to some other situation or to determine when covers occur in 'twin pairs'. So first we have not been able to resist a temptation to try to find a more elementary way of constructing the above twin covers. Our attempt has been successful and we have used only the following observation:

> *Let us take m copies of a graph H, say $(H)_1,\dots,(H)_m$ and let a vertex v_i of H_i corresponds to a vertex v of H. Now, let us choose for each vertex u of a graph G a subset $P(u)$ of vertices of H and join u with v_i for each i and $v \in P(u)$. Then only our labeling of the obtained graph distinguishes the m copies of H.*

We will construct distance-regular covers with parameters $(9,3,3)$ the same way we have constructed the Shrikhande graph and the Klein graph. First, we will label all the 27 vertices of such a cover, and then we will continue by making the list of their adjacencies (starting with empty list and adding a few entries at each step). When we say that the corresponding vertices of some subgraphs have the same neighbours, we mean that our labeling of the vertices and edges, which has been recorded so far, has this property (this will be evident from Table 3.2). As the above subgraphs will have only one or two vertices, it will be easy to find out which vertices correspond to which vertices.

Proof. Let G be a distance-regular cover with parameters $(9,3,3)$.

Step 1: Let $\{a,b,c\}$ be an antipodal class of G and let us denote by $x_1,\dots,x_8$ the neighbours of x, so that $x_1 \sim x_2$, $x_3 \sim x_4$, $x_5 \sim x_6$ and $x_7 \sim x_8$ for $x = a,b,c$. Since $a_1(G) = 1$, these are the only edges in $S(x)$, $x = a,b,c$. Note the following properties of G:

(1) Each vertex lies in exactly four triangles that have pairwise no common edges (i.e., $k = 8$ and $a_1(G) = 1$). This means that we know the graph locally.

(2) Two vertices at distance two have three common neighbours

(i.e., $c_2(G) = 3$).

(3) *There are no antipodal vertices in any neighbourhood. So let $i, j, k \in \{1, \ldots, 8\}$ be such integers that $\{x_i, y_j, z_k\}$ is an antipodal class, when $\{x, y, z\} = \{a, b, c\}$. Then (1) and (2) imply that x_i is adjacent to exactly one vertex of each pair of adjacent vertices in $S(z) \setminus \{z_k\}$, and y_j is adjacent to the remaining vertices of these three pairs (cf. the definition of the generalized quadrangle).*

So far we have recorded 36 edges and the vertices a, b, c already satisfy the condition (1). All the other vertices lie in exactly one triangle. From here on we will record our construction in Table 3.2, so that it will be easier for the reader to follow the construction. Each step will be marked in the table.

Step 2: Without loss of generality we can choose $\{a_1, b_1, c_1\}$ to be an antipodal class. Since the vertices x_{2i-1} and x_{2i} have the same neighbours, for $i = 2, 3, 4$ and $x \in \{a, b, c\}$, we take by (3) that $a_1 \sim b_4, b_6, b_8$; $b_1 \sim c_4, c_6, c_8$; $c_1 \sim a_4, a_6, a_8$. It implies, again by (3) that: $a_1 \sim c_3, c_5, c_7$; $b_1 \sim a_3, a_5, a_7$; $c_1 \sim b_3, b_5, b_7$.

Now we consider two cases:

<u>**Case 1**</u>: $\mathrm{dist}_G(a_2, b_2) = 3$. **Step 3:** By $\mathrm{dist}_G(a_2, b_1) = 2$ and (2), we have $a_2 \sim c_4, c_6, c_8$. So a_2, b_2, c_2 are antipodal. Because of symmetry among a, b, c, we get similarly $a_2 \sim b_3, b_5, b_7$; $b_2 \sim a_4, a_6, a_8, c_3, c_5, c_7$ and $c_2 \sim a_3, a_5, a_7, b_4, b_6, b_8$ as well.

Step 4: Note that for $x \in \{a, b, c\}$ the corresponding vertices of the three edges (x_{2i-1}, x_{2i}), $i = 2, 3, 4$, have the same neighbours, therefore we can choose vertices a_3, b_3, c_3 to be antipodal without loss of generality. Since for $x = a, b, c$ the corresponding vertices of the two edges (x_5, x_6) and (x_7, x_8) have the same neighbours, by (2) and by $a_1(G) = 1$ we take $a_3 \sim b_6, b_7, c_5, c_8$; $b_3 \sim a_5, a_8$. By (3) it follows that $b_3 \sim c_6, c_7$ and $c_3 \sim a_6, a_7, b_5, b_8$.

Step 5: Now we use (1) for a_1, a_2, a_3 to get $c_7 \sim b_6$, $c_5 \sim b_4$; $c_8 \sim b_5$, $c_4 \sim b_7$; $c_5 \sim b_7$ respectively. As there is still a symmetry among a, b and c, we also have $a_7 \sim c_6$, $a_5 \sim c_4$; $a_4 \sim c_7$, $a_8 \sim c_5$; $a_5 \sim c_7$ and $b_7 \sim a_6$, $b_5 \sim a_4$; $b_4 \sim a_7$, $b_8 \sim a_5$; $b_5 \sim a_7$. Therefore $\{a_5, b_5, c_5\}$ and $\{a_7, b_7, c_7\}$ are antipodal classes and (1) holds for all their vertices.

Step 6: Each of the vertices x_4, x_6, x_8, for $x \in \{a, b, c\}$, has to lie in exactly one more triangle, therefore we need three more triangles. The condition $a_1(G) = 1$ implies that the only possible triangles are the triangles (a_i, b_j, c_k), where (i, j, k) is an even permutation of $(8, 6, 4)$, and the triangles (a_i, b_i, c_i), where $i = 4, 6, 8$. (This is equivalent to the fact that there are exactly two mutually orthogonal Latin squares of size three and also that $K_3 \times K_3$ is a self complementary graph.) These three triangles have to be disjoint, thus we either take the first set of triangles, in this case the vertices of the triangles of the second set determine the remaining antipodal classes, or the other way around.

The two obtained graphs are regular, antipodal, have diameter three and satisfy the condition (1). By Godsil and Hensel [68, Lemma 3.1], it remains to check if $c_2 = 3$. These two graphs can be completed to the same graph of diameter two by joining the vertices in each antipodal class. Let us denote this graph by H, then because of (2) we only have to prove that $c_2(H) = 5$. By (1) each vertex of H lies in exactly five triangles therefore $c_2(H) \leq 5$. If $u \in V(H)$, then $|S(u)| = 8$, $|S_2(u)| = 16$ and $b_1(H) = k(H) - 1 - a_1(H) = 8$. By counting the edges between $S(u)$ and $S_2(u)$ we conclude that $c_2(H) = 5$. (Thus H is a strongly regular graph with parameters $\{10, 8; 1, 5\}$ and hence, by Cameron, Goethals and Seidel [46], it is the point graph of the generalized quadrangle $GQ(2,4)$.)

x	$a_1 b_1 c_1$	$a_2 b_2 c_2$	$a_3 b_3 c_3$	$a_4\ b_4\ c_4$	$a_5 b_5 c_5$	$a_6\ b_6\ c_6$	$a_7 b_7 c_7$	$a_8\ b_8\ c_8$	
$S(X)$	$b_4 c_4 a_4$	$b_3 c_3 a_3$	$b_1 c_1 a_1$	$c_1\ a_1\ b_1$	$b_1 c_1 a_1$	$c_1\ a_1\ b_1$	$b_1 c_1 a_1$	$c_1\ a_1\ b_1$	Step 2
	$b_6 c_6 a_6$	$b_5 c_5 a_5$	$c_2 a_2 b_2$	$b_2\ c_2\ a_2$	$c_2 a_2 b_2$	$b_2\ c_2\ a_2$	$c_2 a_2 b_2$	$b_2\ c_2\ a_2$	Step 3
	$b_8 c_8 a_8$	$b_7 c_7 a_7$	$b_6 c_6 a_6$	$b_5\ c_5\ a_5$	$b_3 c_3 a_3$	$c_3\ a_3\ b_3$	$c_3 a_3 b_3$	$b_3\ c_3\ a_3$	Step 4
	$c_3 a_3 b_3$	$c_4 a_4 b_4$	$b_7 c_7 a_7$	$c_7\ a_7\ b_7$	$c_4 a_4 b_4$	$b_7\ c_7\ a_7$	$c_6 a_6 b_6$	$c_5\ a_5\ b_5$	Step 5
	$c_5 a_5 b_5$	$c_6 a_6 b_6$	$c_8 a_8 b_8$	$b_{8,4} c_{8,4} a_{8,4}$	$b_8 c_8 a_8$	$b_{4,6} c_{4,6} a_{4,6}$	$b_4 c_4 a_4$	$b_{6,8} c_{6,8} a_{6,8}$	Step 6
	$c_7 a_7 b_7$	$c_8 a_8 b_8$	$c_5 a_5 b_5$	$c_{6,4} a_{6,4} b_{6,4}$	$c_7 a_7 b_7$	$c_{8,6} a_{8,6} b_{8,6}$	$b_5 c_5 a_5$	$c_{4,8} a_{4,8} b_{4,8}$	

Table 3.2: The distance-regular covers of K_9

<u>**Case 2**</u>: For all $x, y \in \{a, b, c\}$ such that $\mathrm{dist}_G(x_1, y_1) = 3$, we have $\mathrm{dist}_G(x_2, y_2) < 3$. In this case we get a contradiction, by the same approach as in the first case. For the sake of completeness we shall consider this case in detail too.

Step 3: Without loss of generality we choose $\{a_2, b_3, c_4\}$ to be an antipodal class. By (3) and $a_1(G) = 1$ we have $a_2 \sim b_2, b_5, b_7, c_2, c_6, c_8$ and then also $b_3 \sim c_5, c_7$; $c_4 \sim b_6, b_8$. By observing that the corresponding vertices of the edges (a_3, a_4), (a_5, a_6), (a_7, a_8) have the same neighbours, and by (1) for c_1, c_4, we also take $b_3 \sim a_3, a_5, a_8$, thus by (3) $c_4 \sim a_4, a_6, a_7$. Therefore $S_3(b_2) \cap S(a) \subseteq \{a_4, a_6, a_8\}$.

(a) If $\mathrm{dist}_G(b_2, a_8) = 3$ then $c_3 = S_3(b_2) \cap S_3(a_8)$ (for a_8 see paths through b_3 and for b_2 see paths through b_1 or a_2). **Step 4:** Since the corresponding vertices of the edges (a_3, a_4), (a_5, a_6); (b_5, b_6), (b_7, b_8); (c_5, c_6), (c_7, c_8)) have the same neighbours, and by (2), (3) we get $b_2 \sim c_2, c_5, c_7, a_4, a_6$;

$c_3 \sim b_4, b_5, b_7, a_3, a_5$; $a_8 \sim c_6, c_8, b_6, b_8$. **Step 5:** Now because of (1) for a_8 and since the corresponding vertices of the edges (b_7, b_8) and (b_5, b_6) have the same neighbours, we take $c_6 \sim b_8$ and $c_7 \sim b_6$. But then (1) for a_1 yields $b_8 \sim c_5$, so we get two triangles on the edge (c_5, c_6). Contradiction!

(b) Because the corresponding vertices of the edges (a_3, a_4) and (a_5, a_6) have the same neighbours, it remains to consider the case when $\mathrm{dist}_G(b_2, a_4) = 3$. **Step 4:** As the corresponding vertices of the edges (c_5, c_6), (c_7, c_8) have the same neighbours, by the assumption of Case 2, we take $c_5 = S_3(b_2) \cap S_3(a_4)$. Now, the corresponding vertices of the edges (b_5, b_6), (b_7, b_8) have the same neighbours, so by (2) and (3) we get $b_2 \sim c_2, c_3, c_7, a_6, a_8$; $a_4 \sim b_4, b_5, b_8, c_8$; $c_5 \sim a_5, a_7, b_6, b_7$. **Step 5:** Finally (1) for b_2 and b_1 implies $c_3 \sim a_8$; $a_6 \sim c_6$ and $a_3 \sim c_6$; $a_5 \sim c_8$ respectively. But then $\mathrm{dist}_G(a_6, c_7) = 2$ (since b_2 is a common neighbour of them) and thus a_6 has no antipodal vertices in $S(c)$. Contradiction! □

After this construction a quote of Peter Cameron [39, pp. 117-118] is appropriate:

> *"As a general principle, a good construction of an object leads to a proof of its uniqueness (by showing that it must be constructed this way), thence to a calculation of its automorphism group (since the object is uniquely built around a starting configuration, and so any isomorphism between such starting configurations extends uniquely to an automorphism), and gives on the way a subgroup of the automorphism group (consisting of the automorphism group of the starting configuration)."*

We will not try to determine the full automorphism group of the twin covers. As Brouwer's and Wilbrink's construction suggests their automorphism groups are related to projective geometries, our combinatorial setting is probably not appropriate for such a task. But we can still make a meaningful use of the obvious symmetry among a, b and c which leads us to the following section.

4. Cyclic distance-regular covers

If a group of automorphisms of an r-cover G which fixes each fibre, is cyclic of order r, then we call G a *cyclic* cover. Let G be a cyclic cover of a complete graph K_n with parameters (n,r,c_2). Then we can describe G by orienting the edges of K_n and labeling them, for example by using a function f, with the elements of $\{1,2,\ldots,\lfloor\frac{r}{2}\rfloor\}$. So let vertices of G be $(v_1)_i,\ldots,(v_n)_i$, for $i=1,\ldots,r$. For a directed edge (u,v) (with v as its head) of K_n labeled by $f(u,v)$ in the corresponding fibres precisely the vertices $(u)_i$ and $(v)_{i+f(u,v)}$ are adjacent for $i=1,\ldots,r$ (where indices are taken modulo r with $1,2,\ldots,r$ as representatives).

We can assume without loss of generality that all the edges incident with some fixed vertex, u say, are labeled by zero. Sometimes a graph induced by $S((u)_i)$ is determined up to isomorphism by parameters of G. For example, for $a_1(G)=1$ it is a perfect matching. In this case we label by zero also the edges of a perfect matching of $K_n\backslash\{u\}$. The edges of G that are not incident with vertices $u_1,u_2,\ldots,u_r$ and do not have both ends in the set $S((u)_i)$ for some i, connect vertices from different $S((u)_i)$, so the remaining edges of K_n have nonzero labels. If furthermore $r=3$, there is only one label (number one). Thus in this case the covering graph G can already be described by some orientation of K_{n-1} with a perfect matching deleted.

3.4.1 LEMMA. *Let G be a distance-regular cover of K_n. Then $a_1=1$ and $r=3$ implies $n=5$ or 9.*

Proof. Let G be a distance-regular cover of K_n with $a_1=1$ and $r=3$. Then $c_2=(n-3)/2$ and n must be odd. Using Krein condition ($\theta^3\geq n-1$) we get $9\geq n$. We also have $a_1\leq n-3$, so $n\in\{5,7,9\}$. By Godsil and Hensel [68, Theorem 3.4(a)], $n=7$ is impossible too. Therefore $n=5$ and $(n,r,c_2)=(5,3,1)$ or $n=9$ and $(n,r,c_2)=(9,3,3)$. $\square$

We will show that a unique cyclic distance-regular three-fold cover of K_9 exists and we will construct it. Let D be the directed complete graph K_8 minus a matching which determines such a cover. The condition $a_1(G)=1$ is equivalent to the fact that each vertex of D lies in exactly three cyclically oriented triangles with no common edges. Since D is 6-regular this eight triangles partition the edges of D. Now we put back the 'ninth' vertex u and all four triangles incident with it (actually we observe D embedded in K_9). We orient these four triangles cyclically. We now have nine vertices ($V=9$), twelve cyclically oriented triangles ($B=12$), each vertex lies in four cyclically oriented triangles ($R=4$), each triangle has three vertices ($K = 3$) and finally because of $a_1(G) = 1$ each edge lies in exactly one cyclically oriented triangle ($\Lambda = 1$). Therefore the incidence structure of vertices as points and triangles as lines is an affine plane $AG(2,3)$. But an affine plane of order three is unique, so it remains to cyclically orient these triangles. There is a unique way to do this. Without loss of generality we choose $(1,4,5)$ as the orientation of the triangle (v_1, v_4, v_5) (see Figure 3.5). Because of $a_1 = 1$ for the edges in the fibre (v_1, v_2) we also get $(2,5,8)$ and analogously $(2,3,6)$ and $(1,6,7)$. Now because of $a_1 = 1$ for the edges in the fibre (v_7, v_8) we have $(2,7,4)$ and analogously $(1,8,3)$, $(3,7,5)$ and $(4,8,6)$.

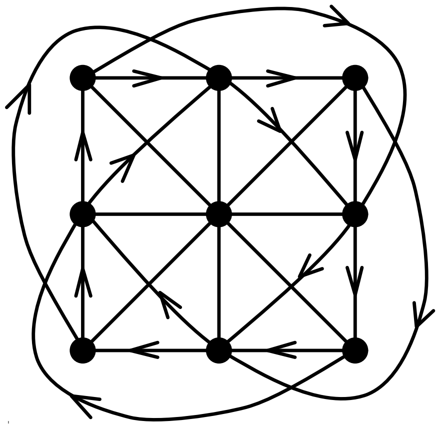

Figure 3.5: The cyclic cover with parameters $(9,3,3)$.

Remarks: As we know there are two kinds of spreads in the generalized quadrangle $GQ(2,4)$, and we can always change one kind of spread to the other kind just by substituting three triangles. So if we substitute the edges of any cyclically oriented triangle with directed loops with label one at its vertices, we get the other distance-regular cover $(9,3,3)$. Using the software package *Graphs and Groups* by Kocay [95], it was verified that the cyclic cover is distance transitive, and the other one is not.

5. Regular spreads and switching

This section is the culmination of the chapter. We will succeed in interpreting the relation between the twin covers of K_9 in a way different from Brouwer and Wilbrink, and will generalize it to a switching on a regular spread (defined below) of any $GQ(s,t)$. For example, all known generalized quadrangles $GQ(q-1,q+1)$ derived from generalized quadrangles $GQ(q,q)$ with a regular point have such spreads.

Let $\mathcal{S} = (P, L)$ be an incidence structure, where the elements of P and L are called points and lines respectively (lines are considered as subsets of points). For $x \in P$ we define a *star* of x, denoted by $x^\perp$, to be the set of points collinear with x. similarly for $A \subseteq P$ we define $A^\perp := \cap\{x^\perp; x \in A\}$. In a generalized quadrangle $GQ(s,t)$ we have $|\{x,y\}^\perp| = s+1$ or $t+1$ depending on whether x and y are collinear or not. For noncollinear points x and y, the set $\{x,y\}^{\perp\perp}$ is called a *hyperbolic line* on x and y and has cardinality at most $t+1$. The point x is *regular* if the set $\{x,y\}^{\perp\perp}$ has size $t+1$ for each $y \neq x$ (i.e., if $s = t$ and all the hyperbolic lines on x have size $t+1$). A *regular spread* is a spread S such that for each line $s \in S$, for any two points x and y on s and for each line ℓ on x, there is a line ℓ' on y so that ℓ and ℓ' intersect the same elements of S.

Payne [109] [113, p. 38, p. 58] used a generalized quadrangle $\mathcal{S}$ of order (q,q) with a regular point x to construct a generalized quadrangle $GQ(q-1,q+1)$ with a spread, denoted by $P(\mathcal{S},x)$. In [112] he showed further that the spread of $P(\mathcal{S},x)$ is a regular spread, and that starting with a generalized quadrangle $GQ(q-1,q+1)$ with a regular spread, we can construct a generalized quadrangle $GQ(q,q)$ with a regular point. Therefore these two objects are equivalent. For yet another approach see De Soete and Thas [124].

3.5.1 CONSTRUCTION (Payne). *Let x be a regular point of a generalized quadrangle $GQ(q,q)$ with the point set $\mathcal{P}$ and the line set $\mathcal{L}$. Then the set of points $\mathcal{P}\backslash x^\perp$ together with all the hyperbolic lines on x and the lines of $\mathcal{L}$ not on x with the points of $x^\perp$ delete, form a generalized quadrangle $GQ(q-1,q+1)$. The set of hyperbolic lines on x is a regular spread of this generalized quadrangle. Conversely, starting with a regular spread of a generalized quadrangle $GQ(q-1,q+1)$ the same generalized quadrangle $GQ(q,q)$ with a regular point can be reconstructed. Therefore a generalized quadrangle $GQ(q,q)$ with a regular point is equivalent to a generalized quadrangle $GQ(q-1,q+1)$ with a regular spread.*

In [110] or [113, p. 48] Payne has shown further that all known infinite families of generalized quadrangles $GQ(q-1, q+1)$ are special cases of the general construction $P(\mathcal{S}, x)$.

The last part of the above proof, which is dealing with general $GQ(s,t)$ containing a regular spread, is due to the author, and its conclusion is summarized in the following lemma.

Proof. Suppose that (P, L) is a generalized quadrangle $GQ(q,q)$ with x as a regular point. Let $P' := P \backslash x^{\perp}$, and let L' consist of all the hyperbolic lines on x and all the lines of L which are not on x with the points of the star $x^{\perp}$ deleted. We omit the proof that (P', L') is a generalized quadrangle $GQ(q-1, q+1)$ and the hyperbolic lines on x form a spread S of it (see Payne and Thas [113, p. 48]), and continue with our own proof of the rest of the statement. Let us prove that S is a regular spread. Let $\ell \in L$ intersect the hyperbolic lines $s_1, \dots, s_q$ on x and the star $x^{\perp}$ in the points $u_1, \dots, u_q$ and y respectively. Then $y \in \{u_i, x\}^{\perp}$ for each i, and therefore, by the definition of a hyperbolic line, the star $y^{\perp}$ contains all the points on the lines $s_1, \dots, s_q$. Thus the lines of L' which correspond to the lines on y are the ones we were looking for.

Conversely, suppose that (P', L') is a generalized quadrangle $GQ(s,t)$ minus a regular spread S (note that this time L' does not contain lines of S). Identify the points of P' which are on the same line of S, and identify the lines of L' which intersect the same elements of S. This way $st+1$ points and $t(st+1)/(s+1)$ lines are obtained. Each line has $s+1$ points, there are t lines on each point and any two points determine a unique line. Hence, this is a 2-design with parameters $(v, k, \lambda) = (st+1, s+1, 1)$. Let us set $s = q-1$ and $t = q+1$ to obtain an affine plane of order q. Let P'' be the set of lines of this affine plane together with a new point ∞, and let L'' be the family of parallel classes (considered as sets of lines of the affine plane) extended with the point ∞. Finally, let L''' be the set of lines L' where we extend each line with its class (a corresponding line of the affine plane). Now, we can easily show that $(P' \cup P'', L'' \cup L''')$ is a generalized quadrangle $GQ(q,q)$ with ∞ as a regular point. □

Let (P, L) be a generalized quadrangle with a spread $S \subset L$. Then we define a new incidence structure, called the *quotient of* (P, L) *over* S, by taking for the points the lines of S, and for the lines the lines of $L \backslash S$ in which we substitute each point with the line of S through it. Alternatively, we can delete the lines of S, identify the points of P which were on the same line of S, and then identify the lines with the same points.

3.5.2 LEMMA. *A generalized quadrangle $GQ(s,t)$ with $t > 1$ quotients over a regular spread to a 2-design with parameters $(st+1, s+1, 1)$.*

We can rephrase this result in graph theory language: the incidence graph of a generalized quadrangle minus a regular spread covers the incidence graph of a 2-design.

Here is now our main result of the Chapter.

3.5.3 THEOREM. *If a regular spread of a generalized quadrangle $GQ(s,t)$, $t > 1$, exists, then a spread which is not a regular one also exists.*

Proof. Choose any line ℓ of $GQ(s,t)$ and let $s_1, \ldots, s_{s+1}$ be the lines of the regular spread whose union U contains ℓ. By the definition of a regular spread there exists for each point p_i of $s_1 \setminus \ell$ a line ℓ_i which is also contained in the union U. Lines ℓ,$\ell_1, \ldots$,ℓ_s are disjoint by the definition of a generalized quadrangle, therefore these $s+1$ lines partition U. The spread obtained from the regular spread by *switching* (substituting) the first set of $s+1$ lines by the second one is again a spread, but it is clearly not a regular one. For, let r be a line on $s_1 \cap \ell$ distinct from s_1 and ℓ. So $|r \cap U| = 1$. Let r' be the line on p_1 which intersects the same elements of the regular spread. Then after the switching the lines r and r' do not intersect the same elements of the new spread. $\square$

We have already seen that generalized quadrangles $GQ(q-1, q+1)$ have regular spreads. Another infinite family of examples will be shown after we consider some small examples.

The above result implies that there is up to isomorphism a unique regular spread of the generalized quadrangle $GQ(2,4)$, which comes from a regular point in $GQ(3,3)$. Hence the two kinds of spreads of $GQ(2,4)$ are related by this switching (cf. the remark after the construction of the cyclic cover with parameters $(9,3,3)$).

There is another small example. A generalized quadrangle $GQ(4,4)$ is isomorphic to the orthogonal generalized quadrangle $Q(4,4)$, which is self dual by Payne and Thas [113, Proposition 3.2.1], and its dual is known also as the symplectic generalized quadrangle $W(4)$, see Payne and Thas [113, p. 129]. By Payne and Thas [113, Propositions 3.3.1(i), 3.4.1], the symplectic generalized quadrangle $W(4)$ has regular points, and spreads which induce spreads in the generalized quadrangle $GQ(3,5)$ coming from Payne's construction 3.5.1, cf. Payne and Thas [113, Proposition 3.4.3]. A generalized quadrangle $GQ(3,5)$ is, by Payne and Thas [113, Propositions 3.2.6 and 6.2.4] and by the uniqueness of the generalized quadrangle $GQ(4,4)$, isomorphic to $P(W(4),x)$, for a regular point x, therefore it comes from the Payne's construction. This implies that the generalized quadrangle $GQ(3,5)$ has a regular spread, and therefore, by Theorem 3.5.3, there are at least two nonisomorphic covers with parameters $(16,4,4)$.

In [112] Payne actually determined all the spreads in the generalized quadrangles $GQ(3,5)$, $GQ(4,4)$ and $GQ(5,3)$. There are 4608 of the first, 120 of the second and 24 of the third kind. The 4608 spreads were divided into 360 ones coming from planar ovoids of $GQ(5,3)$, 216 ones related to a regular spread by the above switching, and 4032 *serendipitous* spreads, constructed by J. A. Thas (whose construction works in any $GQ(2^r,2^r)$). Brouwer and Koolen (private communication, October 1993) have shown there are exactly five non-isomorphic spreads in $GQ(3,5)$, with exactly one of them corresponding to a planar ovoid in $GQ(5,3)$ and exactly one serendipitous spread.

Let S be a spread in a generalized quadrangle $GQ(s,t)$ such that the group of automorphisms of the generalized quadrangle which fixes each component of S is cyclic of order $s+1$. Then S is called a *cyclic* spread. A cyclic spread S is always a regular spread, since for each line $s \in S$ and for any two points x and y on s there is an automorphism which fixes each line of S and maps x to y, so for each line on x there is a corresponding line on y which intersects the same elements of S. We can apply the above switching to any cyclic distance-regular cover of a complete graph coming from a generalized quadrangle $GQ(s,t)$ with a spread in order to obtain a non-isomorphic distance-regular graph with the same intersection array. Since, by Godsil and Hensel [68, Thm. 9.2], in a cyclic r-cover of K_n the index r has to divide n, in this case $s+1$ has to divide $st+1$. Beside the generalized quadrangles with parameters $(q-1,q+1)$ the only other candidates are the generalized quadrangles with parameters (q,q^2). Godsil [65, Lemma 6.1] has shown that for every prime power q there is a generalized quadrangle $GQ(q,q^2)$ with a cyclic spread. In the same article Godsil has mentioned a generalization of the construction of the cyclic cover with parameters $(9,3,3)$. As the proof of this result has not been published, we present it here.

3.5.4 THEOREM (Godsil). *There is up to isomorphism a unique cyclic spread in the dual of the unitary generalized quadrangle $GQ(q,q^2)$.*

Proof. We will prove the dual statement. Let H be a nondegenerate hermitian variety in $PG(3,q^2)$. Then its points and lines form a unitary generalized quadrangle $\mathcal{U}(3,q^2)$ with parameters (q^2,q). Let $\mathcal{O}$ be an ovoid of this generalized quadrangle. Then $|\mathcal{O}| = q^3+1$. Since a line of $PG(3,q^2)$ contains q^2+1 points, we can choose for distinct points x_1 and x_2 of $\mathcal{O}$ the third point $x_3 \in \mathcal{O}\backslash\{x_1,x_2\}$ so that the points x_1,x_2,x_3 generate a plane E. Let $x_4 \in \mathcal{O}\backslash\{x_1,x_2,x_3\}$. Suppose that $x_4 \notin E$. Then the identity is the only linear mapping of $PG(3,q^2)$ fixing the points x_1,x_2,x_3 and x_4. By Tits, see Cameron [39, Theorem 7.6.2], an automorphism of a classical generalized quadrangle, which is not symplectic or orthogonal, extends to a semilinear mapping of the underlying vector space. Therefore if the ovoid $\mathcal{O}$ is cyclic, then $\mathcal{O} \subset E$, i.e., $\mathcal{O} = E \cap H$. By Godsil [65, Lemma 6.1], plane ovoids (i.e., intersections of a nondegenerate hermitian variety in $PG(3, q^2)$ with any non-degenerate plane) are cyclic ovoids. $\square$

6. Spreads of generalized quadrangles of order three

A generalized quadrangle $GQ(3, 3)$ is either the orthogonal generalized quadrangle $Q(4, 3)$ or its dual, the symplectic generalized quadrangle $W(3)$, see Payne and Thas [113, p. 55]. The first one has no spreads and the second one does have them. In this section we prove that there is up to isomorphism a unique spread in the symplectic generalized quadrangle $W(3)$.

In the generalized quadrangle $W(3)$ all points are regular, by Payne and Thas [113, p. 51], so $P(W(3), x)$ is the generalized quadrangle $GQ(2, 4)$ with a spread S for a point x of $W(3)$. Let S' be any spread of the generalized quadrangle $W(3)$ and let S'' be the set of nonempty lines obtained from the lines of S' by deleting the points of $x^\perp$. Then S'' is a spread in $P(W(3), x)$ disjoint from S, see Payne and Thas [113, p. 58]. We have learned in the previous section that S is a regular spread, so no two lines of S'' intersect the same elements of S. By the construction in Theorem 3.5.1 any spread S'' of $P(W(3), x)$ with this property uniquely determines a spread in $W(3)$. Hence the spreads of $W(3)$ and the spreads of $P(W(3), x)$, whose lines do not intersect the same elements of the regular spread, are in one to one correspondence.

3.6.1 THEOREM. *There is up to isomorphism only one spread of the symplectic generalized quadrangle $W(3)$.*

Proof. We start with the generalized quadrangle $GQ(2, 4)$ minus the cyclic spread S (we have shown after Lemma 3.4.1 that this object is unique) and we try to find all possible spreads S'' such that no two elements of S'' intersect exactly the same elements of S. Therefore the nine lines of S'' project one-to-one into the lines of the affine plane $AG(2, 3)$. There are four parallel classes of lines in $AG(2, 3)$ (all together twelve lines). Since the three lines of $AG(2, 3)$, which are not images of lines from S'', have to cover all the points of $AG(2, 3)$, they form a parallel class. There is an automorphism τ of the directed $AG(2, 3)$ which rotates it for $\pi/2$ radians (see Figure 3.5), further there is an automorphism which exchanges the pair of skew parallel classes with the horizontal and the vertical parallel classes (see Figure 3.6).

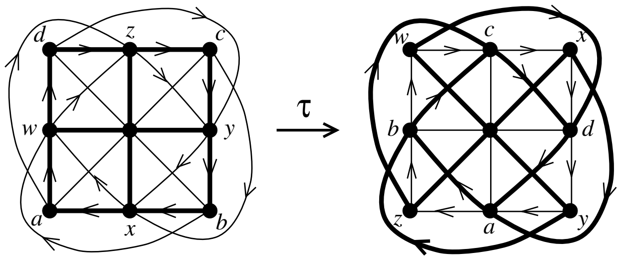

Figure 3.6: The automorphism τ, which exchanges the vertical and the horizontal classes with the skew ones.

Therefore we can assume, without loss of generality, that S'' projects to the vertical, to the horizontal and to one of the two skew parallel classes. Since S is a cyclic spread we can assume that S'' contains the line $\binom{y=x}{z=-1}$ (this is only a convenient form to write two equations, and the top equation determines the line corresponding to the image under projection), see Figure 3.8, where in the two dimensional case the lines of the spread S'' are presented in a different style at different levels, 0 – solid, 1 – dashed, -1 – dotted.

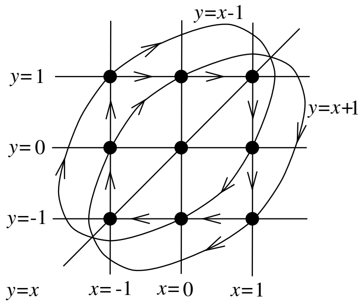

Figure 3.7: The automorphism τ^2.

Since the automorphism τ^2 preserves the sets of skew lines, see Figure 3.7, we can choose the line $\binom{y=0}{z=0}$ to be an element of S''. This implies that the line $\binom{x=0}{z=1}$ is an element of S'' too. We now have two choices where to lift the line $y = -1$: to the line $\binom{y=-1}{z+x=0}$ or to the line $\binom{y=-1}{z+x=-1}$.

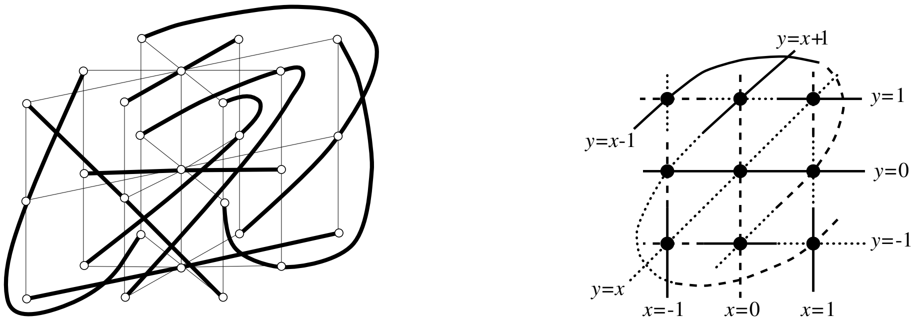

Figure 3.8: The unique spread of the generalized quadrangle $W(3)$ drawn in three and two dimensions.

Let us proceed with the first choice. Then S'' must contain the lines $\binom{x=-1}{z-y=1}$, $\binom{y=1}{z-x=-1}$, $\binom{x=1}{z+y=-1}$, $\binom{y-x=1}{z=x}$, $\binom{x-y=1}{z+x=-1}$, and S'' is uniquely determined. Since the automorphism τ^2 maps the line $\binom{y=1}{z=x-1}$ to the line $\binom{y=-1}{z-x=-1}$ and preserves previously chosen lines for S'', the second choice renders an isomorphic spread S''. $\square$

7. Orthogonal generalized quadrangles

In this section we give an alternative proof, due to Brouwer and Wilbrink (private communication, February 1993), of Theorem 3.6.1. We present an extended version of their proof, as the original version has only eight sentences. We will use the following classical result which can be found, for example, in Hirschfeld [85], or Van Lint and Wilson [101].

3.7.1 THEOREM. *Any nondegenerate quadratic form $Q(x_0, \ldots, x_n)$ over $GF(q)$ is projectively equivalent*

(i) for n even to

$$\mathcal{P}_n(x) = x_0^2 + x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n,$$

in which case the quadric Q contains $(q^n - 1)/(q - 1)$ points and the maximum projective dimension of a flat $F \subseteq Q$ is $n/2 - 1$;

(ii) for n odd to

$$\mathcal{H}_n(x) = x_0x_1 + x_2x_3 + \cdots + x_{n-1}x_n,$$

or

$$\mathcal{E}_n(x) = x_0x_1 + x_2x_3 + \cdots + x_{n-3}x_{n-2} + p(x_{n-1}, x_n),$$

where $p(x_{n-1}, x_n)$ is an irreducible quadratic form, and the quadric Q contains $(q^{(n+\epsilon)/2} - 1)(q^{(n-\epsilon)/2} + 1)/(q - 1)$ points and the maximum projective dimension of a flat F of Q is $(n - 2 + \epsilon)/2$, with $\epsilon = 1$ in the hyperbolic case and $\epsilon = -1$ in the elliptic case. $\square$

Quadratic forms equivalent to P, H and E are called *parabolic*, *hyperbolic*, and *elliptic* respectively. It is worthwhile mentioning that all elliptic quadratic forms are projectively equivalent (i.e., we can choose $p(x, y) = dx^2 + xy + y^2$ with $d = 1$ for q even and $1 - 4d$ a nonsquare for q odd. The above result can be quite easily derived from elementary facts, see the books by Cameron [39, p. 76], and by Van Lint and Wilson [101, Ch. 26]. For the following proof it is crucial to remember that *the intersection of a quadric with a flat F of $PG(n, q)$ is a quadric in that flat.*

Second proof of Theorem 3.6.1 (Brouwer and Wilbrink): The dual statement will be proved:

There is a unique ovoid in the orthogonal generalized quadrangle $Q(4, 3)$.

Brouwer calls this ovoid $Sp(4, 3)$. Let $Q(x)$ be a nondegenerate quadratic form in $PG(4, 3)$ (cf. Van Lint and Wilson [101, Example 26.E]). Then its isotropic points and lines determine the orthogonal generalized quadrangle $Q(4, 3)$. Let X, Y and Z respectively be the set of isotropic points (i.e., $Q^{-1}(0)$), $Q^{-1}(1)$ and $Q^{-1}(2)$. As we know (see Van Lint and Wilson [101, p. 318]), a line ℓ of $PG(4, 3)$ can intersect the quadric X in four different ways:

(1) $|\ell \cap X| = 4$ (ℓ is an isotropic line), i.e., $(x_1, x_2) = 0$ for distinct $x_1, x_2 \in \ell \cap X$.

(2) $|\ell \cap X| = 2$ (ℓ is a hyperbolic line), i.e., $(x_1, x_2) \neq 0$ for distinct $x_1, x_2 \in \ell \cap X$, i.e., $|\ell \cap Y| = 1 = |\ell \cap Z|$.

(3) $|\ell \cap X| = 1$ (ℓ is a tangent line), i.e., either
   (a) $|\ell \cap Y| = 3$, i.e., $(y_1, y_2) \neq 0$ for distinct $y_1, y_2 \in \ell \cap Y$, or
   (b) $|\ell \cap Z| = 3$, i.e., $(z_1, z_2) \neq 0$ for distinct $z_1, z_2 \in \ell \cap Z$.

(4) $|\ell \cap X| = 0$ (ℓ is an elliptic line), i.e., $|\ell \cap Y| = 2 = |\ell \cap Z|$, i.e., $(y_1, y_2) = 0$ for distinct $y_1, y_2 \in \ell \cap Y$, i.e., $(z_1, z_2) = 0$ for distinct $z_1, z_2 \in \ell \cap Z$.

Let $\mathcal{O}$ be an ovoid, i.e., the set of ten isotropic points, no two orthogonal and let E be the plane spanned by distinct points x_1, x_2 and x_3 of $\mathcal{O}$. Then it is nondegenerate, since it contains more than two nonisotropic points and not all the points are isotropic (cf. Van Lint and Wilson [101, Example 26.3]).

Therefore $E \cap X$ is a parabolic quadric of E and it consists of four isotropic points, say x_0, x_1, x_2 and x_3 that are pairwise orthogonal.

Let us now prove that $x_0 \in \mathcal{O}$. Let x be some isotropic point orthogonal to x_1. The space H generated by E and x is not degenerate, i.e., $H^\perp \notin X$, since otherwise the lines on x, x_1 and $H^\perp$ would form a triangle in the generalized quadrangle. Because H contains the isotropic line xx_1, it is a hyperbolic 3-space. This implies that isotropic points of H form the generalized quadrangle $GQ(3,1)$, i.e., 4×4 grid. The points x_0, x_1, x_2, x_3 are pairwise nonorthogonal, so no two of them lie in the same column or row. Since the points on a vertical line through x_0 cannot be in $\mathcal{O}$, x_0 must be.

Let S be a 3-space generated by the plane E and any point of $\mathcal{O} \backslash E$. Since the generalized quadrangle $GQ(3,1)$ does not contain five pairwise noncollinear points, S is not hyperbolic, and since the first neighbourhood of a vertex in the point graph of a generalized quadrangle $GQ(3,3)$ has at most four pairwise noncollinear points, S is not degenerate. Therefore S is an elliptic 3-space (cf. Van Lint and Wilson [101, Example 26.5]). As a subspace U is degenerate whenever $U \cap U^\perp \neq \emptyset$, i.e, whenever its orthogonal complement $U^\perp$ is degenerate, the line $E^\perp$ is either hyperbolic or elliptic. It is not completely trivial (see Higman [84, p. 22]) to show that for any nonisotropic point u there exists a linear transformation preserving $Q(x)$, which maps u to any other point v with $Q(v) = Q(u)$. So the action of the group $PGL(5,3)$ on nonisotropic points has two orbits Y and Z, and the same is true for their orthogonal complements, i.e., elliptic and hyperbolic hyperplanes. In the case when $E^\perp$ is a hyperbolic line, there exists a unique elliptic 3-space containing E. Since an elliptic quadric in $PG(3,3)$ has ten points, the ovoid $\mathcal{O}$ is the elliptic quadric of S determined by $\mathcal{O} \cap X$. In the case when $E^\perp$ is an elliptic line, there are two possibilities for S, and $\mathcal{O} \backslash E$ is an independent set of size six in a bipartite subgraph on 6+6 vertices (six vertices in each choice for S) of the point graph of the orthogonal generalized quadrangle $Q(4,3)$. The valency of this graph is at least four, since the set of ten isotropic points of S intersects each isotropic line. Hence the set $\mathcal{O} \backslash E$ must be one of the two bipartite halves. So the only ovoids are the elliptic quadrics of $PG(3,3)$. $\square$

8. Conclusion

While we determined geometric and locally cyclic covers with parameters $(10,4,2)$, work on nongeometric covers is still not finished (cf. Haemers [73], Peeters [114]). For $n > 10$ a few classical constructions or characterizations (Bondy in Biggs [15], Brouwer [24] (Thm. 3.3.1), Mathon [103], Thas-Somma

[127], [136]) are known, see Table 3.3. For most of these graphs we do not know if they uniquely realize their parameters. Many infinite families of feasible parameter sets also exist, communicated to me by de Caen and Godsil. One has, for example, parameters $(2^t, 2^{t-1}-1, 2)$ and seems to be related to design theory, while another one has parameters $(2^{2t}, 2^{2t-1}, 2)$. For this one a new infinite family of covers was constructed quite recently and it was related to coding theory, see de Caen, Mathon and Moorhouse [38].

We close this chapter by mentioning that the double-covers of K_{10} and K_{16} will be constructed in Section 5.5.

n	r	a_1	c_2	a cover G of K_n	# of G
11	9	1	1	does not exist (PG(2,10))	0
12	5	2	2	Mathon's construction [103]	≥ 1
13	11	1	1	open (PG(2,12))	?
14	2	6	6	equivalent to Paley graph $\{6,3;1,3\}$	1
14	3	4	4	Mathon's construction [103]	≥ 1
14	6	2	2	Mathon's construction [103]	≥ 1
16	2	6	8	[38], [127] and [136]	1
16	2	8	6	unique two-graph	1
16	4	2	4	GQ(3,5)$\backslash$ spread, Thm. 3.3.1, [38]	≥ 5
16	6	4	2	GQ(5,3)$\backslash$ spread, Thm. 3.3.1	≥ 1
16	7	2	2	OPEN	?
16	8	0	2	[38]	≥ 1
17	3	5	5	Mathon's construction [103]	≥ 1
17	5	3	3	GQ(4,4)$\backslash$ unique spread, [103], Thm. 3.3.1	≥ 2
17	15	1	1	equivalent to PG(2,16), Mathon's construction [103]	≥ 1
18	2	8	8	Mathon's construction [103]	1
18	4	4	4	Mathon's construction [103]	≥ 1
18	8	2	2	Mathon's construction [103]	≥ 1
19	4	2	5	[74] (GQ(3,6) does not exist [113, p. 124])	0
19	7	5	2	[66] (GQ(6,3) does not exist [113])	0
19	17	1	1	open (PG(2,18))	?

Table 3.3: List of distance-regular covers of complete graphs K_n for $n = 11, \ldots, 19$.

4

ANTIPODAL DISTANCE-REGULAR GRAPHS WITH $D = 4, 5$

In this chapter we begin to study antipodal distance-regular graphs of diameter 4 and 5. Their antipodal quotients are distance-regular by Theorem 2.4.2 and have diameter two, i.e., they are strongly regular graphs. So we can regard these graphs as distance-regular antipodal covers of strongly regular graphs. This study is a natural step following the previous chapter, where we have been investigating antipodal distance-regular graphs of diameter three, i.e., distance-regular (antipodal) covers of complete graphs.

In general there was not much known about antipodal distance-regular graphs of diameter four and five when we started to work on this problem, with the exception of covers of complete bipartite graphs of course. It is quite difficult to decide even which families of strongly regular graphs are more interesting than others. In Brouwer et al. [27, Ch. 14] a table of parameters of these graphs, which pass certain criteria, is given. We have determined a minimal set of such criteria and then study some of them in detail. This has motivated a parametrization of Q-polynomial covers of diameter four with two parameters. Terwilliger has shown that all P- and Q-polynomial antipodal graphs of diameter at least five are already known. Further he proved that P- and Q-polynomial antipodal graphs are locally strongly regular. Inspired by this we study local structure of distance-regular graphs and find that also in some other cases intersection parameters force a graph to be locally strongly regular. From this we derive some new uniqueness and nonexistence results. In particular a quarter of possible P- and Q-polynomial antipodal graphs of diameter four and many feasible intersection arrays from the above mentioned table are ruled out.

In Section 4.1 we have established basic facts about strongly regular graphs. The intersection numbers and the dual intersection numbers (also called the Krein parameters) are essential when we study feasibility of an intersection array. The intersection numbers have to be integers and the dual intersection numbers have to be, by Krein bounds, non-negative. This is the theme of Section 4.2. In Section 4.3 we devote special attention to bipartite antipodal distance-regular graphs. In Section 4.4 we explain when we call an intersection array of antipodal distance-regular graphs of diameter four and five feasible, and list all known examples. Local structure of antipodal distance-regular graphs is studied in Section 4.5. In the last section we present some infinite families of feasible intersection arrays of antipodal distance-regular graphs of diameter 4 and 5 as future challenges.

1. Strongly regular graphs

We have defined a strongly regular graph in Chapter 2 as distance-regular graphs of diameter two. Here we give more general definition. A graph is *strongly regular* if it is a k-regular graph with the property that the number of common neighbours of two vertices u and v is either λ or μ depending on whether u and v are adjacent or not. Some examples of these graphs are the quadrangle, the pentagon, the direct product of two triangles, the Petersen graph and Paley graphs. Equivalently, a strongly regular graph is an association scheme with at most two classes. We will see that a connected regular graph is strongly regular if and only if it has three eigenvalues. Let $\lambda = a_1$, $\mu = c_2$, and let $\{k, k-\lambda-1; 1, \mu\}$ be an intersection array of a strongly regular graph. Then by Lemma 2.1.1(a) the number of its vertices is equal to

$$n = 1 + k + k(k-\lambda-1)/\mu = 1 + k(k-\lambda-1+\mu)/\mu$$

and (n, k, λ, μ) are the parameters which are traditionally given for a strongly regular graph. It can be easily seen that the complement of a strongly regular graph with parameters (n, k, λ, μ) is also strongly regular and has parameters

$$(\overline{n}, \overline{k}, \overline{\lambda}, \overline{\mu}) = (n, n-k-1, n-2k+\mu-2, n-2k+\lambda).$$

The only disconnected strongly regular graphs are the disjoint unions of a number of isomorphic complete graphs; these are the only strongly regular graphs with $\mu = 0$. We will usually not consider covers of disconnected strongly regular graphs nor their complements (these are the complete multipartite graphs $K_{t(m)}$), in which case

$$\max\{0, 2k-n+1\} \le \lambda \le k-2 \quad \text{and} \quad \max\{1, 2k-n+2\} \le \mu \le k-1.$$

Strongly regular graphs satisfying these inequalities will be called *nontrivial strongly regular graphs*. The following result can be found in [27, Thm. 1.3.1].

4.1.1 PROPOSITION. *A graph G on n vertices is strongly regular if and only if its adjacency matrix A satisfies $A^2 = kI + \lambda A + \mu(J - I - A)$ and $AJ = kJ$ for some integers k, λ and μ. Its eigenvalues are then $\theta_0 = k$ with multiplicity one and*

$$\theta, \tau = \frac{1}{2}\Big[\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\Big],$$

with multiplicities

$$m_\theta = \frac{(n-1)\tau + k}{\tau - \theta} = \frac{(\tau+1)k(k-\tau)}{\mu(\tau-\theta)} \quad \text{and} \quad m_\tau = n - 1 - m_\theta.$$

They satisfy $k > \theta > 0 > \tau > -k$. □

In the next section we study distance-regular antipodal covers of diameter four, for which, by Theorem 2.4.4 (or 2.4.6), the eigenvalues with even indices are also the eigenvalues of the antipodal quotient, which is a strongly regular. So we have $\theta_2 = \theta$ and $\theta_4 = \tau$.

In the case of strongly regular graphs with $\mu > 0$ all the parameters can be determined from their eigenvalues:

$$n = 1 + \frac{k(k-\theta-\tau-1)}{(k+\theta\tau)} = \frac{(k-\theta)(k-\tau)}{k+\theta\tau},$$

$$\lambda = k + \theta + \tau + \theta\tau, \qquad \mu = k + \theta\tau,$$

so we can use the eigenvalues for the classification of strongly regular graphs as well.

The nontrivial Krein bounds (corresponding to q_{22}^2 and q_{44}^4) are:

$$(k-\lambda-1)^2(k^2+\theta^3) \geq \mu^2(\theta+1)^3, \quad \text{and} \quad (k-\lambda-1)^2(k^2+\tau^3) \geq \mu^2(\tau+1)^3,$$

for an alternative form see Seidel [118], Brouwer and Van Lint [31]. The nontrivial absolute bounds are:

$$n \leq m_\theta(m_\theta + 1 + 2s_1)/2, \quad \text{and} \quad n \leq m_\tau(m_\tau + 1 + 2s_2)/2,$$

where s_i is one if the equality holds in the i-th of the above Krein bounds and zero otherwise.

The following statement is a direct consequence of the fact that the sequences $\{b_i\}$ and $\{-c_i\}$ are decreasing, and it is due to Godsil, Schade and the author [69].

4.1.2 PROPOSITION. *Let G be a strongly regular graph with parameters (n,k,λ,μ), i.e., intersection array $\{k, k-\lambda-1; 1, \mu\}$, and suppose that it has a distance-regular antipodal r-cover H. Then*

(i) $k \leq \lfloor \frac{n-1}{2} \rfloor$ *for* $\mathrm{diam}(H) = 5$,

(ii) $k \leq \left\lfloor \frac{r(n-1)}{2r-1} \right\rfloor$ *for* $\mathrm{diam}(H) = 4$. □

In the second case the bound is attained for the octagon, which is the double-cover of the quadrangle. There are no other such examples, so we conjecture that for $k \neq 2$ the following holds:

$$k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{i.e.,} \quad k \geq \lambda + \mu + 1, \quad \text{i.e.,} \quad c_2(G) \leq b_1(G),$$

with only one parameter set attaining this bound, cf. Brouwer et al. [27, Thm 1.5.5] (and remember that $c_2(H) = \mu/r$). For small parameter sets all potential examples are ruled out by a single integrality condition: $m(\theta_1) \in \mathbb{N}$, where θ_1 is the smallest new eigenvalue of the cover. This condition is nontrivial only for $\lambda \neq 0$, in which case it is equivalent to $z = \sqrt{\lambda^2 + 4k} \in \mathbb{N}$, cf. Proposition 4.2.1, and we could use z to parametrize $k = (z^2 - \lambda^2)/4$.

2. Intersection parameters, Krein and absolute bounds

In this section we determine all the intersection numbers of antipodal distance-regular graphs of diameter four and five, and Krein and absolute bounds in diameter four case. Theoretically we could determine all the intersection numbers from the intersection array by using the recurrence relation mentioned at the beginning of Chapter 2. However, calculations become messy quite soon, so it is more convenient to use Theorem 2.4.2 and Figures 4.1 and 4.3 for this purpose.

Throughout this section we will assume that H is a distance-regular antipodal r-cover of a strongly regular graph G with parameters (n,k,λ,μ) and eigenvalues $\theta_0 > \theta_2 > \theta_4$.

We start with the case when the cover H has diameter four. Then all its parameters are determined by (r,k,λ,μ) or if you prefer by $(r,\theta_0,\theta_2,\theta_4)$:

$$\{k, k-\lambda-1, (r-1)\mu/r, 1; 1, \mu/r, k-\lambda-1, k\}.$$

Let us define for $s \in \{0,1,2,3,4\}$ the symmetric 4 by 4 matrix $P(s)$ with its ij-entry being equal to $p_{ij}(s)$. Then:

$$P(0) = \begin{pmatrix} k & 0 & 0 & 0 \\ & k_2 & 0 & 0 \\ & & (r-1)k & 0 \\ & & & r-1 \end{pmatrix}, \quad P(1) = \begin{pmatrix} \lambda & b_1 & 0 & 0 \\ & k_2 - b_1 r & b_1(r-1) & 0 \\ & & \lambda(r-1) & r-1 \\ & & & 0 \end{pmatrix},$$

$$P(2) = \begin{pmatrix} \mu/r & a_2 & b_2 & 0 \\ & k_2 - r(a_2+1) & (r-1)(k-\mu) & r-1 \\ & & b_2(r-1) & 0 \\ & & & 0 \end{pmatrix},$$

$$P(3) = \begin{pmatrix} 0 & b_1 & \lambda & 1 \\ & k_2 - rb_1 & b_1(r-1) & 0 \\ & & \lambda(r-2) & r-2 \\ & & & 0 \end{pmatrix}, \quad P(4) = \begin{pmatrix} 0 & 0 & k & 0 \\ & k_2 & 0 & 0 \\ & & k(r-2) & 0 \\ & & & r-2 \end{pmatrix},$$

where $b_1 = k-1-\lambda$, $k_2 = rkb_1/\mu$, $a_2 = k-\mu$ and $b_2 = (r-1)\mu/r$.

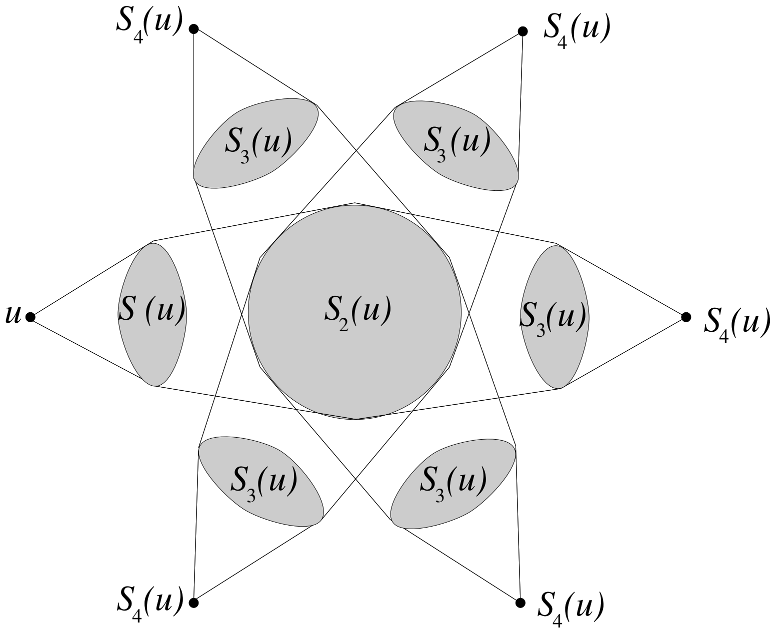

Figure 4.1: A distance-regular graph of diameter four (the distance partition corresponding to an antipodal class).

An immediate consequence of Theorem 2.4.6 is the following result (mentioned already in Van Bon and Brouwer [17]).

4.2.1 PROPOSITION. *Let H be an antipodal distance-regular graph of diameter four. The two new eigenvalues θ_1 and θ_3 of H are the two roots of $x^2 - \lambda x - k = 0$ and they occur with multiplicity*

$$m(x) = \frac{(r-1)n}{2 + \lambda x/k}, \quad \text{i.e.,} \quad m_1 = \frac{n(r-1)\theta_3}{\theta_3 - \theta_1} \quad \text{and} \quad m_3 = n(r-1) - m_1$$

Consequently, either $\lambda = 0$ or $\lambda^2 + 4k$ is a square and these eigenvalues are integral. □

Remark: This result immediately implies that the conference graphs (with the exception of the pentagon) and the lattice graphs (with the exception of the quadrangle) cannot have distance-regular antipodal covers of diameter four (see Godsil, Jurišić and Schade [69, Corollaries 4.5 and 3.10]).

By Theorem 2.4.6 we have $\theta_0 > \theta_1 > \theta_2\ (\geq 0) > \theta_3 > \theta_4$ and these are all the eigenvalues of H (with equality only if H covers the complete bipartite graph). The matrix of eigenvalues of H, defined by $(P)_{ij} = v_i(\theta_j)$ (Lemma 2.2.5), has the following form:

$$P(H) = \begin{pmatrix} 1 & \theta_0 & r\theta_0 b_1/\mu & \theta_0(r-1) & r-1 \\ 1 & \theta_1 & 0 & -\theta_1 & -1 \\ 1 & \theta_2 & -r(\theta_2+1) & \theta_2(r-1) & r-1 \\ 1 & \theta_3 & 0 & -\theta_3 & -1 \\ 1 & \theta_4 & -r(\theta_4+1) & \theta_4(r-1) & r-1 \end{pmatrix}.$$

After straightforward computation we find that there are only three nontrivial Krein bounds: $q_{22}(2) \geq 0$, $q_{44}(4) \geq 0$, and $q_{11}(4) \geq 0$. The first two come from the strongly regular graph G, and the last one translates to $\theta_3^2 \geq -\theta_4$. Furthermore, we find that $q_{12}(2), q_{12}(4), q_{14}(4), q_{22}(3), q_{23}(4), q_{34}(4) = 0$, and $r = 2$ if and only if $q_{11}(1), q_{11}(3), q_{13}(3), q_{33}(3) = 0$. All the other Krein parameters are strictly positive. Recall [131], [27, Thm. 2.11.6], that a d-class association scheme is Q-polynomial if and only if the *representation diagram* Δ_E for $E = E_s$ being a minimal idempotent, is a path. The representation diagram Δ_E is the undirected graph with vertices $0, 1, \ldots d$, where we join two distinct vertices i and j whenever $q_{ij}(s) = q_{ji}(s) \neq 0$. For $s = 1$ and $r = 2$ we get the following graph:

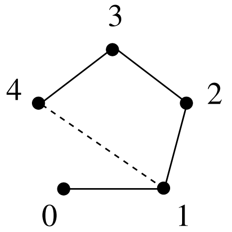

Figure 4.2: The representation diagram Δ_E.

which implies:

4.2.2 PROPOSITION. *Let H be an antipodal distance-regular graph of diameter four. Then $\theta_3^2 \geq -\theta_4$. For $r=2$ the equality holds if and only if H is Q-polynomial.* □

The equality case is a special case of a result of Terwilliger [133, Thm. 3]. We will study further Q-polynomial antipodal distance-regular graphs of diameter four in Section 5.

Based on the above information it is not difficult to determine all the absolute bounds as well. Let $s_{11}(4)$ be one if $q_{11}(4)$ is zero and zero otherwise. Similarly, let s be one if $r-2$ is zero and zero otherwise. Since $m_1 \leq m_3$, $m_2+m_4 = n-1$ and $m_1+m_3 = n(r-1)$ the absolute bounds reduce for $k \geq 5$ to the nontrivial absolute bounds of G and to the additional three inequalities:

$$\begin{array}{rcll}
(r-1)n & \leq & m_1 m_2, & \text{i.e.,}\ \ 1+k/\theta_3^2 \leq m_2,\\
(r-1)n & \leq & m_1(m_4+s_{11}(4)), & \text{i.e.,}\ \ 1+k/\theta_3^2 \leq m_4+s_{11}(4),\\
n+(r-1)ns & \leq & m_1(m_1+1)/2+m_4 s_{11}(4). &
\end{array}$$

In the case when the cover H has diameter five the intersection array is determined by five parameters:

$$\{k, k-\lambda-1, (r-1)t, \mu, 1; 1, \mu, t, k-\lambda-1, k\}.$$

and we have:

$$P(1) = \begin{pmatrix}
\lambda & b_1 & 0 & 0 & 0\\
 & a_2 b_1/\mu & b_1 b_2/\mu & 0 & 0\\
 & & a_3 b_1 (r-1)/\mu & b_1(r-1) & 0\\
 & & & \lambda(r-1) & r-1\\
 & & & & 0
\end{pmatrix},$$

$$P(2) = \begin{pmatrix} \mu & a_2 & b_2 & 0 & 0 \\ & p_{2,2}(2) & (r-1)p_{22}(3) & (r-1)k & 0 \\ & & p_{33}(2) & (r-1)a_2 & r-1 \\ & & & (r-1)\mu & 0 \\ & & & & 0 \end{pmatrix},$$

$$P(3) = \begin{pmatrix} 0 & t & a_3 & \mu & 0 \\ & p_{22}(3) & p_{33}(2)/(r-1) & a_2 & 1 \\ & & p_{33}(3) & (r-2)a_2 & 0 \\ & & & (r-2)\mu & 0 \\ & & & & 0 \end{pmatrix},$$

$$P(4) = \begin{pmatrix} 0 & 0 & b_1 & \lambda & 1 \\ & k_2 & a_2 b_1/\mu & b_1 & 0 \\ & & (r-2)a_2 b_1/\mu & (r-2)b_1 & 0 \\ & & & (r-2)\lambda & 0 \\ & & & & r-2 \end{pmatrix},$$

$$P(5) = \begin{pmatrix} 0 & 0 & 0 & k & 0 \\ & 0 & k_2 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & (r-2)k & 0 \\ & & & & r-2 \end{pmatrix},$$

where $t = c_3$, $b_1 = k-1-\lambda$, $a_2 = k-(r-1)t-\mu$, $a_3 = k-\mu-t$, $k_2 = kb_1/\mu$, $k_3 = (r-1)k_2$, $p^3_{22} = t(a_2+a_3-\lambda)/\mu$, $p^2_{22} = k_2-1-a_2-p_{22}(3)k_3/k_2-(r-1)k$, $p^3_{33} = k_3-a_2-1-t-p_{22}(3)$, and $p^3_{23} = (r-1)kb_1/\mu-a_2-1-t-p_{22}(3)$.

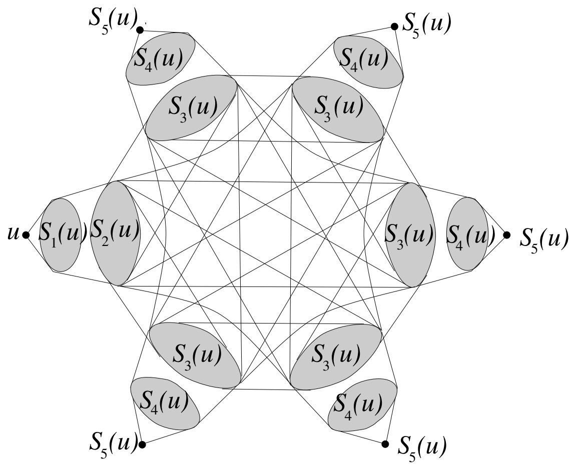

Figure 4.3: A distance-regular graph of diameter five (the distance partition corresponding to an antipodal class).

3. Bipartite antipodal distance-regular graphs

There are cases in which the existence of antipodal distance-regular graph and its antipodal quotient are equivalent. For an illustrative example we need one more definition. A *generalized Odd graph* with diameter d (also called *regular thin near* $(2d+1)$-*gon*) is a distance-regular graph G with diameter d such that $a_1(G) = a_2(G) = \cdots = a_{d-1}(G) = 0$ and $a_d(G) > 0$.

4.3.1 PROPOSITION. *If H is a bipartite antipodal distance-regular graph with odd diameter, then it is the bipartite double of its antipodal quotient G (i.e., $K_2 \otimes G$), which is a generalized Odd graph. Conversely, the bipartite double of a generalized Odd graph G is a bipartite distance-regular antipodal cover of G, with odd diameter.*

Proof. Let H has diameter $2d+1$ and index r. From the fact that $a_d(H) = a_{d+1}(H) = 0$, it follows that $r = 2$ (see Gardiner [60] or [27, p. 142]). If we add $r = 2$ to the assumptions of the statement we get the known result from Brouwer et al. [27, Thm. 4.2.11]. The converse is also known, see Brouwer et al. [27, Thm. 1.11.1(vi)]. Therefore we omit the remainder of the proof. □

Biggs and Gardiner [16] have mentioned in the proof of Proposition 5.9(3) that a bipartite antipodal distance-regular graph of odd diameter must have index two, but nobody concluded that there is a natural bijection between bipartite antipodal distance-regular graphs of odd diameter and their antipodal quotients as implied by the above result (see Jurišić [92, Theorem 6.1]).

The known examples are the Desargues graph as the bipartite double of the Petersen graph, the five-cube, the double Hoffman-Singleton, the double Gewirtz, the double 77-graph (i.e., the bipartite double of the unique strongly regular graph with intersection array $\{21, 20; 1, 4\}$, see Brouwer [25]), and the double Higman-Sims.

If a distance-regular antipodal cover H with diameter four of a strongly regular graph G with intersection array $\{k, k-\lambda-1; 1, \mu\}$ is bipartite, then $a_2(H) = 0$ implies $k = \mu$. It means that G is the complete multipartite graph with t classes of size m, i.e., $K_{t(m)}$, and $a_1(H) = 0$ implies $t = 2$, i.e., G is the complete bipartite graph $K_{k,k}$. In this case H is an incidence graph of a resolvable transversal design, cf. Theorem 6.3.1.

4. Feasibility and list of known covers

Schade and the author have used the properties of parameters of strongly regular graphs mentioned in Section 1 to obtain a list of small intersection arrays of possible nontrivial strongly regular graphs. Then we used the following conditions:

(F1) the upper bound on valency implied by Proposition 4.1.2,

(F2) $2 \le r \le k$,

(F3) the multiplicities of the new eigenvalues of the cover are integers,

(F4) Krein conditions,

(F5) absolute bounds,

and in the case of diameter four also

(F6) $r|\mu$,

(F7) if $\mu = 2$, *and* $k < \frac{1}{2}\lambda(\lambda+3)$, *then* $(\lambda+1)|k$ *(see Brouwer et al. [27, p. 6]),*

to obtain lists of small intersection arrays of possible antipodal distance-regular graphs of diameter four and five, see Godsil, Jurišić and Schade [69]. The following are the only five intersection arrays in our list which were known not to exist.

$$
\begin{aligned}
&1) \quad \{10,6,4,1;1,3,6,10\},\\
&2) \quad \{10,9,1,1;1,1,9,10\},\\
&3) \quad \{10,9,4,2,1;1,2,4,9,10\},\\
&4) \quad \{28,15,8,1;1,3,15,28\},\\
&5) \quad \{56,45,18,1;1,6,45,56\}.
\end{aligned}
$$

The first one should be a distance-regular antipodal cover of the complement of the triangular graph $T(7)$. Van Bon and Brouwer [17] have shown that such a graph has no distance-regular antipodal cover of diameter four (alternatively: such a graph would be locally Petersen graph, but for this parameter set this is not possible by Hall [75], cf. Brouwer et al. [27, Thm. 1.16.5]). The fourth one has $\theta_1 = b_1 - 1$ and $\mu = 3$, so by Brouwer et al. [27, Thm. 4.4.11] it does not exist. The second and the third one should be distance-regular antipodal covers of the Gewirtz graph, but the only distance-regular antipodal cover of this graph is its bipartite double with intersection array $\{10,9,8,2,1;1,2,8,9,10\}$, see Brouwer et al. [27, Proposition 11.4.5]. The nonexistence of the fifth one has been shown by Brouwer (private communication). Modulo these five intersection arrays and Bagchi's parameters our lists agree with the one of Brouwer, Cohen and Neumaier [27, pp. 417-418, 421-425]. We will call the intersection arrays which satisfy the above conditions *feasible* intersection arrays of antipodal distance-regular graphs of diameter four and five. Corresponding to the previous section we list only all known non-bipartite distance-regular antipodal covers of strongly regular graphs.

#	G	n	k	λ	μ	H	r	$r.n$
1	! Folded 5-cube	16	5	0	2	! Wells graph	2	32
2	! $\overline{T(6)}$	15	6	1	3	! 3.Sym(6).2	3	45
3	! $\overline{T(7)}$	21	10	3	6	! 3.Sym(7).2	3	63
4	folded J(4,8)	35	16	6	8	! Johnson graph J(4,8)	2	70
5	! truncated 3-Golay code	81	20	1	6	shortened 3-Golay code	3	243
6	! folded halved 8-cube	64	28	12	12	! halved 8-cube	2	128
7	$S_2(S_2(McL.))$	105	32	4	12	S_2(Soicher graph)	3	315
8	Zara graph (126,6,2)	126	45	12	18	$3.O_6^-(3)$	3	378
9	! S_2(McLaughlin graph) [26]	162	56	10	24	! Soicher graph	3	486
10	hyperbolic pts. of PG(6,3)	378	117	36	36	$3.O_7^-(3)$	3	1134
11	Suzuki graph	1781	416	100	96	Soicher [125]	3	5346
12		306936	31671	3510	3240	$3.Fi_{24}^-$	3	

Table 4.1: Non-bipartite antipodal distance-regular graphs of diameter four.

The strongly regular graph on 81 vertices is unique by Brouwer and Haemers [37], and for the construction of its cover see Brouwer et al. [27, p. 365].

The second subconstituent of the McLaughlin graph on 275 vertices is strongly regular and it is uniquely determined by its intersection array, see Cameron, Goethals and Seidel [46]. Soicher [125] constructed a distance-transitive antipodal three-fold cover of it, and Brouwer [26] has proved that it is uniquely determined by its intersection array. Soicher has shown further that the second subconstituent of this graph is a distance-regular antipodal three-fold cover of the second subconstituent of the second subconstituent of the McLaughlin graph. The third antipodal cover constructed by Soicher is distance-transitive and its antipodal quotient is Suzuki graph. The other five three-fold covers can be obtained from Fisher groups, see Brouwer et al. [27, p. 397]. For their explicit constructions and their uniqueness see Brouwer et al. [27, pp. 397-400]. The two covers on 45 and 63 vertices are unique, see Brouwer et al. [27, Theorem 13.2.1 and 13.2.3], and they are called the halved Foster graph and the Conway-Smith graph respectively.

The Wells graph was first constructed by Biggs and Gardiner [16], and for its uniqueness see Brouwer et al. [27, Thm. 9.2.9]. The remaining two double-covers are the Johnson graph $J(4, 8)$ and the halved eight-cube, whose uniqueness is well known, see Terwilliger [132] and Brouwer et al. [27, Thm. 9.2.7]. However, note that the intersection arrays of the folded halved eight-cube and of the folded Johnson graph $J(4, 8)$ do not uniquely determine them; the block graph of any transversal design $TD(4, 8)$ has the same parameters (but is not isomorphic) as the first graph, see Chapter 6, and there are at least 1853 nonisomorphic strongly regular graphs with the same parameters as the second graph, see Bussemaker, Mathon and Seidel [35] or [27].

#	G	n	k	λ	μ	H	r	t	$r.n$
1	! Petersen graph	10	3	0	1	! Dodecahedron	2	1	20
2	3-Golay code	243	22	1	2	short. ext. 3-Golay code	3	9	729
3	folded Johnson graph J(10,5)	126	25	8	8	! Johnson graph J(10,5)	2	9	252
4	folded halved 10-cube	256	45	16	6	! halved 10-cube	2	15	512

Table 4.2: Non-bipartite antipodal distance-regular graphs of diameter five.

The situation for the last two covers is the same as in the case of diameter four, cf. Brouwer et al. [27, p. 259 and p. 265]. Graphs related to Golay codes will be revisited in the next chapter.

5. Locally constant antipodal distance-regular graphs

Theorem 2.4.2 (cf. Figures 4.1 and 4.2) implies that the neighbourhood of a vertex of a distance-regular antipodal cover of diameter at least four projects to the neighbourhood of the projected vertex. Therefore, if we know locally some graph of diameter at least two, then we know locally its distance-regular antipodal cover as well. As we have seen in Section 3.2, this is a very strong condition on a graph (cf. Blokhuis and Brouwer [21], [22] and Hall [76]), for example, a connected graph that is locally a pentagon must be the icosahedron, see Brouwer et al. [27, p. 5].

We first examine what are the local graphs of all known antipodal distance-regular graphs of diameter four and five. Then we use graph representation to give an alternative proof in diameter four case of Terwilliger's result that Q-polynomial antipodal distance-regular graphs are locally strongly regular. This enables us to extend this result to antipodal distance-regular graphs which are not necessarily Q-polynomial or antipodal.

Sometimes the local strongly regular graph has the same parameters as the point graph of a generalized quadrangle $GQ(q+1, q-1)$. The graph for which the local strongly regular graph is the point graph of a generalized quadrangle forms an incidence structure called an *extended generalized quadrangle (EGQ)*. These combinatorial objects have already been extensively studied for almost ten years by several authors (see Thas [137], Cameron, Hughes and Pasini [47], Cameron and Fisher [44], Hobart and Hughes [88], Del Fra, Ghinelli and Hughes [55], Blokhuis, Kloks and Wilbrink [23], and Del Fra, Ghinelli, Meixner and Pasini [56]) and Cameron [43] has constructed some new antipodal distance-regular graphs of diameter three. Therefore there is a hope that this connection will provide some interesting ideas for new constructions of antipodal distance-regular graphs. At the end of this section we mention one such result, again due to Terwilliger. This section, with exception of Corollary 4.5.4, is joint work with J. Koolen [94].

Here is the status of local graphs for all strongly regular graphs for which an example of a distance-regular antipodal cover is known:

#	G	n	k	λ	μ	locally
1	! Petersen graph	10	3	0	1	$3 \cdot K_1$
2	3-Golay code	243	22	1	2	$11 \cdot K_2$
3	folded Johnson graph $J(10,5)$	126	25	8	8	$GQ(4,1)$
4	folded halved 10-cube	256	45	16	6	$T(10)$

Table 4.3: Local graphs of strongly regular graphs with covers of diameter five.

#	G	n	k	λ	μ	locally
1	! Folded 5-cube	16	5	0	2	$5{\cdot}K_1$
2	! $\overline{T(6)}$,	15	6	1	3	$3{\cdot}K_2$
3	! $\overline{T(7)}$	21	10	3	6	Petersen graph [75]
4	folded J(4,8)	35	16	6	8	GQ(3,1) [27, p. 256]
5	! truncated 3-Golay code	81	20	1	6	$10{\cdot}K_2$
6	folded halved 8-cube	64	28	12	12	T(8) [27, p. 267]
7	$[S_2([S_2(McL.)])]$	105	32	4	12	AG(2,4)$\backslash$parallel class [125]
8	Zara graph (126,6,2)	126	45	12	18	GQ(4,2) [27, p. 399]
9	! $[S_2$(McLaughlin graph)]	162	56	10	24	Gewirtz graph [26] [125]
10	hyperbolic points of PG(6,3)	378	117	36	36	SRG(117,36,15,9)
11	Suzuki graph	1781	416	100	96	SRG(416,100,36,20) [125]

Table 4.4: Local graphs of strongly regular graphs with covers of diameter four.

The seventh graph of Table 4.4 is locally the point graph of $AG(2,4)$ minus a parallel class deleted, which is further a distance-regular antipodal cover of $K_{4,4}$. All the other graphs are locally strongly regular. If one of the Krein parameters q^i_{ii}, $i \in \{2,4\}$, in the antipodal quotient equals zero, then for each vertex the first and the second subconstituent graphs are strongly regular as well, see Cameron, Goethals and Seidel [46]. The results of Terwilliger [133] imply a similar conclusion for Q-polynomial antipodal distance-regular graphs.

4.5.1 THEOREM (Terwilliger). *If an antipodal distance-regular graph H of diameter d and with eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$, is Q-polynomial, then each first subconstituent graph is strongly regular with eigenvalues λ, $b^+ = -1 - b_1/(\theta_d + 1)$ and $b^- = -1 - b_1/(\theta_1 + 1)$.*

Both double-covers from Table 4.1 satisfy the conditions of this theorem. Terwilliger used the theory of Krein modules, see Brouwer et al. [27, Section 2.11] and Terwilliger [134], to prove this result (private communication, March 1992). We will rather use his result from graph representations, see Terwilliger [130] or [27, Thm. 4.4.3 and Thm. 4.4.4], to prove it in the case of diameter four. This simplifies the proof and motivates our generalization.

4.5.2 THEOREM (Terwilliger). *Let G be a distance-regular graph of diameter $d \geq 3$, with eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$ and corresponding multiplicities $1 = m_0, m_1, \ldots, m_d$. Then for each first subconstituent graph the smallest eigenvalue is at least $b^- = -1 - b_1/(\theta_1 + 1)$ and the second largest eigenvalue is $b^+ = -1 - b_1/(\theta_d + 1)$ at most (here the second eigenvalue is taken to be the valency λ in case the local graph is disconnected). If $m_i < k$ for some $i \neq 0$ then $i \in \{1, d\}$ and each first subconstituent graph has eigenvalue $-1 - b_1/(\theta_i + 1)$ with multiplicity at least $k - m_i$.* □

Note that for an antipodal distance-regular graph of diameter four $b^+ = \theta_2$ and $b^- = \theta_3$.

Proof of Theorem 4.5.1 in the case of diameter four: We already know that every subconstituent graph has $n' = k$ vertices and valency $k' = \lambda$. In order to prove that a subconstituent graph G' is strongly regular we basically need to determine one more parameter.

For $\theta_2 = p$ and $\theta_3 = -q$ Proposition 4.2.2 implies $\theta_4 = -q^2$. By Proposition 4.2.1, we have $-k = \theta_1\theta_3$ and $\theta_1 + \theta_3 = \lambda$, and thus $\theta_1 = pq + p + q$, $n' = k = \theta_0 = q(pq + p + q)$, $k' = \lambda = p(q + 1)$, $\mu = q(p + q)$ and $b_1 = (q^2 - 1)(p + 1)$. Straightforward calculations yield $k - m_1 = qp(p + 1)(q + 1)/(p + q) > 0$ (where we recognize the multiplicity of the second largest eigenvalue of the point graph of a generalized quadrangle $GQ(p, q)$). By Theorem 4.5.2, the smallest eigenvalue of the graph G' is $-1 - b_1/(\theta_1 + 1) = -q$ with multiplicity at least $k - m_1$ and the second largest eigenvalue is $-1 - b_1/(\theta_4 + 1) = p$ at most. The trace of the adjacency matrix equals zero, hence:

$$\begin{aligned} 0 &= \sum_{i=0}^{e} \theta'_i m'_i \quad \leq \quad k' + m'_1 p + \cdots - m'_e q \\ &\leq k' + (n' - 1 - (k - m_1))p - (k - m_1)q \end{aligned}$$

Using the above expression for $k - m_1$ we obtain that the last expression in the above inequality equals zero. Therefore the trace of the adjacency matrix of G' equals zero only if p is the second largest eigenvalue of G' with multiplicity $n' - 1 - (k - m_1)$. This means that the graph G' is strongly regular and with eigenvalues λ, p and $-q$, and so the antipodal quotient is locally strongly regular.
□

The parameters (k, λ, μ) of strongly regular graphs whose distance-regular antipodal covers are Q-polynomial and of their local graphs are:

$$\Big(q(pq + p + q), p(q + 1), q(p + q)\big) \quad \text{and} \quad \Big(p(q + 1), 2p - q, p\big).$$

Although we were not able to make use of the fact that $k - m_1$ is an eigenvalue multiplicity of a generalized quadrangle $GQ(p, q)$, there are still situations when the local graph can be the point graph of a generalized quadrangle. Since the eigenvalues of $GQ(s, t)$ are $s(t + 1)$, $s - 1$ and $-t - 1$, we must have $k' = (\theta_1' + 1)(-\theta_2') = (p + 1)q$, and thus $p = q$, in which case the local graph is a generalized quadrangle $GQ(q + 1, q - 1)$.

Studying the literature on extended generalized quadrangles we learn that in many cases local restrictions are still not enough to uniquely determine a graph (or even just to construct some). Sometimes it suffices to forecast the right μ-graphs, i.e., the graphs induced by common neighbours of two vertices at distance two. Terwilliger observes the following.

4.5.3 LEMMA (Terwilliger). *In a distance-regular graph, which is locally strongly regular with parameters (n', k', μ'), the μ-graphs are regular (with the parameter μ' as its valency).* □

We will need one more definition in the proof of the following result. A graph on v vertices is called a *Zara graph* with parameters (v, K, e) when every maximal clique has size K, and for every maximal clique C and every vertex u not in C, there are exactly e vertices in C adjacent to u.

4.5.4 COROLLARY. *A strongly regular graph which is locally $GQ(4, 2)$ has a unique distance-regular antipodal cover (the distance-regular graph with parameters*
$\{45, 32, 12, 1; 1, 6, 32, 45\}$ *constructed in Brouwer et al.* [27, p. 399]*).*

Proof. Let G be a strongly regular graph which is locally $GQ(4, 2)$ and let H be its distance-regular antipodal cover. As H is locally the same as G, the maximal cliques in both graphs have size six, and the Delsarte's clique bound (often called Hoffman's clique bound) is met in it, see Godsil [64, p. 276], [66] or Brouwer et al. [27, Prop. 4.4.6], so each point not in some maximal clique has exactly zero or two neighbours in that maximal clique. We will prove that in the antipodal quotient G the zero case cannot occur and that the covering index r equals three.

Let u be a vertex of the quotient graph, let v be one of its neighbours, and let C be some maximal clique containing u but not v. Then, by the property of generalized quadrangles, v has exactly one neighbour in $C \cap S_1(u)$, hence precisely two neighbours in C. Now let w be a vertex in $S_2(u)$. We want to show that w has two neighbours in $C \cap S_1(u)$. Since this is supposed to be true, for any maximal clique containing u, it suffices to prove that the common neighbours of u and w consist of two disjoint ovoids of the generalized quadrangle $GQ(q+1, q-1)$. Remember that, by Lemma 4.5.3, any μ-graph has valency q, which is three in this case. Thus μ has to be even, and by the tables from Brouwer et al. [27, pp. 421-425], it can only be six, i.e., $r = 3$. Therefore, by Van Bon and Brouwer [17], any μ-graph of the antipodal quotient has to have three components of equal size, which is six in our case. Now, there is only one graph on six vertices, which has no triangles and the valency three, the complete bipartite graph $K_{3,3}$. Therefore the μ-graph must consist of three copies of $K_{3,3}$. But then, by taking three independent vertices from each copy, we get the set of nine independent vertices, which means that we have got an ovoid of $GQ(4, 2)$. The other nine vertices also correspond to an ovoid. Therefore the vertices of μ-graph of u and w correspond to two disjoint ovoids, and thus w has precisely two neighbours in each maximal clique containing the vertex u.

Finally, the antipodal quotient graph G must be the Zara graph with parameters $(v, K, e) = (126, 6, 2)$. Uniqueness of the Zara graph with these parameters has been proved by Blokhuis and Brouwer [20], and in the reference [89] of Brouwer et al. [27] (A. Blokhuis and A. E. Brouwer, Graphs that are locally a generalized quadrangle, with complete bipartite μ-graphs, unpublished manuscript (1983), which unfortunately I was not able to obtain). $\square$

Remarks: (i) As soon as we have realized that $\mu(H) = 6$ and H is locally $GQ(4, 2)$ we could have referred to Brouwer et al. [27, p. 399].
(ii) There is perhaps one more important aspect of this proof, namely, the part where we have ruled out $r = 2$ and $r = 6$ cases. For that we did not use the fact that the local graph is the point graph of some particular incidence structure. The valency of a μ-graph is determined solely by the fact that the local graph is strongly regular. We will use this in the Corollary 4.5.8.

Let us look for distance-regular antipodal q-covers of diameter four which are locally $GQ(q+1, q-1)$. Then the μ-graphs have valency q. For $q = 2$ we get the fourth graph of Table 4.1. The Johnson graph $J(8, 4)$ is a unique example. For $q = 3$ the cover from the above corollary is a unique example again. Since in both cases the μ-graphs are isomorphic to q copies of $K_{q,q}$, we try our luck with the same μ-graphs also for $q = 4$. Unfortunately there are no such examples. We can see this by going in the dual of $GQ(5, 3)$. Note that in the dual the μ-graph corresponds to q copies of the grid graphs on q^2 points. Remember that $GQ(3, 5)$ has been proved to be unique by Dixmier and Zara [113, p. 125]. Now take $T_2^*(\mathcal{O})$ model of a generalized quadrangle, cf. the last section of Chapter 2 or Payne and Thas [113, p. 38]. The ovoid $\mathcal{O}$ (in $PG(2, 4)$)

has six points, the grid graphs have to be planar, and four of them have to lie on planes intersecting in a line through two points of the ovoid, so there are only 15 different $4{\cdot}K_{4,4}$ subgraphs in the first subconstituent graph and we needed 15^2 of them.

Note that our proof of the special case of Theorem 4.5.1 worked mainly because $k - m_1 = m'_2$, which is, of course, equivalent to $q_{11}(4) = 0$ and $r = 2$. Therefore it is not easy to extend Theorem 4.5.1 in order to cover some cases which are not Q-polynomial. However, it seems quite reasonable that antipodal distance-regular graphs with antipodal quotients which have the same intersection array are locally the same. We will show two result in this direction.

4.5.5 THEOREM. *If we have for a distance-regular graph of diameter at least three $b^+ = -b^-$ and $k\lambda = \lambda^2 + (k-1)(b^-)^2$, then the first subconstituent graph is strongly regular with eigenvalues λ, b^+ and b^-.*

Proof. The first condition means that that the lower and the upper bounds for the nontrivial eigenvalues from Theorem 4.5.2 have equal magnitude. Note that the trace of the square of the adjacency matrix of the local graph equals $n'k' = k\lambda$, and it also equals the sum of squares of all eigenvalues (where each eigenvalue appears as many times as it is its multiplicity). But then, by the second condition, all nontrivial eigenvalues have to have the same magnitude, which implies that the local graph has only three distinct eigenvalues, as it cannot be the complete graph, cf. Brouwer et al. [27, Lemma 1.1.7]. $\square$

For the above two-parameter family these conditions are both equivalent to $p = q$.

Problem: Can we use the above theorem to prove that a unique antipodal distance-regular graph with intersection array $\{45, 32, 12, 1; 1, 6, 32, 45\}$ exists, i.e., a distance-regular antipodal cover of a strongly regular graph with intersection array $\{45, 32; 1, 18\}$?

4.5.6 LEMMA. *Let a, b and c be real numbers and $b < c$. If real numbers $x_1, \cdots, x_n$ from the interval $[b, c]$ sum to a and we have*

$$\sum_{i=1}^{n} x_i^2 = \max\Big\{\sum_{i=1}^{n} y_i^2 \mid b \le y_i \le c,\ \sum_{i=1}^{n} y_i = a\Big\},$$

then $(x_j - b)(x_j - c) \neq 0$ for at most one $j \in \{1, \dots, n\}$.

Proof. Suppose that $(x_i-b)(x_i-c)(x_j-b)(x_j-c)\neq 0$ for $i\neq j$ and $x_i\leq x_j$. Then for $\epsilon\in\big(x_j-x_i,\min(x_j-b,x_i-c)\big]$, $x=x_i+\epsilon$ and $x'=x_j-\epsilon$ we have $x_i^2+x_j^2<x^2+x'^2$, and $b\leq x,x'\leq c$, which is impossible. □

4.5.7 THEOREM (Jurišić and Koolen). *Let G be a distance-regular graph of diameter at least three, valency k, n vertices, and let $\xi_1\leq\xi_2\leq\cdots\leq\xi_n$ be its eigenvalues. Then for $\lambda=a_1$, $b^-=-1-b_1/(\xi_2+1)$ and $b^+=-1-b_1/(\xi_n+1)$ the graph G is locally strongly regular with λ, b^+ and b^- as its eigenvalues if and only if there exist non-negative integers s and t such that*

$$s+t=k-1,\quad sb^-+tb^+=-\lambda\quad \textit{and}\quad s(b^-)^2+t(b^+)^2=\lambda(k-\lambda).$$

Proof. Suppose that the above system has a non-negative integral solution $(s,t)=(s',t')$. Let us first prove that in this case

$$\lambda(k-\lambda)=\max\Big\{\sum_{i=1}^{k-1}y_i^2 \mid b^-\leq y_i\leq b^+,\ \sum_{i=1}^{k-1}y_i=-\lambda\Big\}=:M.$$

Suppose that real numbers $x_1,\ldots,x_n$ from the interval $[b^-,b^+]$ sum to $-\lambda$ and that we have $\sum_{i=1}^n x_i^2=M$ and suppose that $M\geq\lambda(k-\lambda)$. Then, by Lemma 4.5.6, we have $x_i=b^-$ for s'' indices i, and $x_i=b^+$ for t'' indices i, where $s''+t''=k-1-\epsilon$ for $\epsilon\in\{0,1\}$. Suppose that $\epsilon=1$. Then we have for $\xi\in(b^-,b^+)$

$$s''+t''=k-2,\qquad s''b^-+t''b^++\xi=-\lambda$$

and

$$s''(b^-)^2+t''(b^+)^2+\xi^2\geq\lambda(k-\lambda),$$

and thus $(s'-s'')+(t'-t'')=1$, and $(s'-s'')b^-+(t'-t'')b^+=\xi$. This implies $\xi-b^-=(t'-t'')(b^+-b^-)$, which is not possible, since $0<\xi-b^-<b^+-b^-$ and $t'-t''$ is an integer. It follows that $\epsilon=0$, and as the first two equations of the above system have a unique solution also $(s',t')=(s'',t'')$.

By Theorem 4.5.2, all the eigenvalues of a local graph, except the eigenvalue λ, lie in the interval $[b^-,b^+]$ and their sum (where each eigenvalue appears as many times as it is its multiplicity) is $-\lambda$. Note that the trace of the square of the adjacency matrix of the local graph equals $k\lambda$, and also equals the sum of squares of all the eigenvalues (where each eigenvalue appears as many times as it is its multiplicity). Therefore all the eigenvalues of the local graph except λ lie in the two-element set $\{b^-,b^+\}$, and the local graph is strongly regular.

The converse follows from the fact that s and t are the multiplicities of the nontrivial eigenvalues of the local graph. □

Remarks: (i) Let a strongly regular graph G with parameters

$$(k, \lambda, \mu) = \Big(q(pq + p + q), p(q + 1), q(p + q)\Big)$$

has a distance-regular antipodal cover H, and let $m = m_1(H)/(r - 1)$ be an integer, i.e., $(p + q)|q^2(q^2 - 1)$. (Note that $(p + q)|q^2(q^2 - 1)(q - 1)$ follows from k_2 being an integer.) Then $(s, t) = (m - 1, k - m)$ solves the above system of equations, and therefore G is locally strongly regular. Since $b^- = -q$ and $b^+ = p$, its parameters are

$$(k', \lambda', \mu') = \Big(p(q + 1), 2p - q, p\Big).$$

For $r = 2$ this shows that an antipodal distance-regular graph, which is Q-polynomial is locally strongly regular.

(ii) By Theorem 4.5.1, for an antipodal distance-regular graph, which is Q-polynomial (i.e., P- and Q-polynomial antipodal cover), the above system of equations has a non-negative solution. By relatively straightforward calculations we can show the same in the case of diameter three (cf. [27, Prop. 8.3.2]) and four (cf. Remark (i)) without using Theorem 4.5.1.

(iii) Another two examples, for which we do not need to use Theorem 4.5.1, are the Johnson graph $J(2q, q)$ and the halved n-cube, n even. The first graph is locally the lattice graph $K_v \times K_v$ (i.e., the Hamming graph $H(2, q)$), cf. [27, p. 256], whose nontrivial eigenvalues are $b^+ = q - 2$ and $b^- = -2$. The second graph is locally the triangular graph $T(n)$ (i.e., the Johnson graph $J(n, 2)$), cf. [27, p. 267], whose nontrivial eigenvalues are $b^+ = n - 4$ and $b^- = -2$.

(iv) Suppose that the assumptions of Theorem 4.5.5 holds. Then the second condition in Theorem 4.5.5 is equivalent to the third equation in Theorem 4.5.7. However, it is not clear how to derive the second equation in Theorem 4.5.7 from the assumptions of Theorem 4.5.5 without using Theorem 4.5.7. So although Theorem 4.5.5 does not give us any examples, which are not also the examples of Theorem 4.5.7, it is sometimes still more convenient to test the assumptions of Theorem 4.5.5, than to look for a non-negative integral solution of Theorem 4.5.7.

Now we derive from Theorem 4.5.7 a nonexistence result for antipodal distance-regular graphs of diameter four.

4.5.8 COROLLARY (Jurišić and Koolen). *A strongly regular graph with parameters*

$$(k, \lambda, \mu) = \Big(q(pq + p + q), p(q + 1), q(p + q)\Big),$$

where $(p+q)|q^2(q^2-1)$, *has no distance-regular antipodal* r*-covers of diameter four when either the numbers* p *and* $q(p+q)/r$ *are odd or* $rp \geq q(p+q)$.

Proof. Suppose that a strongly regular graph with the above parameters has a distance-regular r-cover. Then, by Theorem 4.5.7 and Remark (i), we know that its local graph is strongly regular, with parameters $(k, \lambda, \mu) = (p(q+1), 2p - q, p)$. Since the parameter μ of an r-cover equals $q(p+q)/r$, and its μ-graphs have valency p, at least one of these two integers must be even (as their product is twice the number of edges in a μ-graph), and $p < q(p+q)/r$ (as the valency must be smaller than the number of vertices in a graph). □

The above corollary implies that the ten-th graph of Table 4.4 is locally strongly regular. This was not known before and could serve as a hint to prove the uniqueness of the cover. Many infinite families of feasible intersection arrays and in particular the following parameters from the tables in Brouwer et al. [27, pp. 421-425] are ruled out.

#	intersection array	p	q	r	ruled out by
1)	$\{45, 32, 9, 1; 1, 9, 32, 45\}$	3	3	2	parity
2)	$\{45, 32, 15, 1; 1, 3, 32, 45\}$	3	3	6	parity or bound
3)	$\{81, 56, 24, 1; 1, 3, 56, 81\}$	6	3	9	bound
4)	$\{96, 75, 28, 1; 1, 4, 75, 91\}$	4	4	8	bound
5)	$\{115, 96, 35, 1; 1, 5, 96, 115\}$	3	5	8	parity
6)	$\{117, 80, 27, 1; 1, 9, 80, 117\}$	9	3	4	parity or bound
7)	$\{117, 80, 30, 1; 1, 6, 80, 117\}$	9	3	6	bound
8)	$\{117, 80, 32, 1; 1, 4, 80, 117\}$	9	3	9	bound
9)	$\{175, 144, 25, 1; 1, 25, 144, 175\}$	5	5	2	parity
10)	$\{176, 135, 40, 1; 1, 8, 135, 176\}$	8	4	6	bound
11)	$\{189, 128, 45, 1; 1, 9, 128, 189\}$	15	3	6	parity or bound
12)	$\{189, 128, 27, 1; 1, 27, 128, 189\}$	15	3	2	parity
13)	$\{261, 176, 54, 1; 1, 18, 176, 261\}$	21	3	4	bound
14)	$\{414, 350, 45, 1; 1, 45, 350, 414\}$	9	6	2	parity

Table 4.5: Intersection arrays of antipodal distance-regular graphs of diameter four which are ruled out.

Koolen observes further that for the following intersection array

$$15) \quad \{115, 96, 35, 1; 1, 4, 96, 115\},$$

$\mu = \mu' + 1$, i.e., each μ-graph is complete, which means that this graph is a *Terwilliger graph*, cf. [27, p. 34], and therefore, by [27, Theorem 1.16.3], it cannot exist.

We finish this section with one more result of Terwilliger (private communication, March 1992):

4.5.9 THEOREM (Terwilliger). *If an antipodal distance-regular graph H of diameter four is Q-polynomial, then each second subconstituent graph is either disconnected or of diameter four and it has six eigenvalues at most. In the second case the second subconstituent graphs are antipodal covers (not distance-regular unless c_2 is defined).* □

There is hope that this result can also be extended, since, for example, it does not include the Wells graph whose second subconstituent is the dodecahedron (i.e., the non-bipartite distance-regular antipodal double-cover of the Petersen graph), see Brouwer et al. [27, Thm. 9.2.9].

The above result is one reason to study antipodal covers which are not necessarily distance regular. There are many cases when the second subconstituent graph of a strongly regular graph is strongly regular again, in which case we have an antipodal cover of a strongly regular graph. We will study such covers in Chapter 6.

6. Conclusion

Little is known about antipodal distance-regular graphs of diameter four and five in general. A milestone would be attained if an infinite family of such graphs was constructed. So let us discuss some known infinite families of feasible parameters of such graphs, which are known only in diameter four:

The first family comes from Brouwer et al. [27, p. 421], cf. Cameron and Van Lint [48, pp. 29]. The parameters of their quotient graphs are, for $t \geq 1$,

$$(k, \lambda, \mu) = \Big(t(t^2 + 3t + 1), 0, t(t + 1)\Big),$$

and r is any divisor of μ. For $t = 1$, we have the first graph from Table 4.1, for $t = 2$, the antipodal quotient must be the Higman-Sims graph.

The next family is one of the remaining two open questions stated in Van Bon and Brouwer [17, pp. 155, 156, 164]. These are distance-regular antipodal covers of diameter four of the Hermitean forms graphs of diameter two. Let H be a set of 2×2 Hermitean matrices over $GF(q^2)$ with q a prime power. Then the *Hermitean forms graph* is the graph with elements of H as vertices and two of them being adjacent if the rank of their difference is one. It turns out that this graph is isomorphic to the second subconstituent graph of the point graph of a $GQ(q, q^2)$, see Van Bon and Brouwer [17, p. 156]. Then

$$(n, k, \lambda, \mu) = \Big(q^4, (q^2 + 1)(q - 1), q - 2, q(q - 1)\Big).$$

The feasibility conditions imply for $q \neq 2$ that $4q-3$ is a perfect square, i.e., $q = s^2+s+1$ for some positive integer s. For $q=2$ and $q=3$ we get the first graph and the fifth graph of Table 4.1 respectively. For larger q (e.g., 7, 13, 31) nothing is known.

We derive the next family from the previously mentioned two parameter family of P- and Q-polynomial antipodal covers. The second subconstituent of antipodal quotient has kb_1/μ vertices, therefore $(p+q)|(q-1)^2q^2(q+1)$. The most interesting case is an infinite family of feasible parameters of antipodal distance-regular graphs of diameter four whose local graphs have the same parameters as the point graphs of generalized quadrangles $GQ(q+1, q-1)$. Their antipodal quotients and their local graphs (which must be strongly regular) have the following parameters:

$$(k, \lambda, \mu) = \Big(q^2(q+2), q(q+1), 2q^2\Big),$$

and

$$(n', k', \lambda', \mu') = \Big(q^2(q+2), q(q+1), q, q\Big).$$

We have already mentioned that examples are known only for $q = 2, 3$.

The last family, which we expose here, is again from Brouwer et al. [27, p. 417] and corresponds to double-covers of diameter five and $t = \mu^2$. Parameters of their quotient graphs are,

$$(n, k, \lambda, \mu) = \Big(2\mu^2(2\mu+3), 2\mu^2+\mu, 0, \mu\Big).$$

For $\mu = 1$ we obtain the Petersen graph with the dodecahedron as its cover, and for $\mu = 2, 4$ Brouwer et al. [27, p. 372] and Koolen (private communication, September 1993) respectively have shown that there are no examples.

In the following chapters we will find new, simpler, families of feasible parameters.

5

MERGING

One of the more important results in Chapter 2 was Brouwer's theorem, where the idea of merging was used for the characterization of certain distance-regular covers of complete graphs. In this chapter we study merging in more general context and obtain the main result of this thesis, Theorem 5.3.3, which generalizes Brouwer's theorem. First, let us explain more precisely what we mean by merging. Let $(\mathcal{A}, X)$ be an association scheme formed by the matrices $A_0, \dots, A_d$ and let π be a partition of the set $\{1, \dots, d\}$ with m cells. Let $A'_1, \dots, A'_m$ be the 01-matrices

$$\sum_{i \in C} A_i$$

where C ranges over the cells of π and let $A'_0 = I$. Since these matrices lie in the Bose-Mesner algebra $(\mathcal{A}, X)$, they commute and sum to J. In many cases the matrices $A'_0, \dots, A'_m$ form an association scheme. A scheme constructed this way is said to be obtained from $\mathcal{A}$ by *merging* classes (or by *fusion*). Merging was, for example, used to construct some new strongly regular graphs (see the survey paper about strongly regular graphs by Brouwer and Van Lint [31]). For example, take all three element sets of a seven element set as vertices and define two to be adjacent if they intersect in two elements. We obtain the Johnson graph $J(7, 3)$. Connecting also all the vertices which are at distance three, i.e., merging the first and the third class, results in a strongly regular graph, which is the line graph of $PG(3, 2)$. For general references on merging, subalgebras of Bose-Mesner algebra and subschemes of an association scheme see Bannai [7], Bridges and Mena [36], Faradžev, Ivanov and Klin [58] and Muzychuk [104].

In the first section it is shown that a merging which results in an association scheme always exists in an imprimitive association scheme. In Section 5.2 we study merging in distance-regular graphs of diameter four. This results in the characterization of some diameter four antipodal distance-regular graphs with certain triangle free strongly regular graphs, and we obtain new infinite families of feasible parameters of antipodal distance-regular graphs of diameter four. In Section 5.3 we determine, for any diameter, when merging the first and the last class in an antipodal distance-regular graph produces a distance-regular graph. In the merged graph $b_i = k-(\lambda+1)c_i$ for all i less than diameter d (a property of regular near polygons, cf. Brouwer et al. [27, Thm. 6.4.1]) and $c_i = i$, i less than diameter d, c_d is equal to d or $k/(\lambda+1)$. Conversely, given a distance-regular graph with the same intersection array as the merged graph and a certain clique partition, we construct an antipodal distance-regular graph. In Section 5.4 we investigate merging in antipodal distance-regular graphs of diameter five. It enables us to construct two distance-regular covers of complete graphs. In the last section Brouwer's generalization of Proposition 5.3.2 is mentioned.

1. Imprimitive association schemes

In this short section we explain how to obtain from an imprimitive association scheme two smaller schemes. All the results in this section and their proofs can be found in both Bannai and Ito [8, Propositions 4.5 and 9.4] and Godsil [64, pp. 232-234], however we present them, since we feel that they are important for the motivation and understanding of the whole chapter.

5.1.1 PROPOSITION. *Let $(X,\mathcal{A})$ be a d-class association scheme and let us for a fixed $i \in \{1,\dots,d\}$ define a relation $\mathcal{R}$ on X by $x\mathcal{R}y$ if and only if they are connected in G_i. Then $\mathcal{R}$ is an equivalence relation which is a union of some association relations R_j.*

Proof. (Godsil and Martin [70]) Suppose that G_1 is disconnected. Let x and y be i-related vertices in a component of G_1 and let u be i-related to some v. Since x and y are joined by a path with all its edges in G_1, it follows that u and v are also joined by a path with edges in G_1, i.e., they lie in the same component of G_1. This implies that the graph on X with two vertices adjacent if and only if they are in the same component of G_1 is the union of graphs from $\mathcal{A}$. □

The following argument is taken from Godsil [64, pp. 232-234]. Let $(X,\mathcal{A})$ be a d class imprimitive association scheme, where G_d is not connected. If C is a vertex set of a component of G_d, then the above result implies that the non-empty restrictions of the graphs G_i to the component C form an association scheme. If P is a partition of X corresponding to the connected components of G_d, then the above result means that PP^T is a 01-matrix contained in the

Bose-Mesner algebra of $\mathcal{A}$ and without loss of generality we can also assume that $PP^T = I_q \otimes J_r$. Let [0] denote a subset of $\{0, \ldots, d\}$ such that

$$PP^T = \sum_{i \in [0]} A_i.$$

Let $\approx$ be the relation on $\{0, \ldots, d\}$ defined by $i \approx j$ if and only if $p_{ia}(j) \neq 0$ for some $a \in [0]$. Then this is an equivalence relation and $0 \in [0]$. If $[i]$ is the equivalence class of i, then there exists a matrix B_i such that

$$\sum_{j \in [i]} A_j = B_i \otimes J_r.$$

The matrices B_i define an association scheme which is called the *quotient scheme* of $\mathcal{A}$. Suppose that $B_0 = I_q$. Then the matrices I_{qr}, $(B_0 \otimes J_r) - I_{qr}$ and $B_i \otimes J_r$ for all $i \neq 0$ define an association scheme as well. Since its span is a subspace of the Bose-Mesner algebra of $\mathcal{A}$ which contains I and J, and it is closed with respect to both matrix and Schur multiplication, we call this association scheme a *subalgebra* of $\mathcal{A}$ (or also a *fusion scheme*).

Suppose that $(X, \mathcal{A})$ defines an antipodal distance-regular graph, and that the graph G_d is a union of cliques on the antipodal classes. Then $PP^T = I + A_d$, which means that A_d does not merge in the above merging (i.e., A_d remains a Schur idempotent in the above fusion scheme). Despite this fact there are still some examples when A_d can merge. This is the topic of the next sections.

2. Antipodal distance-regular graphs of diameter four

Since merging A_1 and A_3 in antipodal distance-regular graphs of diameter three has been already studied by Brouwer (see Theorem 3.3.1), and since this is the only nontrivial merging in such graphs, we start by investigating the diameter four case. Merging A_1 with one of the remaining three classes in a distance-regular graph with diameter four forces the other two classes to merge too.

5.2.1 PROPOSITION. *Let H be a distance-regular graph of diameter four. If we merge A_1 and A_x for $x \in \{2, 3, 4\}$ then A_y and A_z, where $\{x, y, z\} = \{2, 3, 4\}$, merge as well. This results in a strongly regular graph if and only if*

(i) $k = \mu$ for $x = 3$,

(ii) $\mu/2 = \lambda + 2$ *and* r *is* 2 *or* $\lambda + 2$ *for* $x = 4$ *or* $x = 2$ *respectively.*

Proof. The above merging results in an association scheme if and only if for $f(i) := p_{11}(i) + 2p_{1x}(i) + p_{xx}(i)$ we have $f(1) = f(x)$ and $f(y) = f(z)$. This can be obtained by a direct verification of the condition (b) in the definition of an association scheme and by the use of Lemma 2.3.1(c) in order to get down to only two conditions. The translation of these conditions to the above is now straightforward. Since the new association scheme has two classes, it determines a strongly regular graph. □

Suppose that A_1 is the adjacency matrix of an antipodal distance-regular graph H of diameter four, and that G is its antipodal quotient. Suppose that M is the merged graph $H_1 \cup H_x$. Let us use the same notation for the parameters of H and G as in the previous chapter. In the case when $x = 3$ the merged graph M is strongly regular if and only if $k = \mu$, i.e., G is a complete multipartite graph $K_{t(m)}$. Since among complete multipartite graphs only the complete bipartite graphs allow distance-regular antipodal covers, see Proposition 6.1.7, we have $t = 2$. In this case the merged graph is the complete bipartite graph with valency rk. Here "unmerging" is deleting the edges of the distance three graph H_3 in M. However the identity $A_3(H) = A_1(H)A_4(H)$, where H_4 is a union of cliques (i.e., $A_4(H) = (J_r - I_r) \otimes I$), suggests that unmerging most probably does not help to find $A_1(H)$. These covers have already been characterized by resolvable transversal designs, see Drake [57] or Theorem 6.3.1, and many constructions are known.

When x is equal to 2 or 4 the merging produces a strongly regular graph for many feasible parameters of antipodal distance-regular graphs of diameter four from tables in Brouwer et al. [27, p. 421] or Godsil, Jurišić and Schade [69]. We list those with valency of G at most 100 in Tables 5.1 and 5.2.

G	n	k	λ	μ	H	$\overline{M}_{x=2}$, $M_{x=4}$	n	k	λ	μ
$!K_{4,4}$	8	4	0	4	!4-cube	!halved and folded 5-cube	16	5	0	2
// Krein	28	9	0	4	//	!Gewirtz graph	56	10	0	2
?	176	25	0	4	?	?	352	26	0	2
?	352	36	0	4	?	?	704	37	0	2
?	638	49	0	4	?	?	1276	50	0	2
?	1702	81	0	4	?	?	3404	82	0	2
?	2576	100	0	4	?	?	5152	101	0	2

Table 5.1: Merging in distance-regular graphs with $\lambda = 0$.

G	n	k	λ	μ	H $x{=}2$	$\overline{M}$ $x{=}2$	n	k	λ	μ	H $x{=}4$	M $x{=}4$	n	k	λ	μ
!Golay	81	20	1	6	?	?	162	21	0	3	Golay	BLS	243	22	1	2
$S_2(S_2(McL.))$	105	32	4	12	?	?	210	33	0	6	?	?	630	37	4	2
?	196	45	4	12	?	?	392	46	0	6	?	?	1176	50	4	2
?	325	54	3	10	?	?	650	55	0	5	?	?	1625	58	3	2
$!S_2(McL.)$	162	56	10	24	//	?	324	57	0	12	//	?	1944	67	10	2
?	276	75	10	24	?	?	552	76	0	12	?	?	3312	86	10	2
?	540	77	4	12	?	?	1080	78	0	6	?	?	3240	82	4	2
?	400	84	8	20	?	?	800	85	0	10	?	?	4000	93	8	2
?	825	96	4	12	?	?	1650	97	0	6	?	?	4950	101	4	2

Table 5.2: Merging in distance-regular graphs with $\lambda \neq 0$.

In the case $x = 2$ the complement of M (denoted by $\overline{M}$) has intersection array $\{k+1, k; 1, \lambda+2\}$, so M is the complement of a triangle free strongly regular graph. In the case $x = 4$ the merged graph M has intersection array $\{k+\lambda+1, k; 1, 2\}$. The integrality of the multiplicities of a strongly regular graph G with $\mu/2 = \lambda+2$ implies that the discriminant of the quadratic equation which determines the nontrivial eigenvalues of G, i.e.,

$$(\lambda - \mu)^2 + 4(k - \mu) = \lambda^2 + 4k$$

is a perfect square, say $(\lambda + 2y)^2$ for some integer $y \geq 2$ (since $y = 1$ would imply $b_1 = 0$). Therefore $k = y(\lambda + y)$. The discriminants of the quadratic equations which determine the "new" eigenvalues of H and the nontrivial eigenvalues of M are also equal to $\lambda^2 + 4k$. No wonder we get so many feasible parameter sets when $\mu/2 = \lambda + 2$.

Now, we find some infinite families of feasible parameters of strongly regular graphs which allow feasible parameters of covers. (They seem to be "simpler" than the families from the conclusion of the previous chapter, but on the other hand it can be even more difficult to construct them.) For example, $\lambda = 0$ renders the family $\{y^2, y^2-1; 1, 4\}$, where y is an integer not equal to three and not divisible by four, cf. Cameron and Van Lint [48, p. 27], and the family of intersection arrays of their distance-regular antipodal double-covers. When x is 4 or 2, merging gives a graph M with intersection array $\{y^2+1, y^2; 1, 2\}$ or the graph with intersection array of $\overline{M}$ respectively (see Table 5.1). For $y = 2$, the four-cube is the double-cover of its antipodal quotient, the folded four-cube. The merged graph is the folded five-cube $\{5, 4; 1, 2\}$ (i.e., the complement of the halved 5-cube, also known as the Clebsch graph). Note that in this case

$A_1 \cong A_3$, see Figure 5.1 and Brouwer et al. [27, p. 265, Remark (i)]. Therefore $M_{(x=4)} \cong \overline{M}_{(x=2)}$. This is actually true for any triangle-free double-cover of diameter four, by Brouwer et al. [27, Prop. 4.2.13]. Unfortunately, apart from the case $y = 3$, when we get the Gewirtz graph $\{10, 9; 1, 2\}$, no other examples are known.

For $\lambda = 4$ we get another infinite family $\big\{(y+4)y, (y-1)(y+5); 1, 12\big\}$ where $y + 2$ is an integer greater or equal to six and not divisible by four, with distance-regular antipodal double- and six-covers. Merging yields a strongly regular graph with intersection array $\big\{y(y+4)+1, y(y+4); 1, 6\big\}$ in the case $x = 2$ and $\big\{(y+2)^2, y(y+4); 1, 2\big\}$ in the case $x = 4$. An example of G is known for $y = 4$, see Hubaut [90, S.4] and no examples of H or M are known. Note that if we know M does not exist, this implies that antipodal distance-regular graph H does not exist either.

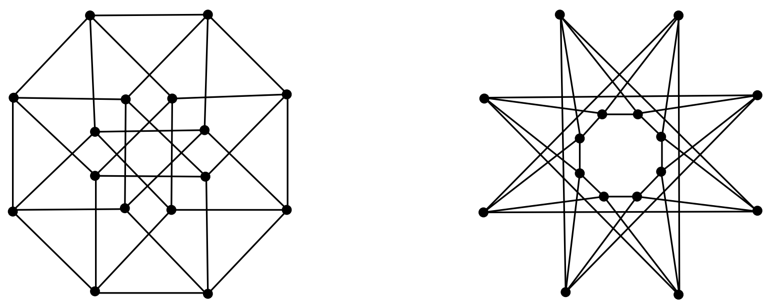

Figure 5.1: The four-cube and its distance three graph, which is again the four-cube.

Unmerging in the case $x = 2$ is equivalent to finding the distance two graph H_2 as a subgraph of M. At this moment this does not seem promising. And even if it did there are only six triangle free strongly regular graphs known which are not complete multipartite graphs: the 5-cycle, the Clebsch graph, the Petersen graph, the Gewirtz graph, the Higman-Sims graph $\{22, 21; 1, 6\}$ and the second subconstituent of the Higman-Sims graph $\{16, 15; 1, 4\}$. Intersection array of H is feasible only for the Clebsch graph and the 4-cube is the double-cover of $K_{4,4}$. Unmerging in the case $x = 4$ is described in the following theorem:

5.2.2 THEOREM. *Let M be a strongly regular graph with intersection array $\{k+\lambda+1, k; 1, 2\}$, $\lambda+1 < k > 2$, and a partition of its vertex set into $(\lambda+2)$-cliques such that there is a perfect matching or nothing between any pair of these cliques. Then the graph H obtained from M by deleting the edges of these cliques is a distance-regular antipodal $(\lambda+2)$-cover of a strongly regular graph G with intersection array $\{k, k-\lambda-1; 1, 2(\lambda+2)\}$. Conversely, any graph with the same parameters as H arises this way.*

The assumption that in a partition of $V(M)$ into $(\lambda+2)$-cliques there is either a perfect matching or nothing between any two of these cliques is necessary. For example the Gewirtz graph $\{10,9;1,2\}$ is ten-regular and ten-edge-connected by Brouwer and Mesner [32], therefore by Tutte's theorem it contains a perfect matching, i.e., a partition into two-cliques, but it cannot be unmerged since there is no strongly regular graph with intersection array $\{9,8;1,4\}$ (the absolute and the Krein bounds both fail). On the other hand, if a strongly regular graph with intersection array $\big\{(t+2)(\lambda+1),(t+1)(\lambda+1);1,t+2\big\}$ has a partition of $V(M)$ into $(\lambda+2)$-cliques, then by Brouwer's theorem the above condition always holds.

Proof. Let $C_1,\ldots,C_q$ be a clique partition of M as required and let G be the quotient graph of M corresponding to this partition. For the first part it is, by Theorem 2.4.2 enough to prove that the graph G has the required parameters, that H is an antipodal graph of diameter four with the vertex sets of the cliques as its antipodal classes, and that in H each geodesic of length two can be extended to a geodesic of length four. The required property for the clique partition implies $k(G)=k(M)-(\lambda+1)=k$. Let $u\in V(C_1)$ and relabel the cliques $C_2,\ldots,C_q$ so that $C_2,\ldots,C_{k+1}$ are the neighbours of C_1 in G). Then each of the cliques $C_2,\ldots,C_{k+1}$ contains exactly one vertex of $S_1(u)\backslash V(C_1)$ and this vertex is not adjacent to any vertex in $V(C_1)\cap S_1(u)$. Since $c_2(M)=2$ no vertex in $V(C_j)\cap S_2(u)$ for $i=2,\ldots,k+1$ has a neighbour in $S_1(u)$ in the graph H. Thus $\mathrm{dist}_H(u,v)>3$ and also $\lambda(G)=\lambda$. The set

$$S_2(u)\backslash\bigcup_{i=2}^{k+1}V(C_i)$$

is non-empty since $a_2(M)-2\lambda=k-\lambda-1>0$. Since a vertex in this set has no neighbours in $V(C_1)$ we have $c_2(G)\geq c_2(M)|V(C_i)|=2(\lambda+2)$ and G has diameter two. The number of edges between $S_1(C_1)$ and $S_2(C_1)$ is at least

$$|S_2(C_1)|(\lambda+2)2=k(k+\lambda+1)-2k(\lambda+1)=k(k-\lambda-1)=|S_1(C_1)|b_1(G)$$

But this is the number of all edges between these two sets, therefore $c_2(G)=2(\lambda+2)$ and G is strongly regular. The required properties of H are now easy to be seen. The converse follows from Theorem 2.4.2 and the fact that the parameters of H imply that $A_1(H)+A_4(H)$ is the adjacency matrix of a strongly regular graph. $\square$

There are only four strongly regular graphs with $\mu=2$ known: the four-cycle, the folded five-cube, the Gewirtz graph and the Berlekamp- Van Lint-Seidel graph with intersection array $\{22,20;1,2\}$ (the coset graph of the ternary

Golay code), see Berlekamp, Van Lint and Seidel [12], Brouwer et al. [27, pp. 360]. The last one has the above partition into cliques, because the coset graph of the shortened ternary Golay code is a distance-regular antipodal three-fold cover with diameter four of the point graph of the truncated ternary Golay code. This point graph is the only graph with intersection array $\{20, 18; 1, 6\}$, see Brouwer and Haemers [37].

5.2.3 COROLLARY. *Distance-regular antipodal double-covers of strongly regular graphs with intersection array* $\{y^2, y^2 - 1; 1, 4\}$, *where* y *is an integer not equal to three and not divisible by four, are equivalent to strongly regular graphs with intersection array* $\{y^2 + 1, y^2; 1, 2\}$ *and a perfect matching such that any two edges of it induce a four-cycle or* $2K_2$. *All these intersection arrays with* $y \neq 3$ *are feasible.* □

5.2.4 COROLLARY. *Distance-regular antipodal six-covers of strongly regular graphs with intersection array* $\{(y+4)y, (y-1)(y+5); 1, 12\}$ *where* $y+2$ *is an integer greater or equal to six and not divisible by four, are equivalent to strongly regular graphs with intersection array* $\{(y+2)^2, y(y+4); 1, 2\}$ *and a partition into four-cliques such that there is a perfect matching or nothing between any pair of these cliques. All these intersection arrays are feasible.* □

3. Characterization of certain antipodal covers

Let $(X, \mathcal{A})$ be a d-class P-polynomial association scheme and suppose that merging is applied in order to get a new association scheme. If A_d is a polynomial of degree d in A_1, the matrix A_1 has to be merged with at least one of the matrices $A_2, \ldots, A_d$ in order to reduce the dimension of Bose-Mesner algebra $\mathcal{A}$ which is of course smaller after merging. It is known when merging of A_1 and A_2 in a distance-regular graph produces a distance-regular graph, see Brouwer et al. [27, Prop. 4.2.18]:

5.3.1 PROPOSITION. *Let* H *be a distance-regular graph with diameter* D. *Then the graph* M *determined by the adjacency matrix* $A_1(H) + A_2(H)$ *is distance-regular if and only if we have*

$$b_j + c_{j+1} = k + 1 \ (j \text{ even}, 0 \leq j \leq D-1)$$

and

$$b_j + c_{j+1} = b_1 + \mu \ (j \text{ odd}, 1 \leq j \leq D-1)$$

(or equivalently $b_{j-1}+c_{j+1}-a_j=k+\mu-\lambda$). If this is the case, then H has diameter $\lfloor (D+1)/2 \rfloor$, and parameters

$$\begin{aligned} b_j(M) &= b_{2j-1}b_{2j}/\mu \ \ (= k_1+k_2 \ \ \textit{if} \ \ j=0), \\ c_j(M) &= c_{2j-1}c_{2j}/\mu \\ &\quad (= c_D(k+\mu-\lambda+a_D-b_{D-1})/\mu \ \ \textit{if} \ \ j=(D+1)/2). \square \end{aligned}$$

We determine when merging of A_1 and A_D in an antipodal distance-regular graph with diameter D results in a distance-regular graph.

5.3.2 PROPOSITION. *Let H be a distance-regular antipodal r-cover with diameter $D>2$ of a graph G with diameter d and $t=c_{d+1}(H)$. Then the graph M with the adjacency matrix $A_1(H)+A_D(H)$ is distance-regular if and only if $r=\lambda+2$ and G has the following intersection array*

$$\big\{k, k-(\lambda+1), \ldots, k-(d-1)(\lambda+1); 1, 2, \ldots, d-1, (\lambda+2)d\big\} \qquad \textit{for } D \textit{ even},$$

$$\big\{(t+d)(\lambda+1), (t+d-1)(\lambda+1), \ldots, (t+1)(\lambda+1); 1, 2, \ldots, d\big\} \qquad \textit{for } D \textit{ odd}.$$

The graph M has valency $b_0(G)+r-1$ and diameter m equal to $\lfloor (D+1)/2 \rfloor$. When M is distance-regular its intersection numbers are

$$\begin{aligned} b_i(M) &= b_0(M)-i(\lambda+1) \quad &&\textit{for } i=1,\ldots,m-1 \\ c_i(M) &= i &&\textit{for } i=1,\ldots,d \end{aligned}$$

and for D odd also $c_m(M)=b_0(M)/(\lambda+1)$.

Proof. To simplify the notation we will use X_i instead $A(X_i)$. First note that $M_1=H_1+H_D$ implies that the diameter d' of M is $\lfloor (D+1)/2 \rfloor$ and $M_i=H_i+H_{D+1-i}$ for $i=1,2,...,d'-1$ (for D odd also $M_{d'}=H_{d'}$). Hence, for $1 \le i \le d'$ and for ϵ equal to two when $i=(D+1)/2$ and 1 otherwise, we have:

$$\begin{aligned} \epsilon M_1 M_i &= (H_1+H_D)(H_i+H_{D+1-i}) \\ &= H_1H_i+H_1H_{D+1-i}+H_{D+1-i}H_D+H_iH_D. \end{aligned}$$

Since H is an antipodal graph with diameter larger than two, a vertex u at distance $i \leq \lfloor D/2 \rfloor$ from a vertex v is at distance $D-i$ from all the other vertices in a fibre containing v, cf. Theorem 2.4.2. Therefore $H_i H_D$ is equal H_{D-i} for $i < D/2$, $(r-1)H_{D-i}$ for $i = D/2$ and $(r-1)H_{D-i} + (r-2)H_i$ for $i > D/2$. This implies that for intersection numbers A_i, B_i and C_i of the graph H

$$\begin{aligned} \epsilon M_1 M_i &= B_{i-1}H_{i-1} + A_i H_i + C_{i+1}H_{i+1} \\ &\quad + B_{D-i}H_{D-i} + A_{D+1-i}H_{D+1-i} + C_{D+2-i}H_{D+2-i} \\ &\quad + (r-1)H_{i-1} + (r-2)H_{D+1-i} \\ &\quad + H_{D-i}\left\{ \begin{array}{ll} 1 & : i < D/2 \\ r-1 & : i \geq D/2 \end{array} \right\} + H_i \left\{ \begin{array}{ll} 0 & : i \leq D/2 \\ r-2 & : i > D/2 \end{array} \right\} \end{aligned}$$

First, let us suppose that D is even (i.e., $D = 2d$, i.e., $d = d'$). Since $i \leq d'$ (i.e., $i + 1 < D - i + 2$) for $i > 1$ the coefficients at H_{i-1} and H_{D-i+2} are equal, thus for $i = 2, \ldots, d$:

$$B_{i-1} + (r-1) = C_{D+2-i}. \tag{1}$$

If $i \leq d' - 1$ (i.e., $i + 1 < D - i$) then the coefficients at H_i and H_{D-i+1}; H_{i+1} and H_{D-i} are pairwise equal, therefore for $i = 1, \ldots, d-1$:

$$A_i = A_{D+1-i} + (r-2) \quad \text{and} \quad C_{i+1} = B_{D-i} + 1 \tag{2}$$

For $i = d'$ the coefficients at $H_{d'+1}$ and $H_{d'}$ are also equal, so (1) holds in this case too. By Theorem 2.4.2 H is distance-regular antipodal if and only if $B_i = C_{D-i}$ for $i = 0, \ldots, d-1, d+1, \ldots D$ (and $r = 1 + B_d/C_{D-d}$), so the conditions obtained are equivalent to:

$$B_i = B_0 - i(r-1) \text{ for } i = 1, \ldots, d-1 \quad \text{and} \quad C_i = i \text{ for } i = 1, \ldots, d,$$

which means that H is a $(\lambda + 2)$-cover of a distance-regular graph G with intersection arrays as desired.

Now, suppose that D is odd (i.e., $D = 2d + 1$, i.e., $d = d' - 1$). For $i = d'$ the coefficients at $H_{d'-1}$ and $H_{d'+1}$ are equal which implies (1) for $i = d'$. Since $i \leq d' - 1$ is equivalent to $i + 1 \leq D - i$ the equation (1) also holds for $i = 2, ..., d' - 1$, the first equation in (2) holds for $i = 1, \ldots, d' - 1$ and the second for $i = 1, \ldots, d' - 2$. The obtained conditions are equivalent to

$$B_i = B_0 - i(r-1) \text{ and } C_i = i \text{ for } \quad i = 1, \ldots, d,$$

If we denote C_{d+1} with t ($\geq d$ by monotonicity of C_i), then $r = 1 + B_d/C_{D-d}$ implies $B_i = (t + d - i)(r - 1)$ and therefore H is an $(\lambda + 2)$-cover with parameter t of a distance-regular graph G (i.e., $((\lambda + 2) \cdot G)_t$) with intersection arrays as desired. (Brouwer has noted that the use of matrices can be avoided by observing the distance partition of H corresponding to a vertex and, as above, using Theorem 2.4.2) $\square$

Now, we reverse merging. This provides, together with Proposition 5.3.2, a characterization of the antipodal distance-regular graphs in which merging of the first and the last class gives a distance-regular graph. This generalizes Brouwer's theorem (Theorem 3.3.1) and our Theorem 5.2.2.

5.3.3 THEOREM (Jurišić [93]). *If M is a distance-regular graph with the same intersection array as in Proposition 5.3.2, $b_0(M)/(\lambda+1) > m \geq 2$ and a partition $\mathcal{P}$ of its vertex set into $(\lambda+2)$-cliques such that there is a perfect matching or nothing between any pair of these cliques, then the quotient graph $M/\mathcal{P}$ is distance-regular with the same intersection array as G. Deletion of the edges contained in the members of $\mathcal{P}$ produces a distance-regular antipodal cover of the quotient graph with the same intersection array as the graph H in Proposition 5.3.2.*

Proof. Let M be a distance-regular graph of diameter $m \geq 2$ with intersection array and clique partition $C_1, \dots, C_q$ as required. Let G be the quotient graph of M corresponding to the clique partition. We use induction to prove that G is distance-regular. Consider the distance partition of M corresponding to a vertex $u \in V(C_1)$

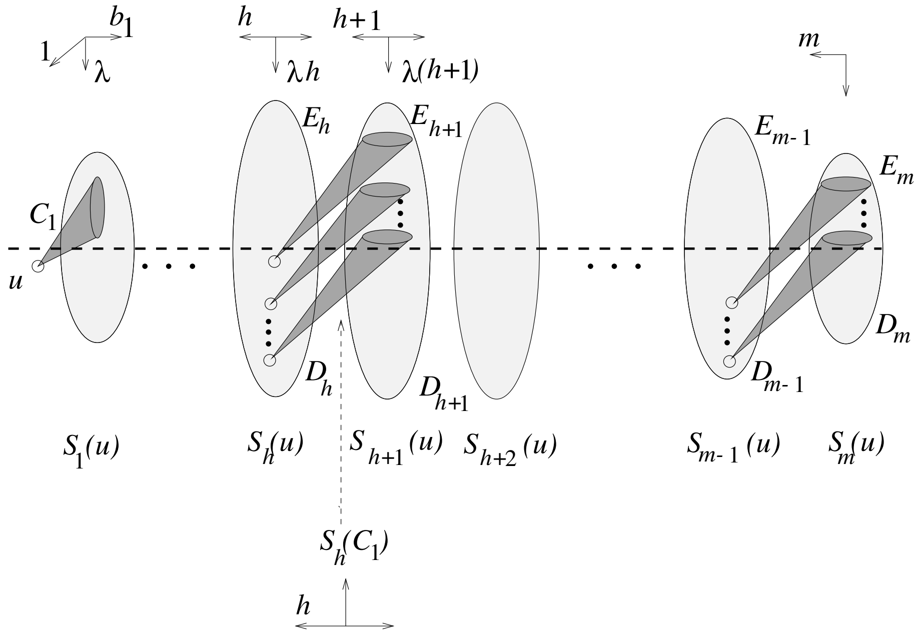

Figure 5.2: The distance partition of the graph M corresponding to the vertex u.

and define

$$D_i := S_i(u) \cap \bigcup_{C \in S_i(C_1)} V(C) \quad \text{and} \quad E_i := S_i(u) \cap \bigcup_{C \in S_{i+1}(C_1)} V(C)$$

for $i = 0, ..., m$. Let us state:

(i) $|V(C_j) \cap S_i(u)| = 1$ and $|V(C_j) \cap S_{i+1}(u)| = \lambda + 1$ for each clique $C_j \in S_i(C_1)$,

(ii) there are no edges between D_i and E_i,

(iii) $c_i(G) = i$.

For $i = 0$ these conditions evidently hold. Since $a_1(M) = |V(C_1)| - 2$ we have $b_0(G) = b_1(M)$. Now, suppose that the above conditions hold for $i = 0, \ldots, h \leq m - 2$ (see Figure 5.2). Since $c_h(G) + 1 = c_{h+1}(M)$, all edges between E_{h+1} and D_h lie in the cliques of $S_h(C_1)$. Therefore for $i = h$ the assumption (ii) implies that $a_h(G) = \lambda h$. From this and from $a_{h+1}(M) = \lambda(h + 1)$ the condition (ii) follows for $i = h + 1$. Since $b_{h+1}(M) \neq 0$ we have $S_{h+1}(u) \backslash E_{h+1} = D_{h+1}$. There are clearly no edges between the sets $S_{h-1}(C_1)$ and $S_{h+1}(C_1)$, so no vertex of D_{h+1} has neighbours in E_h. Thus, by (ii) for $i = h + 1$, it follows $c_{h+1}(G) = c_{h+1}(M) = h + 1$, i.e., the condition (iii) is satisfied for $i = h+1$. If $|V(C) \cap S_{h+1}(u)| \geq 2$ for some $C \in S_{h+1}(C_1)$, then $c_{h+1}(G) \geq 2c_{h+1}(M)$. Therefore (i) is satisfied for $i = h + 1$ as well. By induction $c_i(G) = i$ for $i = 1, \ldots, m - 1$ and $b_i(G) = b_0(G) - i(\lambda + 1)$ for $i = 0, \ldots, m - 2$.

For $s := \lambda + 1$ and $p := b_0(M)/s$ the cardinality of the set D_m equals

$$\begin{aligned}
& 1 + \sum_{i=1}^{m} k_i(M) - (s + 1)\left[1 + \sum_{i=1}^{m-1} k_i(G)\right] \\
= \quad & 1 + \sum_{i=1}^{m-1} s^i \binom{p}{i} + s^m \binom{p-1}{m-1} \frac{p}{c_m(M)} \\
& - (s + 1)\left[1 + \sum_{i=1}^{m-1} s^i \binom{p-1}{i}\right] \\
= \quad & \sum_{i=1}^{m-1} s^i \binom{p}{i} + s^m \binom{p-1}{m-1} \frac{p}{c_m(M)} \\
& - \sum_{i=1}^{m-1} s^i \left[\binom{p-1}{i} + \binom{p-1}{i-1}\right] - s^m \binom{p-1}{m-1} \\
= \quad & s^m \binom{p-1}{m-1} \left[p/c_m(M) - 1\right].
\end{aligned}$$

When D is odd we have $c_m(M) = p$, hence $D_m = \emptyset$, i.e., G has diameter $m-1$ and is distance-regular. When D is even $c_m(M) = m$. Similarly as before $a_{m-1}(G) = \lambda(m-1)$. The assumption $b_0(M) > m(\lambda+1)$ is equivalent to $p > m$ therefore $D_m \neq \emptyset$, i.e., the diameter of G is m. Clearly $c_m(G) \geq |C_1|c_m(M) = (\lambda+2)m$. Since $D_m = \cup\{V(C) : C \in S_m(C_1)\}$ the number of edges between $S_{m-1}(C_1)$ and $S_m(C_1)$ is at least

$$|S_m(C_1)|(\lambda+2)m = |D_m|m = s^m\binom{p-1}{m-1}(p-m) = |S_{m-1}(C_1)|b_{m-1}(G).$$

Since this is the number of all edges between these two sets, we have $c_m(G) = (\lambda+2)m$, which means that G is distance-regular in the case when D is even as well. (Note that this implies $b_1(M) = b_0(G) \geq c_m(G)$, i.e., $b_0(M) \geq (m+1)(\lambda+2)-1$.)

Let H be a graph obtained from M by deletion of the edges contained in $C_1, \ldots, C_q$. Remember that (ii) states that no vertex in E_i is adjacent to a vertex in D_i for $i = 0, \ldots, m-1$. Further, in the graph H no vertex of E_i is adjacent to a vertex in D_{i-1} or D_{i+1} for $i = 0, \ldots, m-1$ and $i = m$ (with $D_{m+1} = \emptyset$) for D even. Therefore the diameter D of H is at least $2m$ for D even and at least $2m-1$ for D odd. Since the assumption $b_0(M)/(\lambda+1) > m$ implies $c_m(M) > c_{m-1}(G)+1$ for D odd and $c_m(G) \neq 0 \neq b_{m-1}(G)$ for D even, the diameter D reaches the lower bound, H is antipodal with $V(C_1), \ldots, V(C_q)$ as its antipodal classes and each geodesic in H of length at least m can be extended to a geodesic of length D. Hence by Theorem 2.4.2 the graph H is distance-regular. $\square$

Remarks. (i) For $\lambda = 0$ and $k = D$ we have the D-cube as the double-cover of the folded D-cube and the folded $(D+1)$-cube as the merged graph.

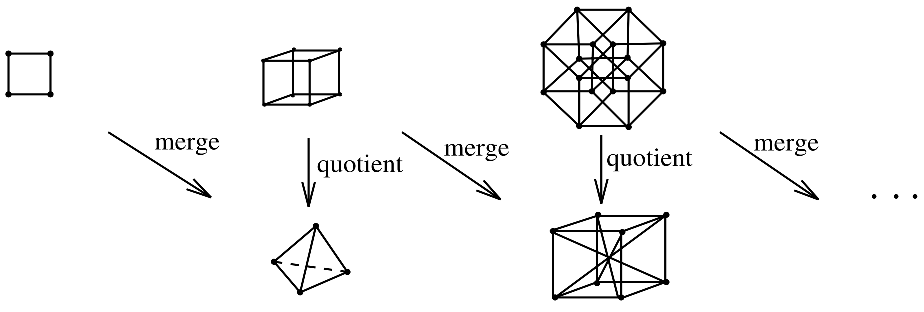

Figure 5.3: Merging in n-cubes.

(ii) In the case when D is odd the assumption $b_0(M)/(\lambda+1) > m$ is not needed for the construction of G.

(iii) For all four intersection arrays mentioned above $b_i = k - (\lambda + 1)c_i$ for all $i < d$, so they include regular near polygons with $c_i = i$, $i \leq d$, see Brouwer et al. [27, Thm. 6.4.1]. When D is odd note also $k(M) = (\lambda(M) + 1)c_d(M)$, therefore in this case the regular near polygons are even. If $D \geq 4$ then M is a regular near polygon if and only if G is a regular near polygon. For, triangles of G lift to triangles in H, so the same is true for $K_{1,2,1}$.

(iv) All these graphs are of Hamming type (i.e., $c_i = i$ for $i = 1, \ldots, e$ and $a_i = i\lambda$ for $i = 1, \ldots, e-1$, where e is an integer less or equal to the diameter), cf. Nomura [107], [108]. For $D \geq 6$ Nomura [107, Corollary] implies $\lambda \in \{0, 2\}$. Rifà and Huguet [116] classified all graphs of Hamming type with diameter δ at least three, $\lambda = 0$ and $e \geq \delta - 1$. The only known example beside (i) is provided by the binary Golay code, cf. Brouwer et al. [27, pp. 360–362]. Its coset graph is the only distance-regular graph with intersection array $\{23, 22, 21; 1, 2, 3\}$. Its bipartite double merges to the coset graph of the extended binary Golay code with intersection array $\{24, 23, 22, 21; 1, 2, 3, 24\}$. Koolen [97] pointed out that there are no examples for $\lambda = 2$.

I would like to thank Brouwer for careful reading of an earlier version of the above proof. He encouraged me to shorten the proof, overcome some inconsistencies and add a few remarks.

4. Antipodal distance-regular graphs of diameter five

In this section we consider merging in diameter five case. An immediate corollary of Theorem 2.4.2 and Theorem 5.3.1 is the following result:

5.4.1 PROPOSITION. *Let H be such an antipodal distance-regular graph that merging H_1 and H_2 results in a distance-regular graph M. Then M is antipodal distance-regular graph (of the same covering index as H) if and only if the diameter of H is odd.* □

In Brouwer et al. [27, p. 150] it is remarked that the assumptions of the above proposition imply that the covering index must be two, and that the merging of the first two classes in the antipodal quotient of H results in a distance-regular graph again.

In particular, if H in the above proposition has diameter five, then merging H_1 and H_2 results in a strongly regular graph M if and only if $rt = k + 1$. If this is the case then the merged graph is a distance-regular graph of diameter three, i.e., a distance-regular cover of a complete graph.

$$\begin{tikzcd} H \arrow[d] & & M(H) \arrow[d] \\ G & \overset{\text{merge}}{\Longrightarrow} & M(G) \end{tikzcd}$$

Figure 5.4: Merging in antipodal distance-regular graphs of odd diameter.

Let us further assume that H is bipartite. Then it must be, by Proposition 4.3.1, the bipartite double of its antipodal quotient. This implies $r = 2$, $\lambda = 0$ and $t = c_3 = k - b_3 = k - \mu$ and thus $k = 2\mu + 1$. Now, the integrality of multiplicity $m_2 = (2\mu+1)(3\mu+2)/(\mu+2)$ of G implies that $\mu \in \{1, 2, 4, 10\}$. For $\mu = 1$ the quotient is the Petersen graph which has the Desargues graph as the bipartite double-cover and the merged graph is the Johnson graph $J(6, 3)$, i.e., the unique double-cover of K_{10} (this is a special case of merging in the bipartite doubles of Odd graphs, see Remark (ii) after Proposition 4.2.18 by Brouwer et al. [27, p. 150]). For $\mu = 2$ the quotient is the folded five-cube, and the merged graph is the unique double-cover of K_{16}, see Bussemaker, Mathon and Seidel [35], i.e., halved 6-cube. This is a special case of merging in n-cubes, see Remark (i) after Proposition 4.2.18 by Brouwer et al. [27, p. 150]. The last graph also corresponds to the unique strongly regular graph $\{8, 3; 1, 4\}$ which is the complement of the point graph of the generalized quadrangle $GQ(2, 2)$). In the remaining two cases ($\mu = 4, 10$) intersection arrays of H are not feasible.

The lists of antipodal distance-regular graphs of diameter five in Brouwer et al. [27] or Godsil, Jurišić and Schade [69] suggest that the graph H must be bipartite.

When we merge H_1 and H_5 we get a distance-regular graph if and only if G has intersection array $\{(\lambda + 1)(t + 2), (\lambda + 1)(t + 1); 1, 2\}$ and the covering index of H is $\lambda + 2$. The graph M then has the following intersection array:

$$\{(\lambda + 1)(t + 3), (\lambda + 1)(t + 2), (\lambda + 1)(t + 1); 1, 2, t + 3\} \quad \text{for } t \geq d.$$

Merging H_1 and H_4 cannot result in a distance-regular graph, since otherwise $\mu = k$, i.e., G is a complete multipartite graph and these graphs have no distance-regular antipodal covers of diameter five, see Proposition 6.1.7. Similarly merging H_1 and H_3 cannot yields a distance-regular graph. For, if we suppose that $r > 2$, we get $2 + b_1 = t + 2 = (r - 2)\mu + k \geq 2 + b_1$ from $M_1 M_1 \in \mathcal{A}'$, but then equality has to hold, i.e., $t = b_1$, $k = b_1$ and by the monotonicity of intersection array $b_1 \geq (r - 1)b_1 \geq 2b_1$, we obtain a contradiction. In the case $r = 2$ the product $M_1 M_3$ cannot be a linear combination of M_1, M_2, M_3 and we get a contradiction again.

5. Conclusion

Let H be a distance-regular graph of diameter D. Brouwer (private communication, 1992) noticed that when we define $M = H_1 + H_D$ in order to conclude that $M_i = H_i + H_{D+1-i}$ it is not necessarily to require that H is antipodal. It suffices that $p_{D,D}(i) = 0$ for $i \neq 0, 1, 2, D-1, D$.

5.5.1 PROPOSITION (Brouwer). *Let H be a distance-regular graph of diameter D with $p_{D,D}(i) = 0$ for $i \neq 0, 1, 2, D$. Then $M = H_1 + H_D$ is distance-regular if and only if*

$$\begin{aligned} b_i + p_{D,D-i}(i) &= p_{D,i+1}(D+1-i) + c_{D+1-i} && \text{for } i = 1, \ldots, m-1, \\ c_i + p_{D,D+2-i}(i) &= p_{D,i-1}(D+1-i) + b_{D+1-i} && \text{for } = 2, \ldots, d, \end{aligned}$$

where $m = \lfloor D/2 \rfloor$ and $d = \lfloor (D+1)/2 \rfloor$. If this holds, then M has parameters

$$\begin{aligned} &b_i(M) = b_i + p_{D,D-i}(i) && \textit{for } i = 1, \ldots, m-1, \\ &c_i(M) = c_i + p_{D,D+2-i}(i) && \textit{for } i = 1, \ldots, d, \\ &c_m(M) = c_m + p_{D,m-1}(m) + b_m + p_{D,m+1}(m) && \textit{if } D = 2m-1. \end{aligned}$$ □

This generalizes Proposition 4.2.17(i) in Brouwer et al. [27] and the merging in Section 2 of this chapter when $x = 4$.

6

ANTIPODAL COVERS OF STRONGLY REGULAR GRAPHS

Perhaps the problem of looking for distance-regular antipodal covers is easier when studied in a more general setting. For example, many questions about distance transitive graphs can be answered more elegantly by techniques developed for distance-regular graphs (which contain all distance transitive graphs). In Chapter 4 we have seen Terwilliger's result which demonstrates that in some cases knowledge of antipodal covers (not necessarily distance-regular) could be used to construct distance-regular antipodal covers of diameter four. In addition, M. Brown (private communication, 1994) showed that subquadrangles in the generalized quadrangle $Q(4,q)$ with parameters (q,q) are equivalent to certain antipodal double-covers (not necessarily distance-regular) of certain strongly regular graphs.

We will demonstrate that very often the condition for covers to be antipodal is restrictive enough to rule them out or to give their characterizations or some particular constructions. On the other hand, maybe there are too many strongly regular graphs and our problem of saying something about their (distance-regular) antipodal covers in general does not have a solution. For example, to determine all distance-regular covers of complete graphs is still an open problem. Therefore it would be good to identify which families of strongly regular graphs are of particular interest to us. We have mentioned some of them at the end of Chapter 4. In this chapter we will study antipodal covers of some large families of strongly regular graphs.

Let us introduce two such infinite families of strongly regular graphs which come from designs. The *line graph* (also called the *block graph*) of a design is the graph with lines (i.e., blocks) as vertices and two of them being adjacent whenever there is a point incident to both lines. The line graph of a 2-$(v,s,1)$ design with $v-1 > s(s-1) > 0$ is strongly regular. As these designs are also called Steiner systems, their line graphs $S(s,v)$ are known as the *Steiner* *graphs*. The point graph of a Steiner system is a complete graph, thus the line graph of the Steiner system $S(2, v)$ is the line graph of the complete graph K_v, also called the *triangular graph* $T(v)$. The line graph of a transversal design $TD(s, v)$ is also strongly regular for $2 \leq s \leq v$. For $s = 2$ we get the *lattice graph* $K_v \times K_v$.

Strongly regular graphs with the same parameters as their complements are called *conference graphs*. They are the only strongly regular graphs that could have irrational eigenvalues. Since the multiplicities of eigenvalues are integral, the only distance-regular antipodal cover of a conference graph is the decagon. This has been first proved by Van Bon, see Brouwer et al. [27, p. 180]. Furthermore, it can be shown that the smallest eigenvalue of a strongly regular graph must be negative and cannot be -1. The strongly regular graph with the smallest eigenvalue $-m, m \geq 2$ integral, is with finitely many exceptions, either a complete multipartite graph, a Steiner graph, or the line graph of a transversal design, see Neumaier [105].

In Section 1, the structure of short cycles in an antipodal cover is investigated. It provides a tool to determine when the above two infinite families of strongly regular graphs allow antipodal covers. With one trivial exception, none of these covers is distance-regular. Analysis of this cycle structure also allows us to construct some antipodal covers. Under a mild restriction a bijective correspondence between antipodal covers of a graph and its line graph is established in Section 2. Antipodal covers of the lattice graphs and the complete bipartite graphs are characterized in Section 3.

1. Cycles and nonexistence of antipodal covers

In this section an argument that implies new existence conditions for antipodal covers is given. We apply it to the complete multipartite graphs, the line graphs of transversal designs and the Steiner graphs.

It is sometimes convenient to record an r-cover of a graph by arbitrarily orienting its edges and then defining an *arc function* from the set of arcs to a set of permutations of order r, cf. Section 3.4. We can change orientation of any edge, if we replace the corresponding permutation with its inverse. We can choose this function to be the identity on a spanning tree. The following two results have been pointed out to me by Godsil.

6.1.1 LEMMA (Godsil). *Let G be a graph and H its antipodal cover of diameter D. Let C be a cycle in G with length less than D. Then the matchings* *between the fibres of H corresponding to the edges of C form a disjoint union of r cycles, all of the same length.*

Proof. The definition of an r-cover implies that the matchings corresponding to the edges of any path in G form in H a disjoint union of r paths of the same length as the path from G.

Now, suppose that there is an edge e between the beginning and the end of some path P of length less than $D-1$ in G. Then the matching in H corresponding to e matches the vertices of fibres of the beginning of P with the fibre of the end of P. This matching cannot extend the paths in H corresponding to P to paths, since they would begin and end in the same fibre and they would have length less than D. Hence these paths in H must be extended to the cycles we have wanted. □

In other words this lemma states that the product of the values of an arc function which determines H on the edges of any cycle of length less than D in G is the identity.

Van Bon and Brouwer [17], [27, p. 144] found necessary conditions for a distance-regular graph to have a distance-regular antipodal cover. They have dealt with the even and the odd diameter cases separately. They have shown that most of the classical distance-regular graphs have no distance-regular antipodal covers, cf. Terwilliger [135]. We first used their results together with Godsil and Schade [69] to rule out almost all distance-regular antipodal covers of Steiner graphs and the line graphs of transversal designs. However, the above tool allows more general study and shorter proofs.

6.1.2 COROLLARY (Godsil). *Let G be a graph of diameter d, girth g and with a spanning tree of diameter s. If H is an antipodal cover of diameter D, then $g \le D \le s+1$. In particular, G has no antipodal covers of diameter greater than $2d+1$.* □

Gardiner [59], [60] has shown that in an antipodal distance-regular graph the size of an antipodal class is bounded by the valency, and studied the cases when the bound is attained. Recall that a_1 (resp. c_2) denotes the number of common neighbours of two adjacent vertices (resp. two vertices at distance two). For antipodal covers with diameter three we obtain the following result.

6.1.3 LEMMA. *Let G be a strongly regular graph with an antipodal r-cover H of diameter three. Then $r \le a_1(G)+1$. If the diameter of G is at least two, then also $r \le c_2(G)$.*

Proof. Let u and v be adjacent vertices of H and let F be the fibre containing v. Since the diameter of H is three, the vertex u is at distance two from each of the $r-1$ vertices in $F\backslash\{v\}$. The middle vertices of the corresponding paths of length two between u and vertices of $F\backslash\{v\}$ induce distinct common neighbours of the two adjacent vertices of G, thus $r-1 \le a_1(G)$. The proof of the second part is similar. □

This lemma implies that the index of an antipodal cover with diameter three of the complete graph K_v is $v-1$ at most. When the covering index is $v-1$ the antipodal cover is distance-regular. This forecasts that antipodal covers with maximum covering index are interesting objects.

6.1.4 PROPOSITION. *An antipodal cover of the line graph G of a transversal design $TD(s,v)$, $s \le v$, has diameter four when $s=2$, and diameter three otherwise.*

Proof. Let H be an antipodal r-cover of G determined by an arc function f on G. Suppose that $s=2$, i.e., G is the lattice graph $K_v \times K_v$, and H has diameter three. Then $v>2$, and Lemma 6.1.3 implies $r=2=c_2(G)$. Moreover (remember the proof of Lemma 6.1.3) this implies that each four-cycle in G has to lift to an eight-cycle in H. Hence there exists a colouring of the edges of G with red and blue colours (corresponding to the identity and the nonidentity permutations of f) such that each quadrangle contains an odd number of red edges. Then $K_3 \times K_3 \subseteq G$ is coloured this way as well. This graph has nine quadrangles and each edge lies in two of them (for, $C_3 \times C_3$ is naturally embedded on the torus). If x is the number of quadrangles with three red edges, then $(3x+9-x)/2$ is the number of all the red edges. Contradiction!

Now, suppose that $s \ge 2$ and H has diameter five. Then by Lemma 6.1.1 the product of values of f on any triangle or quadrangle is the identity. Let us choose a spanning tree of diameter two in all the 'horizontal' and one 'vertical' copy of K_v. Then their edges determine a spanning tree T of G. We choose an arc function f so that it is the identity on T, therefore it must be the identity on all the edges of those copies of K_v. Further, since each edge lies in a quadrangle with at least three edges in those copies of K_v, f must be the identity on all the edges.

Finally, let $s>2$, the diameter of H be at least four, and f be the identity on T. Then f is the identity on all the copies of K_v which contain edges of T, i.e., on all the horizontal and one vertical copy of K_v. Any skew edge of $E(G)$ (i.e., not horizontal or vertical) lies in a triangle with two edges in those copies of K_v, so f is the identity on all the skew edges. But now, for the same reason, f has to be the identity on all the vertical edges as well. □

It is easy to construct many antipodal double-covers with diameter three of the line graph G of a transversal design $TD(s, v)$, $s > 2$, We can accomplish this by assigning the identity permutation to the horizontal and the vertical edges of G and the non-identity permutation to all the other edges of G.

Let us verify this construction. By the definition of a cover any two vertices from the same fibre are automatically at distance three. Consider vertices u and v of G. There exists a path of length two at most, which uses only identity edges. On the other hand, choose a skew edge incident with u whose other end, w say, has at least one coordinate equal to v. Since such an edge always exists we can extend the path u, w to the path between u and v of length two at most. The product of permutations assigned to the edges of this path is not the identity since it contains exactly one skew edge. Thus any two vertices from distinct fibres are at distance two at most and our cover must be antipodal.

Alternatively, if we switch the permutations on some small set of vertical edges from the same clique we can get another antipodal cover. For example, a switching on the edges of any matching of some vertical clique will do. Therefore it seems, that there are many nonisomorphic antipodal double-covers with diameter three and that there is no sense in classifying them. Note that a line graph of $TD(s, v)$ is a conference graph when $v = 2s - 1$.

The two-colouring of edges mentioned in the first part of the above proof exists for the n-cube, $n > 1$. We show how this can be used to obtain a very straightforward and elementary proof of the following result of Cohen and Tits [51], [27, Prop. 9.2.10(i)], who used the fundamental group and the homology group of the n-cube.

6.1.5 PROPOSITION (Cohen and Tits). *There is the unique double-cover of the n-cube Q_n, $n > 1$, having no quadrangles.*

Proof. Let us have $Q_1 \subset Q_2 \subset \cdots \subset Q_n$ and let T be the spanning tree of Q_n with the following edges: $E(Q_1)$, the edges of Q_2 with one end in Q_1, the edges of Q_3 with one end in Q_2, ..., and the edges of Q_n with one end in Q_{n-1}. Let f be an arc function on Q_n which determines a double-cover having no quadrangles and is the identity on T. It suffices to prove that there exists a unique two-colouring of edges of Q_n (black for the identity permutation and white for the nonidentity permutation) such that each quadrangle contains an odd number of edges of each colour (i.e., each quadrangle lifts to an eight-cycle) and that the edges of $E(T)$ are black. Note that if we know the colours of three edges of some quadrangle, then we know the colour of the remaining edge as well. Q_2 has already three black edges, so the remaining edge is white. Suppose that we already know the colour of each edge in Q_i, $1 < i < n$. Each edge of the set $E(Q_{i+1})\backslash(E(Q_i) \cup E(T))$ lies in a unique quadrangle with the remaining edges in $E(Q_i) \cup E(T)$, thus we know a colour of each edge in Q_{i+1}. Note that the two-colouring of $Q_{i+1}\backslash V(Q_i)$ is the opposite of the two-colouring of Q_i. By induction we know the whole two-colouring of Q_n. Since we have checked along the way that all quadrangles contain an odd number of edges of each colour, this is a unique double-cover of Q_n having no quadrangles. □

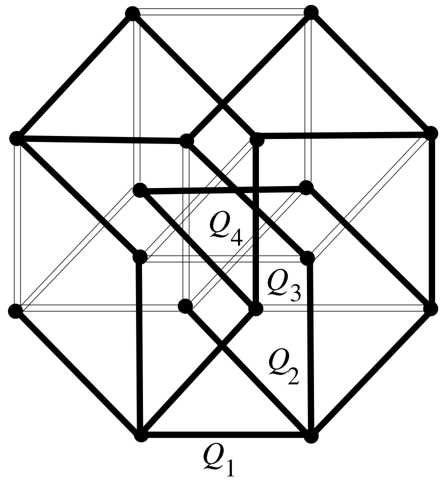

Figure 6.1: A two-colouring of the four-cube.

Note that the first part of the proof of Proposition 6.1.4 implies that no other Hamming graph allows a double-cover, having no quadrangles.

6.1.6 PROPOSITION. *Antipodal covers of Steiner graphs have diameter three.*

Proof. Let us choose a spanning tree T of a Steiner graph G which contains all the edges incident with $\ell \in V(G)$ and no edges from $S_2(\ell)$. Let an arc function f which determines an antipodal cover with diameter at least four be the identity on the edges of this tree. By Lemma 6.1.1 f is the identity on all the edges with both ends in $S_1(\ell)$.

Now, let ℓ' be any element of $S_2(\ell)$ and $m \in S_1(\ell) \cap S_1(\ell')$. Denote the intersections of the line m with the lines ℓ and ℓ' by A and B respectively. If $\ell' m$ is not an edge of T, then there is $\ell'' \in S_1(\ell)$ such that $\ell'\ell''$ is an edge of T. Denote $\ell \cap \ell''$ and $\ell' \cap \ell''$ by C and D respectively. The line through A and

D, denoted by m', is adjacent to ℓ, ℓ', m (or equal to m when $B = D$) and ℓ'' (or equal to ℓ'' when $A = C$). Since $m' \in S_1(\ell)$, we conclude that, in all these cases, f is the identity on edges $\ell'm'$ and $\ell'm$. This implies that f is the identity on all the edges between $S_1(\ell)$ and $S_2(\ell)$.

Finally, since every edge in $S_2(\ell)$ lies in a triangle with a vertex in $S_1(\ell)$, Lemma 6.1.1 implies that f is the identity on all the edges of G. □

An infinite family of antipodal covers of the Steiner graphs will be constructed in the next section. A straightforward consequence of Lemma 6.1.1 is also the next statement.

6.1.7 PROPOSITION. *An antipodal cover of the complete multipartite graph $K_{t(m)}$, with $t, m \geq 2$, has diameter four for $t = 2$ and diameter three otherwise.* □

Antipodal double-covers with diameter three of $K_{t(m)}$ with $t > 2$, can be constructed easily as well. For example, for natural labeling of the vertices of $K_{t(m)}$ with (i, j) for $i = 1, \ldots, t$ and $j = 1, \ldots, m$, choose f to be the identity on all the edges except $(i, j)(i', j)$ for $i \neq i'$, $j = 1, \ldots, m$, when $m > 2$ and $(i, 1)(i + 1, 2)$ for $i = 1, \ldots, t$ when $m = 2$.

The octagon is the only distance-regular antipodal cover of Hamming graphs, see Van Bon and Brouwer [17]. Therefore it is also the only possible distance-regular antipodal cover of the lattice graphs. This follows from Lemma 6.2.2 and the characterization of distance-regular line graphs in Brouwer et al. [27, pp. 148] or, if you prefer, from the fact that multiplicities of eigenvalues are integral.

Gardiner [60] has proved that the diameter of distance-regular antipodal covers of a graph with diameter d can only be $2d$ or $2d + 1$. Hence the graphs with diameter two can only have distance-regular antipodal covers of diameter four or five.

Results in this section are particularly interesting because they yield the following conclusion: only finitely many strongly-regular graphs with the smallest integral eigenvalue $-m$, $m \geq 2$, can have distance-regular antipodal covers.

Let G be a graph and S a subset of its edges. The *closure* of S, denoted by $\overline{S}$, is the subset of $E(G)$ obtained from S by recursively adding edges from $E(G) \backslash S$ which form a triangle with edges of S until no such edge remains. This closure is closely related to that defined in Bondy and Murty [18, pp. 56], which was used to study hamiltonicity of graphs, and whose proof that it is well defined is very similar. We say that a subset of edges S is *independent* if the edges of S do not induce a cycle.

6.1.8 LEMMA. *Let S be an independent set of edges of the complete graph K_n. Then $\overline{S} = E(K_n)$ is equivalent to $|S| = n-1$.*

Proof. Let $P_{k-1} : p_1, p_2, \dots, p_k$ be a path in K_n. Then $\overline{E(P_{k-1})}$ contains the edges $\{p_1, p_3\}, \{p_1, p_4\}, \dots, \{p_1, p_k\}$. But any edge $\{p_i, p_j\}, i, j \in \{2, \dots, k\}$, forms a triangle with these edges, so $\overline{E(P_{k-1})} = E(K_k)$. Now let the edges of S induce a tree T in K_n. Then $\overline{S}$ consists of the edges of a complete graph with the vertex set $V(T)$. For, take any two vertices u, v of T, and notice that the edge uv lies in the closure of edges of the path of T between u and v. Therefore the condition $|S| = n-1$ on independent set S implies that its closure $\overline{S}$ contains all the edges of K_n. If the set S induces a graph consisting of at least two connected components then also the closure $\overline{S}$ induces a graph consisting of at least two connected components. So for the independent set S with $\overline{S} = E(K_n)$ the edges of S induce a spanning tree and thus $|S| = n-1$. $\square$

Lemma 6.1.1 is not enough to prove that there are no antipodal covers with diameter four of the lattice graphs $K_v \times K_v$. For, by Lemma 6.1.8, there should be at least $v-1$ edges in each horizontal and each vertical copy of K_v on which an arc function is the identity. But these edges are distinct, so there would be at least $2v(v-1)$ edges on which an arc function is the identity. Since the maximum number of independent edges is v^2-1, there is no independent subset of edges whose closure would give us the set of all edges.

This encourages us to continue to look for examples of antipodal covers with diameter four of line graphs of transversal designs $TD(2, v)$.

2. Antipodal covers and line graphs

Lemma 6.1.1 implies that in the case of antipodal covers of diameter at least four the cliques of its antipodal quotient "lift" to cliques (see Lemma 6.2.4). This suggests a study of antipodal covers of line graphs.

A graph is a line graph if and only if its edges can be partitioned into cliques in such a way that no vertex lies in more than two cliques, see Harary [79, pp. 74]. We shall usually identify the vertices of a graph with the cliques of its line graph. The distance between two sets of vertices is the minimum distance over all pairs of vertices consisting of one vertex from the first set and another from the second one.

6.2.1 LEMMA. *Let M be a graph, and let C and C' be two of its vertices, i.e., two cliques of its line graph $L(M)$. Then* $\mathrm{dist}_M(C, C') = \mathrm{dist}_{L(M)}(C, C') + 1$.

Proof. Let $C = C_0, C_1, \ldots, C_s = C'$ be a geodesic Q in a graph M. Let us denote the intersection $C_i \cap C_{i+1}$ by q_i, $i = 0, 1, \ldots, s-1$. Then $q_0, q_1, \ldots, q_{s-1}$ is a path of length $s-1$ between the vertex of C and the vertex of C' in $L(M)$. Further, let u and v be such vertices of $L(M)$ from C and C' respectively that $\mathrm{dist}_{L(M)}(u, v)$ is minimum over all the choices of $u \in C$ and $v \in C'$ and let $u = p_0, p_1, \ldots, p_t = v$ be a geodesic P in $L(M)$. Clearly $P \cap (C \cup C') = \{p_0, p_t\}$. Denote by C'_i the clique of $L(M)$ which contains the edge $p_{i-1}p_i$, $i = 1, \ldots, t$, and $C'_{t+1} = C'$. Therefore $C, C'_1, \ldots, C'_t, C'$ is a path from C to C' in M of length $t+1$. Since the path Q is a geodesic of length s, we have $s \leq t+1$. Further, since the path P is a geodesic of length t we also have $t \leq s-1$ and we can finally conclude that $s = t+1$. $\square$

This result implies that the diameters of a graph and its line graph differ by one at most. Furthermore, if one of them is antipodal with at least two antipodal classes, then the other one cannot have larger diameter than the first one.

6.2.2 THEOREM. *Let H be an antipodal r-cover with diameter $D > 1$ of a graph G. Then $L(H)$ is an r-cover with diameter D of $L(G)$. If $S_{D-1}(u) \cap S_{D-1}(v) = \emptyset$ for any two adjacent vertices u and v of H, then $L(H)$ is an antipodal cover of $L(G)$.*

$$\begin{tikzcd} H \arrow[d] \arrow[dr, phantom, "\overset{L}{\Longrightarrow}"] & L(H) \arrow[d] \\ G & L(G) \end{tikzcd}$$

Figure 6.2: Antipodal covers and their line graphs.

Proof. Evidently the line graph $L(H)$ is an r-cover of $L(G)$. Since H is antipodal, the distance between any two vertices of $L(H)$ is D at most, and the vertices in the same fibre of $L(H)$ are at distance D. It remains to show that for any two edges $e_1 = uv$ and $e_2 = tw$ which are not in the same fibre, there is a path in H of length $D-2$ at most between a vertex incident with e_1 and a vertex incident with e_2. We can assume that the distances from w and t to u are at least $D-1$. Since H is antipodal, we can assume that $w \in S_{D-1}(u)$. The condition

$S_{D-1}(u) \cap S_{D-1}(v) = \emptyset$ is equivalent to $S_{D-1}(u) \subseteq S_D(v) \cup S_{D-2}(v)$, so we can assume that $w \in S_D(v)$. Then $t \in S_{D-1}(u)$, since otherwise e_1 and e_2 would be in the same fibre. Finally, as H is antipodal, $t \in S_{D-2}(v)$, which implies the existence of a desired path. □

If H is either distance-regular with $a_{D-1} = 0$ or bipartite, then $S_{D-1}(u) \cap S_{D-1}(v) = \emptyset$ for any two adjacent vertices u and v of H.

6.2.3 COROLLARY. *Let H be a distance-regular antipodal r-cover with diameter D of a graph G. Then $L(H)$ is an r-cover with diameter D of $L(G)$. The line graph $L(H)$ is an antipodal r-cover of $L(G)$ if and only if $a_{D-1}(H) = 0$.*

Proof. If $a_{D-1}(H) \neq 0$, then for two adjacent vertices u and v of H there exists $w \in S_{D-1}(u) \cap S_{D-1}(v)$. If t is a neighbour of w in $S_D(u)$, then the edges uv and tw are at distance D and they are not from the same fibre. Therefore the fibres of $L(H)$ are not antipodal classes. The converse follows from directly from Theorem 6.2.2. □

If $D > 4$ or $D = 3$ and $r = 2$, the condition $a_{D-1}(H) = 0$ translates to 'H being a triangle free'. Most of the known infinite families of feasible intersection arrays of distance-regular antipodal covers of strongly regular graphs are triangle free, see Brouwer et al. [27, pp. 417–425]. The complete bipartite graph $K_{m,m}$ with a perfect matching deleted is a triangle free distance-regular antipodal double-cover of K_m and, by Corollary 6.2.3, its line graph is an antipodal double-cover of $L(K_m) = S(2, m)$. The following statement is an obvious consequence of Lemma 6.1.1 and the characterization of a line graph mentioned above:

6.2.4 LEMMA. *Let G be a graph and A an antipodal cover of the line graph $L(G)$. If A has diameter at least four, then A is a line graph as well.* □

The condition on the diameter of an antipodal cover is necessary, since, for example, the octahedron (i.e., $L(K_4)$) has an antipodal double-cover which is not a line graph.

The line graph operator is injective for the graphs with the minimum valency at least two. Restricting to the line graphs which are obtained from the graphs with the minimum valency at least two, we can define L^{-1} and apply it to a line graph which is an antipodal cover of a line graph.

$$\begin{array}{ccc} L(H) & & H \\ \downarrow & \xRightarrow{L^{-1}} & \downarrow \\ L(G) & & G \end{array}$$

Figure 6.3: The other direction.

In this case we get a result similar to Theorem 6.2.2:

6.2.5 THEOREM. *Let G and H be two graphs with minimum valency at least two. If $L(H)$ is an antipodal r-cover with diameter $D \geq 4$ of $L(G)$, then H is an r-cover with diameter D of G. If the diameter of G is two or if any geodesic of length $D-1$ in $L(H)$ can be extended to a geodesic of length D, then H is an antipodal cover of G.*

Proof. Since $D \geq 4$, by Lemma 6.1.1, the subgraph of $L(H)$ induced by the fibres of vertices of a clique in $L(G)$ consists of disjoint cliques of $L(H)$. Such sets of cliques of $L(H)$ induce a partition of $V(H)$. The graph H is an r-cover of the graph G for this partition. Since the line graph $L(H)$ is antipodal, the graph H has diameter D at most. By Lemma 6.2.1 and the fact that the minimum valency is at least two, the distance of the cliques which are in the same fibres of vertices equals D.

To prove that the graph H is antipodal and that its antipodal classes are fibres we need to show that for any two vertices C and C' from different fibres of H their distance is strictly less than D.

First, let us suppose that there exists a pair u, u' of vertices from C and C' respectively, and that these two vertices are at distance D. Since C and C' are not in the same fibre of H this is the only antipodal pair of vertices in these two cliques. Denote by C'' the clique which intersects C and belongs to the same fibre of H as C'. Then C'' intersects C in the vertex u. There exists a vertex v in $C' \backslash \{u'\}$ which is at distance $D-1$ from u. Let $u = p_0, p_1, \ldots, p_{D-1} = v$ be a geodesic in $L(H)$. If $p_1 \notin C''$, then, by Lemma 6.2.1, $\text{dist}_H(C', C'') \leq D-1$, which is impossible. Since the vertex u lies in exactly two cliques which correspond to vertices of H, we have $p_0p_1 \in C$ and therefore $p_1 \in C$. Thus $\text{dist}_H(C, C') \leq D-1$.

It remains to prove that cliques C and C' which contain no antipodal pairs of vertices are at distance less than D in H. Suppose the opposite. Then the shortest path between the sets C and C' has length $D-1$ by Lemma 6.2.1. Therefore any pair of vertices, one from C and the other from C', are at distance $D-1$. Let us choose a vertex $u \in C$ and two vertices $v_1, v_2 \in C'$. Since any geodesic of length $D-1$ can be extended to a geodesic of length D, there are two different neighbours u_1 and u_2 of u which are at distance D from v_1 and v_2 respectively. Let C'' be the clique which contains vertices u, u_1 and u_2. Then C'' is in the same fibre as C', and hence C' contains an antipodal vertex of u. Contradiction! □

The condition on the diameter of an antipodal cover is again necessary. Otherwise the line graph of the Petersen graph as distance-regular antipodal three-fold cover of K_5 would imply that the Petersen graph is an antipodal three-fold cover too, which is evidently not true.

3. Antipodal covers of $K_{n,n}$ and $K_n \times K_n$

Drake [57] has proved that a distance-regular antipodal cover with index r of $K_{v,v}$ is equivalent to a resolvable transversal design $TD(v, v/r; r)$. In the extremal cases of r, Shad [121] and Delorme [53] have shown that distance-regular antipodal double-covers of $K_{v,v}$ are equivalent to Hadamard matrices, cf. also Shawe-Taylor [122], and Gardiner [59] has shown that distance-regular antipodal covers with maximal index, i.e., $r = v$, of $K_{v,v}$ are equivalent to affine planes of order v with a parallel class deleted. We will give a similar characterization of antipodal covers of the complete bipartite graphs $K_{v,v}$ and the lattice graphs $K_v \times K_v$.

A *'weak' resolvable transversal design* $WRTD(v, r)$ is an incidence structure on rv points, partitioned into v groups of size r, and rv distinguished subsets, called lines, such that

(1) every line intersects each group in exactly one point,
(2) parallelism (i.e., being either equal or disjoint) is an equivalence relation on the lines,
(3) there are v parallel classes, each consisting of r lines (i.e., v *resolution classes*), and
(4) there exists a line through any two points if and only if they are from different groups.

Note that (4) implies $r \leq v$. If in a weak resolvable transversal design $WRTD(v, r)$ a number of lines through any two points from different groups is constant, then this design is a resolvable transversal design $TD(v, v/r; r)$. Any resolvable transversal design $TD(v, v/r; r)$ satisfies the property (2). The dual of a resolvable transversal design $TD(v, v/r; r)$ is actually a resolvable transversal design with the same parameters. We are now ready to characterize antipodal covers of the complete bipartite graphs and the lattice graphs.

6.3.1 THEOREM. *Antipodal r-covers of $K_{v,v}$ and $K_v \times K_v$ are equivalent to 'weak' resolvable transversal designs $WRTD(v,r)$. In the extremal case, when $r = v$, they are equivalent to affine planes of order v with a parallel class deleted.*

Proof. By Propositions 6.1.7 and 6.1.4 the diameter of an antipodal cover must be in both cases equal to four. By Lemma 6.2.4, an antipodal cover of $K_v \times K_v$ is the line graph $L(H)$ of a graph H and, by Theorem 6.2.5, the graph H is an antipodal cover of $K_{v,v}$. Conversely, an antipodal cover H of $K_{v,v}$ is bipartite, thus by Theorem 6.2.2 the line graph $L(H)$ is an antipodal cover of $K_v \times K_v$. Therefore antipodal covers of $K_{v,v}$ and $K_v \times K_v$ are equivalent.

Let $(\mathcal{P}, \mathcal{L})$ be the colour partition of H into 'points' and 'lines' and $\mathcal{D}$ the design with H as the incidence graph. Therefore the fibres of H partition $\mathcal{P}$ into v groups of size r and $\mathcal{L}$ into v classes each consisting of r disjoint lines. Then $\mathrm{diam}(H) > 2$ implies (1) and (3) for these classes. The definition of H implies that any two points (resp. lines) from different fibres have a common neighbour, i.e., lie on a line (resp. intersect in a point) therefore (4) and (2) are satisfied and $\mathcal{D}$ is a weak resolvable transversal design $WRTD(v,r)$. If $r = v$ then any two lines are either parallel or they intersect in exactly one point, therefore the design is an affine plane with a parallel class deleted.

Now, we have to verify that the line graph A of the incidence graph B of a 'weak' resolvable transversal design $\mathcal{D}$ is an antipodal cover of $K_v \times K_v$. In the case $r = v$, this follows immediately from Gardiner's result and Corollary 6.2.3. In general the design $\mathcal{D}$ has rv points and each point is on v lines. Hence rv^2 is the number of edges of B and also the number of vertices of A.

Let $p,\ q$ be two points of $\mathcal{D}$ and ℓ, ℓ' two lines of $\mathcal{D}$ incident with p and q respectively. Then the flags (p,ℓ) and (q,ℓ') are the edges of B and the vertices of A, denoted by v_1 and v_2 respectively. By the definition of a line graph, these two vertices are adjacent if and only if $p = q$ or $\ell{=}\ell'$ (since an incidence graph is a bipartite graph with the bipartition by points and lines). Let us denote $\mathrm{dist}_A(v_1, v_2)$ by d and assume $p \neq q$, $\ell{\neq}\ell'$, i.e., $d > 1$. Let us determine the possible values of d.

(1) p and q are not in the same group. Then let ℓ_{pq} be a line of $\mathcal{D}$ through p and q.
   - (a) At least one of the lines ℓ and ℓ' contains both points p and q. Then at least one of the vertices (p,ℓ') and (q,ℓ) from A is a common neighbour of vertices v_1, v_2, and $d = 2$. In all the other cases $d > 2$.
   - (b) None of the lines ℓ,ℓ' contains both points p and q. Then (p,ℓ), (p,ℓ_{pq}), (q,ℓ_{pq}), (q,ℓ') is a path of length three in A and $d = 3$.

(2) p and q are in the same group.

(c) ℓ and ℓ' are not parallel. If x is a point of their intersection, then (p,ℓ), (x,ℓ), (x,ℓ'), (q,ℓ') is a path of length three in A and $d = 3$.

(d) ℓ and ℓ' are parallel. In this last possible case there is no path of length 3, but for $y \in \ell' \setminus q$ there is a path of length four (p,ℓ), (p,ℓ_{py}), (y,ℓ_{py}), (y,ℓ'), (q,ℓ'), where ℓ_{py} is a line through p and y, accordingly $d = 4$.

Therefore vertices (p,ℓ) and (q,ℓ') of A are antipodal if and only if $p \neq q$, $\ell \neq \ell'$, if the points p, q are in the same group and if the lines ℓ, ℓ' are parallel. Since this is a transitive relation, A is an antipodal graph with diameter four. Its antipodal quotient has the pairs, consisting of a group and a parallel class of $\mathcal{D}$, as vertices, two vertices (P_1, C_1), (P_2, C_2) being adjacent whenever $P_1 = P_2$ or $C_1 = C_2$ but not both, thus it is the lattice graph on v^2 vertices.

Between the antipodal classes corresponding to two adjacent vertices (P_1, C_1), (P_2, C_2) we have exactly the edges

$$\{(p,\ell_1)(p,\ell_2) :\ p \in P_1,\ \ell_1 \in C_1 \text{ s.t. } p \in \ell_1;\ \ell_2 \in C_2 \text{ s.t. } p \in \ell_2\} \quad \text{for } P_1 = P_2$$
$$\text{and} \quad \{(p,\ell)(q,\ell) : \ell \in C_1,\ p \in P_1 \cap \ell,\ q \in P_2 \cap \ell\} \quad \text{for } C_1 = C_2.$$

Finally we can conclude that A really is an antipodal cover of $K_v \times K_v$ with index r. □

We close this section with a construction of a family of antipodal double-covers of $K_{v,v}$ which are not distance-regular. It is not hard to see that a non-distance-regular graph $K_2 \times C_6$ is the only antipodal double-cover of $K_{3,3}$. (Its line graph is the only antipodal double-cover of $K_3 \times K_3$.)

We generalize this example. We start with two copies of the complete bipartite graph on $2v$ vertices with a matching deleted. Each copy is an antipodal double-cover with diameter three of K_v. Now, we connect each vertex of one copy with the vertex of the other copy which corresponds to its antipodal vertex and we get an antipodal double-cover of $K_{v,v}$. The existence of this family implies that antipodal double-covers of the complete bipartite graphs are not equivalent to Hadamard matrices although distance-regular antipodal double-covers are.

4. Conclusion

The requirement for a cover to be antipodal is restrictive enough that these graphs have a nice combinatorial structure. Our examples show that the structure gets even richer for larger diameter or larger covering index.

Lemma 6.1.1 and Corollary 6.2.3 seem to be reasonable tools for the study of infinite families of feasible arrays of distance-regular antipodal covers of strongly regular graphs, however difficulties arise when there are not enough triangles outside maximal cliques, like in the point graph of a generalized quadrangle.

Perhaps the correspondence between the antipodal covers of the point and the line graphs of the same incidence structure can be extended to other graphs derived from incidence structures.

7

ANTIPODAL DISTANCE-REGULAR GRAPHS

Godsil [67] [64, Lemma 13.5.1] has proved that for a distance-regular graph G of diameter d, not complete multipartite, and non-trivial eigenvalue θ of G with multiplicity m, $d \geq m$, implies $b_{m-1} = 1$. From this he derived that G is either a cycle or $d < 3m - 3$. This result is very important for the classification of distance-regular graphs with an eigenvalue of small multiplicity (as opposed to a dual classification of distance-regular graphs with small valency). Very recently Koolen [96, Theorem 7.17] has lowered the above upper bound on diameter to $d \leq 2m - 1$, with equality if and only if G is a dodecahedron. So far the distance-regular graphs with an eigenvalue of multiplicity eight or less have been determined by Zhu [143], Martin and Zhu [102], Martin and Koolen [96, Section 7.3] (work on multiplicity nine is in progress), while in the valency case only the distance-regular graphs of valency three and four have been determined.

Brouwer et al. [27, p. 182] stated the following problems:

(i) Does in a distance-regular graph G the condition $b_2 = 1$ together with $k > 2$ imply that G is antipodal?

(ii) Does in a distance-regular graph G of diameter d the condition $b_{d-1} = 1$ together with $k > 2$ and $\lambda > 0$ imply that G is antipodal?

Motivated by the above project of classifying graphs with small multiplicity and these two problems we study distance-regular graphs with $b_i = 1$ in Section 7.1. The main result of this section is the following statement, which is joint work with Araya and Hiraki [6]. If G is a distance-regular graph of diameter d, valency $k > 2$ and $b_t = 1$ for $t \leq d/2$, then G is an antipodal double-cover. If we exclude antipodal double-covers from our study (or assume that we already know all of them), then this result translates to: $b_t = 1$ implies $d < 2t$. The above result implies that if $m > 2$ is the multiplicity of an eigenvalue of the adjacency matrix of G, and if G is not an antipodal double-cover, then $d \leq 2m - 3$.

We say that a path P of a distance-regular graph G is *dependent* if its vertices in the graph representation corresponding to some eigenvalue of G are dependent vectors. Let θ be an eigenvalue of a distance-regular graph G and m_θ its multiplicity. If we assume that for some fixed integer e all the paths on e vertices are dependent, then the dimension of corresponding eigenspace is $e-1$ at the most, i.e., we have $m_\theta \leq e-1$. A project of determining all the graphs with small multiplicity is going on for some time but no new distance-regular graphs have been encountered. It might be better to look for distance-regular graphs which satisfy the conditions that there is a path P on e vertices which is dependent and $m_\theta > e-1$ (which implies that at least one path on e vertices is not dependent). Geodesics are good candidates for dependent paths. We search for distance-regular graphs whose geodesics on e vertices are dependent in Section 7.2. As in such graphs $b_{e-2} = 1$ (see Lemma 7.2.1), this study is closely connected with the previous section. Computations in this search are simplified, if we can factor the determinant of the Gram matrix corresponding to the representations of vertices of a geodesic. This motivates us to obtain an auxiliary result, a factorization of the determinant of a Töplitz matrix into the product of two determinants of approximately half the original size.

1. Distance-regular graphs with $b_t = 1$

We start this section by demonstrating in one particular example how useful distance distribution diagrams are when we study distance-regular graphs. Then we state the main result of this section, which is motivated by the above example. Along the way we report the progress in determining distance-regular graphs with $b_2 = 1$ and distance-regular graphs with $b_3 = 1$. A byproduct is a new infinite family of feasible intersection arrays of antipodal distance-regular graphs of diameter four. This section is joint work with Araya and Hiraki.

Suppose that G is a distance-regular graph of diameter d with $b_2 = 1$. By Godsil's diameter bound [64, Lemma 13.5.3], [27, Lemma 5.3.1], the diameter d of G is five at the most. By Lambeck [99], [27, p. 172] (cf. Brouwer et al. [27, p. 156]), only the dodecahedron meets this upper bound. Suppose that $d = 4$. Then by monotonicity of the sequence $\{b_i\}$ we have $b_3 = 1$ and, by Lemma 2.1.1 (d), $c_{4-2} \leq b_2 = 1$, hence $c_2 = 1$.

Let us now assume only $b_2 = c_2$, $b_3 = 1$ and $d = 4$. Let u and v be vertices of G which are at distance four and let us observe the *distance distribution diagram* corresponding to $\{u\} \cup \{v\}$, see Figure 7.1 ($D_i^j := S_i(u) \cap S_j(v)$, arcs denote corresponding b_i's and c_i's, and only filled circles are known to contain nonempty sets). For $S \subseteq V(G)$ we denote by ∂S the set of edges with exactly one end in S. Then $b_3 = 1$ implies

$$\partial D_3^1 \cap \partial(D_4^1 \cup D_4^2) = \emptyset \quad \text{and} \quad \partial D_1^3 \cap \partial(D_1^4 \cup D_2^4) = \emptyset.$$

Similarly $c_2 = b_2$ implies

$$\partial D_2^2 \cap \partial(D_2^3 \cup D_3^3 \cup D_3^2) = \emptyset.$$

Suppose $a_4 = 0$. Then $D_1^4 = D_4^1 = \emptyset$, and hence $D_2^4 = D_4^2 = \emptyset$. As $D_3^2 = \emptyset$ implies that G is antipodal with index two, we assume additionally $D_3^2 \neq \emptyset$. For $x \in D_3^2 = S_3(u) \cap S_2(v)$ we have $S(x) \cap S_4(u) \subseteq D_4^3$ and $S(x) \cap S_3(v) \subseteq D_2^3 \cup D_3^3 \cup D_4^3$. From $b_3 = 1$ it follows $|S(x) \cap D_4^3| = 1$, so by $b_2 = c_2$ we have $c_2 \geq c_3(u, x) + 1$, which contradicts monotonicity of the sequence $\{c_i\}$.

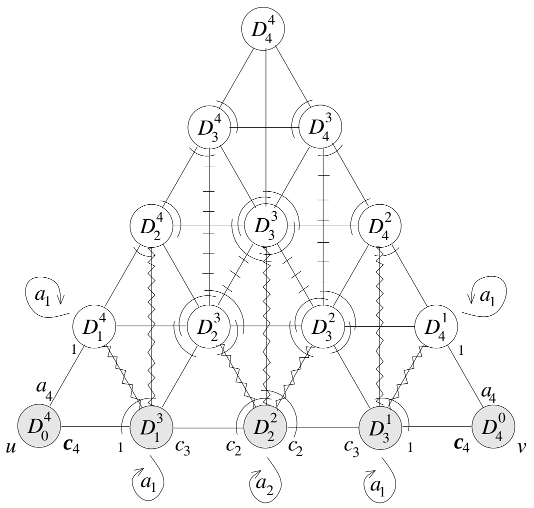

Figure 7.1: A distance distribution diagram.

Now we assume $a_4 \neq 0$, i.e., $D_1^4 \neq \emptyset \neq D_4^1$. A vertex in D_1^4 has c_4 neighbours in D_3^2. For $x \in D_3^2$ we have

$$S(x) \cap S_2(u) \subseteq S(x) \cap D_2^3 \subseteq S(x) \cap S_3(v),$$

which implies $c_3 \leq b_2$ and so $c_3 = b_2 = c_2$. By Brouwer et al. [27, Thm. 5.4.1], we have $c_2 = 1$. But since $S(x) \cap S_3(v) = D_2^3 \cup D_3^3 \cup D_4^3$, we have

$$\partial D_3^2 \cap \partial(D_3^3 \cup D_4^3) = \emptyset \quad \text{and} \quad \partial D_2^3 \cap \partial(D_3^3 \cup D_3^4) = \emptyset.$$

Since D_2^2 has, by $c_2 = c_3 = 1$, exactly c_4 vertices and valency $a_2 = k-b_2-c_2 = k-2$, we have $c_4 = k-1$ and $a_4 = 1$. This implies (see D_4^1) that $\lambda = 0$ (this is implied in $c_3 = 1$ by [27, Lemma 5.3.1 (3d)] as well), $D_4^i = D_i^4 = \emptyset$ for $i = 2, 3, 4$, and, by $c_2 = 1$ and $b_3 = 1$ for a vertex in D_2^3 also $\partial D_2^3 \cap \partial D_1^3 = \emptyset$. Finally a vertex in D_3^1 has valency two and G must be a cycle on nine vertices.

We have learned that for a distance-regular graph G with $b_2 = c_2$, $b_3 = 1$ and $d = 4$ one of the following statements is true:

(a) $a_4 \neq 0$ and G is a cycle on nine vertices,

(b) $a_4 = 0$ and G is an antipodal graph with index two and $c_2 = 1$.

Lambeck [98, Prop. 2.8] has first shown that a distance-regular graph of diameter $d = 4$ and valency $k > 2$ with $b_2 = 1$ must be antipodal double-cover. The above proof was derived in a collaboration with Araya, Hiraki during the author's visit to Japan (Algebraic Combinatorics Conference in Fukuoka, 1993). The graphic method of deriving our proof has motivated us to obtain the main result of this section.

7.1.1 THEOREM (Araya, Hiraki and Jurišić [6]). *Let G be a distance-regular graph of diameter d and valency $k > 2$. If $b_t = 1$ and $2t \le d$, then G is an antipodal double-cover.*

We postpone presenting the proof of this result until the end of this section. This result improves the main result of Suzuki [128], generalizes Lambeck [98, Prop. 2.8] (cf. Brouwer et al. [28, p. 11]) and gives a partial answer to Problem (ii) in Brouwer et al. [27, p. 182]. As an immediate consequence we have the following result:

7.1.2 COROLLARY (Araya, Hiraki and Jurišić [6]). *Let G be a distance-regular graph of diameter d. Let $m > 2$ be the multiplicity of an eigenvalue of G. If G is not an antipodal double-cover, then we have $d \le 2m-3$.*

Proof. Straightforward. See Brouwer et al. [27, Proposition 4.4.8 or Theorem 5.3.2], Godsil [67] or Godsil [64, Lemma 13.5.1 and Theorem 13.5.3]. □

Let us now consider the diameter four case more carefully.

7.1.3 PROPOSITION. *Let G be a distance-regular graph of diameter $d = 4$, valency $k > 2$, $b_3 = 1$ and $b_2 = c_2$. Then $a_1 = 0$ and G is an antipodal double-cover of a strongly regular graph with parameters $(k, \lambda, \mu) = (n^2+1, 0, 2)$, for an integer n not divisible by four.*

Proof. If $\lambda \neq 0$ then, by Proposition 4.2.1, antipodality of G implies $\lambda^2 + 4k = y^2$, for an integer $y > 2$. As $k > \lambda + 1$ we have $y > \lambda + 2$. Now consider the antipodal quotient of G, which is a strongly regular graph with $\mu = 2$. Among strongly regular graphs only the conference graphs allow noninteger eigenvalues, see Brouwer et al. [27, Theorem 1.3.1(ii)]. As J.T.M van Bon [27, p. 180] observed that conference graphs have no antipodal covers of diameter four (cf. the remark after Proposition 4.2.1), we obtain from the expression for the nontrivial eigenvalue in Proposition 4.1.1 that

$$\lambda^2 + 4k - 4\lambda - 4 = (\lambda - 2)^2 + 4(k - 2) = x^2,$$

for some integer $x > \lambda$. Therefore $y^2 - x^2 = (y + x)(y - x) = 4(\lambda + 1)$, and $x + y > 2\lambda + 2$ implies $2y = 4\lambda + 5$, which is impossible. Thus $\lambda = 0$ and $4(k-1) = x^2$, so $k = n^2 + 1$, for an integer n. Hence G is an antipodal double-cover with diameter four of a strongly regular graph with valency $n^2 + 1$, $\lambda = 0$ and $\mu = 2$. If the antipodal quotient of G has v vertices, then its eigenvalues are $\theta_0 = n^2 + 1$, $\theta_2 = n - 1$, $\theta_4 = -n - 1$, with multiplicities $m_0 = 1$, $m_2 = (n^2 + 1)(n^2 + n + 2)/4$, and $m_4 = v - 1 - m_2$ respectively. Therefore n is not divisible by four. □

Remark: G has a feasible intersection array

$$\{n^2 + 1, n^2, 1, 1; 1, 1, n^2, n^2 + 1\}.$$

For $n = 2$ we get the Wells graph and for $n = 3$ there is no such graph G, since it would cover the Gewirtz graph, see Brouwer et al. [27, Proposition 11.4.5]. We have already met the quotients of this family in Section 5.2.

In the remaining part of this section we prove Theorem 7.1.1. Let F_1 and F_2 be antipodal classes such that there is a matching between them. Choose $u, y \in F_1$ and $v, x \in F_2$ so that u is adjacent to v and y is adjacent to x. Then for the quadruple (u, v, x, y) the following equations hold dist$(u, y) =$ dist$(v, x) = d$, dist$(u, x) =$ dist$(v, y) = d - 1$ and dist$(u, v) =$ dist$(x, y) = 1$ (see Figure 7.2). A quadruple of vertices with these properties will be called a *box*.

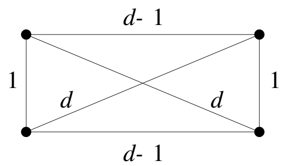

Figure 7.2: Distances in a box.

For vertices u and v in a graph G at distance i, let $A_i(u, v) = S_i(u) \cap S_1(v)$, $B_i(u, v) = S_{i+1}(u) \cap S_1(v)$, and $C_i(u, v) = S_{i-1}(u) \cap S_1(v)$. Now we shall derive some results about boxes.

7.1.4 LEMMA. *Let G be a distance-regular graph of diameter d. If G has no boxes, then we have*

$$c_d \times b_{d-1} \leq {a_d}^2.$$

Proof. Let α and β be vertices in G at distance d. Let $N = C_d(\alpha, \beta)$, $T = C_d(\beta, \alpha)$ and $S = A_d(\beta, \alpha)$. Note that $|N| = |T| = c_d$, $|S| = a_d$ and $S_1(\alpha) = T \cup S$, since $k = a_d + c_d$, i.e., $b_d = 0$. Let $P = \{ (z, w) \mid z \in S, w \in N, \mathrm{dist}(z, w) = d \}$. We count the elements of P in two ways. Take $w \in N$ and consider the set $S_d(w) \cap S_1(\alpha)$. Since $\mathrm{dist}(\alpha, w) = d - 1$ and $S_1(\alpha) = T \cup S$, we have $b_{d-1} = |S_d(w) \cap S_1(\alpha)| = |S_d(w) \cap T| + |S_d(w) \cap S|$. Suppose there exists a vertex x in $S_d(w) \cap T$. Then the quadruple (α, x, w, β) is a box, which contradicts our assumption. Hence we have $b_{d-1} = |S_d(w) \cap S|$, and therefore $|P| = |N| \times b_{d-1} = c_d \times b_{d-1}$. On the other hand, take $z \in S$. Since $\mathrm{dist}(z, \beta) = d$ and $N \subseteq S_1(\beta)$, we have $a_d = |S_d(z) \cap S_1(\beta)| \geq |S_d(z) \cap N|$. This implies $|P| \leq |S| \times a_d = {a_d}^2$. □

The following result provides an important inequality for a graph with $b_t = 1$ if it contains a box.

7.1.5 PROPOSITION. *Let G be a distance-regular graph of diameter d with $b_t = 1$ for some $t \geq 1$. If G contains a box, then we have*

$$b_{d-(h-1)} - c_{h-1} \geq b_{d-h} - c_h \quad \text{for} \quad t+1 \leq h \leq d.$$

Proof. Let u, v, x, y be vertices in G such that the quadruple (u, v, x, y) is a box. Take any $z \in A_1(x, y)$, then $z \in C_d(v, x)$, i.e., $\partial(v, z) \neq d$ (as otherwise $\{z, x\} \subseteq B_{d-1}(v, y)$ is contradicting $b_{d-1} = 1$), and $\partial(v, z) \geq \partial(v, x) - \partial(z, x) = d - 1$. This implies

$$\{y\} \cup A_1(y, x) \subseteq C_d(v, x) - C_{d-1}(u, x).$$

Thus we have $b_0 - b_1 = 1 + a_1 \leq c_d - c_{d-1}$. Hence our assertion holds for $h = d$ and we may assume now that $h < d$. Take a vertex $p \in S_{h-1}(u) \cap S_{d-h}(x)$. By $c_{d-h} \leq b_h = 1$, we set $\{w\} = C_{d-h}(x, p)$. It is clear that $\mathrm{dist}(y, p) = d - h + 1$, $\mathrm{dist}(v, p) = h$, $\mathrm{dist}(y, w) = d - h$, $\mathrm{dist}(u, w) = h$ and $\mathrm{dist}(v, w) = h + 1$. In order to prove the statement, it is sufficient to show

$$B_{d-h}(x, p) - B_{d-h+1}(y, p) \subseteq C_h(v, p) - C_{h-1}(u, p). \qquad (*)$$

Take any $z \in B_{d-h}(x, p) - B_{d-h+1}(y, p)$. Note that $\mathrm{dist}(x, z) = d - h + 1$ and $\mathrm{dist}(y, z) \neq d - h + 2$. Since $\mathrm{dist}(x, w) = d - h - 1$, we have $z \neq w$.

Claim 1. $\mathrm{dist}(y,z) = d-h+1$. Since $\mathrm{dist}(y,x) = 1$ and $\mathrm{dist}(x,z) = d-h+1$, we have $\mathrm{dist}(y,z) \in \{d-h, d-h+1\}$. If $\mathrm{dist}(y,z) = d-h$, then $\{w,z\} \subseteq C_{d-h+1}(y,p)$. This contradicts the inequality $c_{d-h+1} \le b_{h-1} = 1$. Hence we have $\mathrm{dist}(y,z) = d-h+1$.
Claim 2. $\mathrm{dist}(u,z) = h-1$. Since $\mathrm{dist}(u,p) = h-1$ and $\mathrm{dist}(p,z) = 1$, we have $\mathrm{dist}(u,z) \in \{h-2, h-1, h\}$. Suppose $\mathrm{dist}(u,z) = h$. Then we have $\{z,w\} \subseteq B_{h-1}(u,p)$, which contradicts $b_{h-1} = 1$. Suppose $\mathrm{dist}(u,z) = h-2$. Then we have

$$d = \mathrm{dist}(u,y) \le \mathrm{dist}(u,z) + \mathrm{dist}(z,y) = (h-2) + (d-h+1) = d-1.$$

This is a contradiction. Hence we have $\mathrm{dist}(u,z) = h-1$ and therefore also $z \notin C_{h-1}(u,p)$.
Claim 3. $\mathrm{dist}(v,z) = h-1$, i.e., $z \in C_h(v,p)$. Since $\mathrm{dist}(v,p) = h$ and $\mathrm{dist}(p,z) = 1$, we have $\mathrm{dist}(v,z) \in \{h-1, h, h+1\}$. Suppose $\mathrm{dist}(v,z) = h+1$. Then we have $\{z,w\} \subseteq B_h(v,p)$, which contradicts $b_h = 1$. Let $z = z_0 \sim z_1 \sim \cdots \sim z_{d-h+1} = y$ be a shortest path connecting z and y. It is clear that $\mathrm{dist}(u,z_j) = h-1+j$ for $0 \le j \le d-h+1$. Since $b_{h-1+j} = 1$, we have $B_{h-1+j}(u,z_j) = \{z_{j+1}\}$ for $0 \le j \le d-h$. Suppose $\mathrm{dist}(v,z) = h$. Then $u \in C_h(z,v)$ and $B_h(v,z) \subseteq B_{h-1}(u,z) = \{z_1\}$, which implies $B_h(v,z) = B_{h-1}(u,z)$. Inductively, we have $u \in C_{h+j}(z_j,v)$ and $B_{h+j}(v,z_j) = B_{h-1+j}(u,z_j) = \{z_{j+1}\}$ for $0 \le j \le d-h-1$. In particular, we have $\mathrm{dist}(v,z_{d-h}) = d$. Note that $z_{d-h} \ne x$, since $\mathrm{dist}(x,z) = d-h+1$. Then we have $\{z_{d-h}, x\} \subseteq B_{d-1}(v,y)$. This contradicts $b_{d-1} = 1$. Therefore, we obtain $(*)$. □

Next lemma shows that a graph which satisfies the assumption of Theorem 7.1.1 contains a box.

7.1.6 LEMMA. *Let G be a distance-regular graph of diameter d and valency $k > 2$. If $b_t = 1$ and $2t \le d$, then the following statements hold:*
(1) $a_h \le 1$ *for* $2t \le h \le d$,
(2) G contains boxes.

Proof. (1) Let $2t \le h \le d$. Let u and v be vertices of G at distance h and take $x \in S_{h-t}(u) \cap S_t(v)$. Let $A = A_h(u,v)$. Now we show that $A \subseteq S_{t+1}(x) \cap S_1(v)$. Let $z \in A$. It is clear that $\mathrm{dist}(x,z) \in \{t, t+1\}$. Suppose $\mathrm{dist}(x,z) = t$. Let $x = v_0 \sim v_1 \sim \cdots \sim v_t = v$ be a path connecting x and v and $x = z_0 \sim z_1 \sim \cdots \sim z_t = z$ such a path connecting x and z. It is easy to see that vertices v_i and z_i are in $S_{h-t+i}(u)$ for $0 \le i \le t$. By monotonicity of the sequence $\{b_i\}$ and

the condition $t \leq h - t$, we have $1 = b_t = b_{t+1} = \cdots = b_{h-t} = \cdots = b_{d-1}$. Thus we have $z_i = v_i$ for $0 \leq i \leq t$. This contradicts $v \neq z$. Hence we have $A \subseteq S_{t+1}(x) \cap S_1(v)$. Therefore $a_h \leq b_t = 1$.
(2) Note that $a_d \leq 1$. Suppose G contains no boxes. Then from Lemma 7.1.4 we have $c_d \leq {a_d}^2 \leq 1$. This contradicts $a_d + c_d = k > 2$. □

Proof of Theorem 7.1.1. Let G be a distance-regular graph with valency $k > 2$, $b_t = 1$ and diameter $d \geq 2t$. We need to show that

$$b_{d-h} = c_h \qquad (*)$$

holds for $1 \leq h \leq d$. We have $1 = b_t = \cdots = b_{h-t} = \cdots = b_{d-1}$. By $c_{d-i} \leq b_i$ for $i < d$, we also have $1 = c_1 = \cdots = c_{d-t}$. Hence $(*)$ holds for $1 \leq h \leq d - t$. Since, by Lemma 7.1.6, the graph G contains a box, it follows from Proposition 7.1.5 that

$$0 \leq b_0 - c_d \leq b_1 - c_{d-1} \leq \cdots \leq b_{d-t} - c_t.$$

But $b_{d-t} = 1 = c_t$ and the equalities must hold. □

We close this section by mentioning that the work on the case $b_3 = 1$ is in progress. In a collaboration with Araya, Hiraki (private communication, April 1994) we have shown:

7.1.7 PROPOSITION. *If G is a distance-regular graph of diameter d and $b_3 = 1$, then $d \leq 6$. If $d = 6$ then $\lambda \neq 0$.* □

2. Dependent geodesics in distance-regular graphs

We start this section by deriving some constraints on distance-regular graphs which contain a dependent geodesic. This enables us to search for small examples of such graphs. All the known small examples are collected in Table 7.1. This section is joint work with Godsil.

7.2.1 LEMMA. *Let G be a distance-regular graph of diameter $d > 2$ and valency $k > 2$, for which a shortest dependent geodesic has $e > 2$ vertices. Then $b_{e-2} = 1$ and $e - 1 \leq d < 3(e - 2)$.*

Proof. Let G be a distance-regular graph of diameter d and let $e \geq 3$ be the number of vertices in a shortest dependent geodesic. Let θ be the corresponding eigenvalue and m its multiplicity. As the diameter of a geodesic on e vertices is $e-1$ we have $e-1 \leq d$.

Suppose that $b_{e-2} > 1$, and let us extend a geodesic P, with ends v and $a \in S_{e-2}(v)$, to two distinct geodesics P_1 and P_2 with v as one end and with the other end in $S_{e-1}(v)$. Let $\{b\} = P_1 \cap S_{e-1}(v)$ and $\{c\} = P_2 \cap S_{e-1}(v)$ for some vertices b and c. By assumption the geodesics P_1 and P_2 are dependent, so the representations $u(b)$ and $u(c)$ of b and c can be expressed as linear combinations of the representations of the vertices of P. However, the scalar products of $u(b)$ with the representations of the vertices of P are the same as the scalar products of $u(c)$ with the representations of the vertices of P, hence $u(b) = u(c)$. As the graph representation is locally injective, by the remark after Theorem 2.2.10, it follows that $b_{e-2} = 1$. The upper bound follows directly from [64, Thm. 13.5.3]. $\square$

Note that the above lower bound $e-1 \leq d$ is always attained for an antipodal distance-regular graph of index two (for an eigenvalue which is not an eigenvalue of the antipodal quotient). The main result of the previous section, Theorem 7.1.1, enables us to lower the above upper bound on diameter to $2e-5$ if we exclude antipodal distance-regular graphs of index two.

Here is the list of all the examples which appear in Brouwer et al. [27] for $e < 8$, beside the double-covers of complete graphs.

distance-regular graph	$c_1, c_2, \ldots, c_d$	r	d	min m
Dodecahedron	$1,1,1,2,3$	2	5	3
Wells graph (double folded Q_5)	$1,1,4,5$	2	4	5
Johnson graph J(8,4)	$1,4,9,16$	2	4	7
Halved 8-cube	$1,6,15,28$	2	4	8
Desargues graph (double O_5)	$1,1,2,2,3$	2	5	4
Johnson graph J(10,5)	$1,4,9,16,25$	2	5	9
Halved 10-cube	$1,6,15,28,45$	2	5	10
Hamming 5-cube	$1,2,3,4,5$	2	5	5
double Gewirtz	$1,2,8,9,10$	2	5	20
double 77-graph	$1,4,12,15,16$	2	5	21
double Higman-Sims	$1,6,16,21,22$	2	5	22
short., ext. 3-Golay code	$1,2,9,20,22$	3	5	24
related to ternary Golay code [27, p. 365, A15]	$1,2,6,16,20,21$	2	6	21
related to binary Golay code [27, p. 365, A14]	$1,2,3,20,21,22$	2	6	77
related to binary Golay code [27, p. 365, A16]	$1,2,3,16,20,21$	4	6	21
Johnson graph J(12,6)	$1,4,9,16,25,36$	2	6	11
Halved 12-cube	$1,6,15,28,45,66$	2	6	12
Hamming 6-cube	$1,2,3,4,5,6$	2	6	6
double O_7	$1,1,2,2,3,3,4$	2	7	6

Table 7.1: List of small distance-regular graphs with dependent geodesics.

As all examples are antipodal covers, we list only parameters $c_1, c_2, \ldots, c_d$, the index r of the cover, diameter d and a minimal multiplicity m.

A *Töplitz matrix* T is an $n \times n$ matrix in which two entries are equal if magnitude of the difference of their indices is equal, i.e.,

$$|i-j| = |k-h| \qquad \text{implies} \qquad (T)_{ij} = (T)_{kh}.$$

So the Gram matrix of the representations of the vertices of a geodesic path is a Töplitz matrix. In order to make this list we had to check, for many intersection arrays, if the Töplitz matrix corresponding to its standard sequence equals zero.

The calculations are simpler if we know how to factorize the determinant of a Töplitz matrix. For example, we have factored

$$\begin{vmatrix} 1 & a & b & c & d & e \\ a & 1 & a & b & c & d \\ b & a & 1 & a & b & c \\ c & b & a & 1 & a & b \\ d & c & b & a & 1 & a \\ e & d & c & b & a & 1 \end{vmatrix}$$

into

$$\begin{vmatrix} 1+a & a+b & b+c \\ a+b & 1+c & a+d \\ b+c & a+d & 1+e \end{vmatrix} \cdot \begin{vmatrix} 1-a & a-b & b-c \\ a-b & 1-c & a-d \\ b-c & a-d & 1-e \end{vmatrix}.$$

Factorizations of the determinants of up to six-by-six Töplitz matrices can be obtained simply by using Maple, although Maple will not actually give the determinants as factors (but it certainly helps to do verifications). The next case, the factorization of the determinant of seven-by-seven Töplitz matrix, is already too big problem for Maple. However if we guess one factor, then we can use Maple to perform the division and to obtain the other factor. In this case we have factored

$$\begin{vmatrix} 1 & a & b & c & d & e & f \\ a & 1 & a & b & c & d & e \\ b & a & 1 & a & b & c & d \\ c & b & a & 1 & a & b & c \\ d & c & b & a & 1 & a & b \\ e & d & c & b & a & 1 & a \\ f & e & d & c & b & a & 1 \end{vmatrix}$$

into

$$\begin{vmatrix} 1+b & a+c & b+d & 2a \\ a+c & 1+d & a+e & 2b \\ b+d & a+e & 1+f & 2c \\ a & b & c & 1 \end{vmatrix} \cdot \begin{vmatrix} 1-b & a-c & b-d \\ a-c & 1-d & a-e \\ b-d & a-e & 1-f \end{vmatrix}.$$

After seeing a few of these small factorizations we were able to guess a factorization in a general case.

7.2.2 PROPOSITION.

$$\det T \quad = \quad \det P \cdot \det N$$

where

$$\begin{aligned} (T)_{ij} &= w_{|i-j|} && \text{for} \quad i,j = 1,2,\ldots,n, \\ (N)_{ij} &= w_{|i-j|} - w_{i+j-\epsilon} && \text{for} \quad i,j = 1,2,\ldots,\lfloor n/2 \rfloor, \\ (P)_{ij} &= w_{|i-j|} + w_{i+j-\epsilon} && \text{for} \quad i,j = 1,2,\ldots,\lfloor n/2 \rfloor, \end{aligned}$$

where ϵ is one if n is even and zero otherwise, and when n is odd, i.e., $n = 2m+1$ for some integer m, also

$$\begin{aligned} (P)_{m+1,j} &= w_j && \text{for} \quad j = 1,2,\ldots,m, \\ (P)_{i,m+1} &= 2w_i && \text{for} \quad i = 1,2,\ldots,m, \\ (P)_{m+1,m+1} &= 1. \end{aligned}$$

□

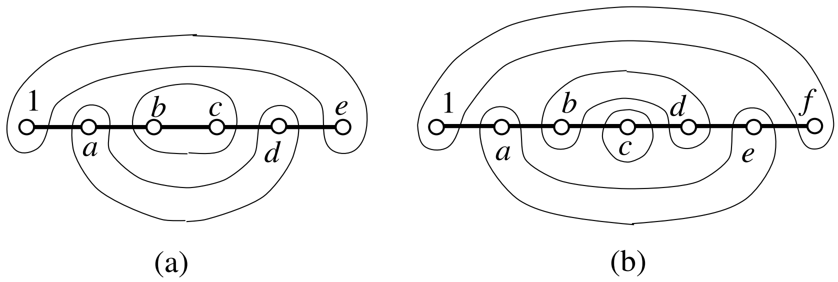

Figure 7.3: Equitable partitions of paths.

Instead of giving a complete proof of this result, let us show one application of equitable partitions. In Figure 7.3 we see two equitable partitions of a path on six and a path on seven vertices. This implies that the set of columns and the set of rows of an n-by-n Töplitz matrix T can be partitioned into $\lceil n/2 \rceil$ parts each, so that the column sums in the corresponding submatrices are all equal (e.g., in the case of six-by-six matrix we take the symmetric partition $\{3,4\}, \{2,5\}, \{1,6\}$ and in the case of seven-by-seven matrix we take the symmetric partition $\{3,5\},\{2,6\},\{1,7\},\{4\}$). When we substitute these submatrices with their column sums we obtain the matrix P. So its eigenvalues are also eigenvalues of T. Therefore $\det P \mid \det T$.

Let us now list some known infinite families of distance-regular graphs which contain dependent geodesics.

(1) Johnson graph $J(2n,n)$ has diameter $d = n$ and it has a dependent geodesic on $d+1$ vertices.
(2) Halved $2n$-cube has diameter $d = n$ and it has a dependent geodesic on $d+1$ vertices.
(3) Double Odd graph O_{2n+1} has diameter $d = 2n+1$ and it has a dependent geodesic on d vertices.

We finish this section by posing two questions.

(1) Are there any primitive distance-regular graphs which contain dependent geodesics?
(2) Are there any distance-regular graphs of diameter d which contain dependent geodesics on $d-2$ vertices?

APPENDIX

Here we add some material, which is not included in the thesis, hoping that it will be of some help to a reader unfamiliar with combinatorics. In Section 1 we collect basic definitions from graph theory. In Section 2 we introduce incidence structures and relate them to graphs. Section 3 is intended for those who have never worked with distance-regular graphs. Finally, we list small feasible parameters of antipodal distance-regular graphs of diameter four in Section 4.

1. Graph Theory

We follow Thomassen [139] and Godsil [64]. A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite set of *vertices*) and $E(G)$ is a set of unordered pairs xy of vertices called *edges*. Unless explicitly stated otherwise graphs have neither loops nor multiple edges. (A *directed graph* G is a pair $(V(G), E(G))$, where $E(G)$ is a subset of $V(G) \times V(G)$. We sometimes view an edge $\{u, v\}$ as being formed from two arcs (u, v) and (v, u). Thus some directed graphs can be viewed as undirected graphs. We say that the edge xy *joins* x and y, that it is *incident* with x and y, and that x and y are *neighbours* (or also that they are *adjacent*), denoted by $x \sim y$. The *valency* (also called the *degree*) of a vertex x is the number of neighbours of x in G. A graph G is *k-regular* if all the vertices have valency k. The set of neighbours of x is denoted by $S(x)$ (or also $N(x)$) and we set $B(x) = S(x) \cup \{x\}$ (or $\overline{N}(x) = N(x) \cup \{x\}$).

A *subgraph* H of a graph G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and we call it a *spanning subgraph* if $V(H) = V(G)$. For $A \subseteq V(G) \cup E(G)$ a graph $G - A$ is obtained from G by *deleting* A and all the edges incident with $A \cap V(G)$. If $A \subseteq V(G)$, then the graph $G(A)$ *induced* by A is defined as $G - (V(G) \backslash A)$. The graph induced by $S(x)$ is the *local graph* of x (also called the *neighbourhood graph*).

An *n-path* is a graph with vertices $x_0, x_1, \ldots, x_n$ and edges $x_{i-1}x_i$, $i = 1, 2, \ldots, n$. A *n-cycle* is obtained from an $(n-1)$-path by adding the edge between the two ends (i.e., vertices of valency one). We say that n is the *length* of the n-path and the n-cycle. A graph is *connected* if any two vertices are joined by a path. The *distance* between two vertices is the length of a shortest path between them. The *diameter* of a graph is the maximum distance between any two vertices. The *girth* of a graph is the length of a shortest cycle. A connected graph with no cycles is called a *tree*. A cycle through all the vertices of G is a *Hamiltonian cycle* of G. A graph G is *bipartite*, if there is a partition $V(G) = A \cup B$ such that all edges of G join A and B. The *complete graph* K_n is a graph on n vertices with all possible edges (when it is considered as a subgraph, it is also called a *clique*). If $|V(G)| = n$ for a graph G and K_n has the same vertex set, then the *complement* of G is obtained by deleting all the edges of G in K_n. The *multipartite graph* $K_{t(m)}$ is the complement of t complete graphs on m vertices, and it is called a *complete bipartite graph* when $t = 2$. A *perfect matching* of a graph G (also called a 1-*factor*) is a spanning subgraph of G with valency one. A graph is *geometric* when it does not contain $K_{1,2,1}$ as a subgraph, i.e., when we can partition the edges with the maximal cliques.

The *line graph* $L(G)$ of a graph G is the graph whose vertex set is $E(G)$ such that two vertices are adjacent in $L(G)$ if and only if they have a common end in G. For example, the lattice graph $K_n \times K_n$ is the line graph of the complete bipartite graph $K_{n,n}$. The *Cartesian product* of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ such that (x, u) and (y, v) are adjacent if and only if either $x = y$ and u and v are adjacent in H, or $u = v$ and x and y are adjacent in G. The n-cube is the Cartesian product of n copies of K_2.

A *homomorphism* (also called a *graph morphism*) from a graph G to a graph H is a mapping $h : V(G) \longrightarrow V(H)$ such that, for any two vertices x and y, x and y are adjacent in G if and only if $h(x)$ and $h(y)$ are adjacent in H. If a homomorphism h is also 1-1 and onto, then it is called an *isomorphism*, and the graphs G and H are said to be *isomorphic*. If further $G = H$, then the isomorphism is called *automorphism*. The automorphism group of G partitions $V(G)$ and $E(G)$ into *orbits*. A graph is *vertex transitive* if $V(G)$ has only one orbit.

A *surface* is a connected compact topological Hausdorff space which is locally homeomorphic to a disc. By the classification theorem for surfaces, see for example Thomassen [139], Andrews [4] or Gross and Tucker [72], the only surfaces are S_g, the sphere with g handles, and N_k, the sphere with k crosscaps. If G is a graph drawn on a surface such that the edges are pairwise simple noncrossing curves, then G partitions S_g or N_k into regions, called *faces*. If each face is homeomorphic to a disc, then the embedding is called a *2-cell embedding*. The *genus* $g(G)$ (respectively *crosscap number* $c(G)$) is the smallest number m such that G has an embedding in S_m (respectively N_m). Assume now that a connected graph G has a 2-cell embedding in a surface S, where $S = S_g$ or $S = N_k$. If a number of faces is denoted by f, $n = |V(G)|$, and $e = |E(G)|$, then the *Euler's formula* says:

$$n - e + f = 2 - 2g, \quad \text{when } S = S_g,$$

and

$$n - e + f = 2 - k, \quad \text{when } S = N_k.$$

2. Incidence structures

We follow Cameron and Van Lint [48]. A t-(v, s, λ_t) *design* is a collection of s-subsets called *blocks* of a set with v elements called *points*, such that each t-set of points lies in exactly λ_t blocks. We assume that $t < s$ to exclude degenerate cases. If $\lambda_t = 1$, then a t-design is called a *Steiner system*, denoted by $S(t, s, v)$.

In a t-design let λ_i denote the number of blocks containing a given set of i points, with $0 \leq i \leq t$. Let S be some i-set. Then S is contained in λ_i blocks and each of them contains $\binom{s-i}{t-i}$ distinct t-sets with S as subset. On the other hand the set S can be enlarged to t-set in $\binom{v-i}{t-i}$ ways and each of these t-sets is contained in λ_t blocks. So we conclude:

$$\lambda_i \binom{s-i}{t-i} = \lambda_t \binom{v-i}{t-i}$$

Therefore λ_i is independent of S. This actually means that a t-design is also an i-design, for $0 \leq i \leq t$. The number of blocks in a design is equal to λ_0 and is denoted by b. Every point in a 1-design lies in λ_1 blocks and this number is often denoted by r. When $t \geq 2$ we get, from the upper identity for $i = 0, t = 1$ and $i = 1, t = 2$, the following:

$$bs = rv \quad \text{and} \quad r(s-1) = \lambda_2(v-1)$$

or

$$r = \lambda_2 \frac{v-1}{s-1} \quad \text{and} \quad b = \lambda_2 \frac{v(v-1)}{s(s-1)}$$

For any design with more than one block we have $b \geq v$, which is the well known *Fisher's inequality*. Designs with $b = v$ are called *square* (or *symmetric*) and have a property that any two blocks meet in exactly λ_t points.

A *finite projective plane of order* n is a Steiner system $S(2, n+1, n^2+n+1)$, and a *finite affine plane of order* n is a Steiner system $S(2, n, n^2)$. Arbitrary projective and affine planes of order n are denoted by $PG(2,n)$ and $AG(2,n)$, respectively. For $d > 2$, $PG(d,q)$ denotes the d-dimensional *projective space* over $GF(q)$, i.e., the lattice of subspaces of $(d+1)$-dimensional vector space over $GF(q)$. Let us define a relation on blocks of a design by two blocks being related if they are either the same or have no points in common. Than this relation is an equivalence relation in an affine plane $AG(2,n)$ and its equivalence classes are called *parallel classes*.

A *transversal design* $TD(s,v)$ is a design with blocks of size s and points partitioned into s groups of size v, such that two distinct points are contained in a unique block if and only if they are in distinct groups. By theorem of Chowla, Erdös and Straus [50] a $TD(s,v)$ exists for $v \geq v_1(s)$ and $v_1(s) = s-1$ for $s \leq 3$. Similarly, by theorem of Wilson [141] and Hanani [78], 2-$(v,s,1)$ design exists for all $v \geq v_0(s)$ with $s-1|v-1$, $s(s-1)|v(v-1)$, and $v_0(s) = s^2-s+1$ for $s \leq 5$.

We introduce two important infinite families of strongly regular graphs which come from the above designs. Let $\mathcal{D}$ be a design with a set of *points* P, a set of *blocks* B (also called *lines*), and an *incidence relation* $I \subseteq P \times B$ (called also a set of *flags*). Then $\mathcal{D}$ is equivalently described by the *incidence graph*, i.e., the bipartite graph with vertex set $P \cup B$, and with a point p adjacent to a block b whenever they are incident. By interchanging the roles of points and blocks (lines) we get the so-called *dual* incidence structure. We can also derive other kinds of graphs from a design, for example the *point graph* (called also the *collinearity graph*), with points as vertices and two of them being adjacent whenever they are collinear, and the *line graph* with lines as vertices and two of them being adjacent whenever there is a point incident to both lines.

The line graph of a 2-$(v,s,1)$ design with $v-1 > s(s-1)$ is strongly regular with parameters

$$n = \frac{\binom{v}{2}}{\binom{s}{2}}, \quad k = s(\frac{v-1}{s-1} - 1), \quad \lambda = \frac{v-1}{s-1} - 2 + (s-1)^2, \;\; \mu = s^2.$$

When in a design $\mathcal{D}$ the block size is two, the number of edges of the point graph equals the number of blocks of the design $\mathcal{D}$. In this case the line graph of the design $\mathcal{D}$ is the line graph of the point graph of $\mathcal{D}$. This justifies the usual name: the line graph of a graph. The line graph of a transversal design $TD(s,v)$ is also strongly regular for $s \leq v$, with parameters

$$n = v^2, \quad k = s(v-1), \quad \lambda = (v-2) + (s-1)(s-2), \;\; \mu = s(s-1).$$

For $s = 2$ we get the lattice graph $K_v \times K_v$, which is also known as the *grid graph* $v \times v$.

3. Motivation of some definitions

General subjects of this thesis are *symmetry* and *regularity*. The first one is usually described by an automorphism group of an object, and the second one is expressed by an arithmetical property. Surprisingly the second often implies the uniqueness of the object and the existence of many symmetries. There is an intriguing interplay of these two quantities.

Intuitively we think of symmetry as a property of an object to look the same when observed from different sides/angles. In this statement we assume some kind of real picture of the object. Since we draw two or three dimensional objects, we usually present them only corresponding to one, two, three,... or at least to a small number of their automorphisms. But this means that small or at least not too large automorphism group of an object implies the existence of many different symmetric pictures of the same object. It is time for an illustration of these thoughts.

Let us consider the Petersen graph. One way to describe this graph is to define its vertex set to be all two-element subsets of a set $\{1, 2, 3, 4, 5\}$, and two vertices to be adjacent if and only if they are disjoint. Similarly, a k-element subsets of a $(2k + 1)$-set in general define the *Odd graph* O_k. Permuting the symbols $1, 2, 3, 4, 5$ evidently does not change the graph, and it is not too difficult to verify that the symmetric group of five symbols $\mathcal{S}_5$ is the automorphism group of the Petersen graph. We are all familiar with the following two presentations of the Petersen graph:

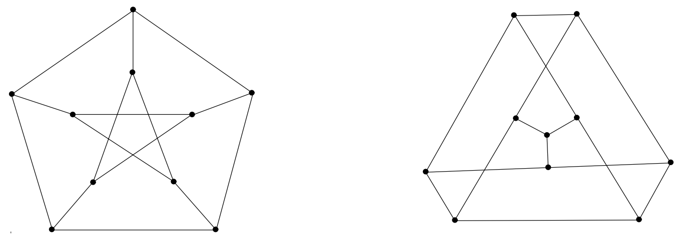

Figure A.1: The Petersen graph.

These two presentations exhibit (among other symmetries) a symmetry of order five, and a symmetry of order three respectively. Here are three more presentations which correspond to symmetries of order four, three and two:

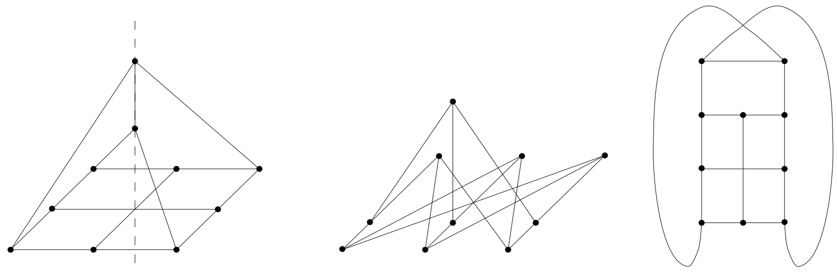

Figure A.2: The Petersen graph.

Despite all these nice pictures we still have not found one from which it would be easy to ‘see’ that the Petersen graph is *vertex transitive*, i.e., each vertex can be mapped to any other vertex by an element of the automorphism group. To make this evident we ‘embed’ the Petersen graph into the dodecahedron drawn ‘symmetrically’ on a sphere.

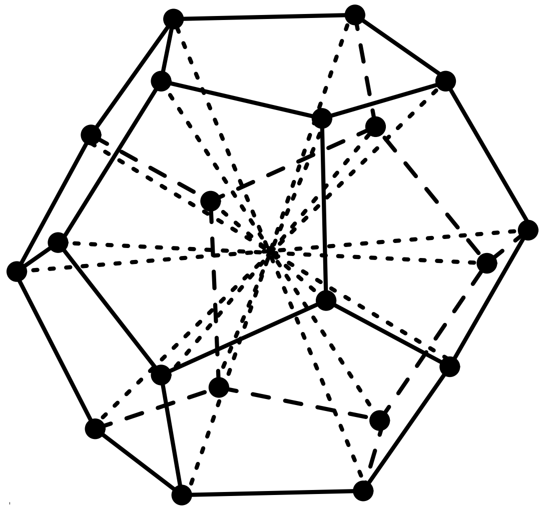

Figure A.3: The dodecahedron.

Let us define a graph with the diagonals (or, if you prefer the antipodal pairs of vertices) of the dodecahedron as vertices, and let two of them be adjacent if and only if they are joined with an edge of the dodecahedron. Then this graph is again the Petersen graph. The vertex transitivity is now obvious. It is also evident that any two vertices can be mapped to any other two vertices at the same distance by an element of the automorphism group. A graph with this property is called *distance transitive*. All these properties can be deduced directly from the abstract definition of the Petersen graph and from the fact that we know its automorphism group, but we want to make a point that it is sometimes possible to obtain the same conclusions by purely combinatorial approach. This turns out to be very important when we do not know the automorphism group of the object we study, or the automorphism group is trivial and we cannot get any help from algebraic or group theoretical approach.

Let us now give some examples of regularities. For instance, we can require that each vertex has the same number of neighbours, in which case we say that the graph is *regular*. A variation of this would be to require that the number of vertices at distance i from a vertex is independent of the choice of a vertex. This certainly has to be true for vertex transitive graphs, since an automorphism of a graph preserves the distance. Another example of regularity is a requirement that a graph has no triangles (or some other cycles of given length). We can rephrase this in the following way: any two adjacent vertices have no common neighbours. Two natural variations of this are:

(a) any two adjacent vertices have exactly λ common neighbours,
(b) any two nonadjacent vertices have exactly μ common neighbours.

A graph is called *strongly regular* when it satisfies these two properties. Strongly regular graphs can also be treated as extremal graphs and have been studied extensively (for basic properties see [48], [118] and [31]). Since the condition (b), for $\mu \neq 0$ implies that the diameter of a graph satisfying this property is at most two, we substitute (b) with:

(c) any two vertices at distance two have exactly μ common neighbours.

All these properties are implied by distance transitivity. To come even closer to distance transitivity we can require that given any two vertices u, v of G and any integers i and j, the number of vertices at distance i from u and j from v depends only on i, j and the distance between u and v. A connected graph satisfying this property is called *distance-regular*. Connected strongly regular graphs are exactly distance regular graphs of diameter two. Distance-regular graphs were introduced in late 1950's by Biggs [13] as a generalization of distance transitive graphs. As soon as we require enough regularity for some object, there exists, almost as a rule, a distance-regular graph which is related to such an object. Some examples of distance-regular graphs are complete and complete multipartite graphs, cycles, n-cubes, 1-skeletons of Platonic solids, the Petersen graph and its line graph. All the above examples are distance transitive. Obviously, any distance transitive graph is distance-regular, but there are also distance-regular graphs which are not distance transitive (see for example [123]). However, there are not many known such examples with diameter greater than eight, see Brouwer et al. [27, p. 136]. The opposite is true for small diameters. For example, it is believed that almost all strongly regular graphs are not distance transitive, and for them combinatorial approach becomes quite important. There are also other occasions when it is easier to work with numerical regularity conditions instead of symmetry conditions.

Distance-regular graphs are divided into primitive and imprimitive ones. The latter are either antipodal or bipartite (or both) and they give rise to primitive graphs of at most half the diameter. Therefore the big project of classifying distance-regular graphs has two stages:

(a) find all primitive distance-regular graphs (see Bannai and Ito [8], [9])

(b) given a distance-regular graph G find all imprimitive graphs, i.e., bipartite distance-regular graphs or antipodal distance-regular graphs, called distance-regular antipodal covers of G, which give rise to G.

For example, the dodecahedron is a distance-regular antipodal cover of the Petersen graph. The first part of (b) was studied by Hemmeter [81], [82]. A lot of work has been done on distance-regular antipodal covers of complete and complete bipartite graphs. Van Bon and Brouwer [17] used simple geometric arguments to show that most classical distance-regular graphs have no antipodal covers. More precisely, for diameter at least eight, the only known antipodal distance-regular graphs are

0. $2m$-gons,
1. Johnson graphs $J(2k, k)$,
2. doubled Odd graphs,
3. n-cubes, and
4. folded $2n$-cubes.

Many antipodal covers of complete graphs indicate that the situation is different (and more interesting) in small diameter cases, and this is the central theme of this thesis.

4. List of feasible parameters for D=4

In Section 4.4 we mentioned how a list of feasible parameters of distance-regular antipodal covers of strongly regular graphs can be obtained. Here we list the intersection arrays with valency at most 100 in the diameter four case. The uniqueness of the Soicher graph, the distance-regular antipodal cover of the strongly regular graph with parameters $\{56, 45; 1, 24\}$ (i.e., the second subconstituent of the McLaughlin graph), has been shown by Brouwer et al. [28, Theorem 11.4.6].

	distance partition	eigenvalues and multiplicities	intersection array
!	v = 32 = 1+5+20+5+1	$5^1\ 2.2^8\ 1^{10}\ -2.2^8\ -3^5$	$\{5, 4, 1, 1; 1, 1, 4, 5\}$
!	v = 45 = 1+6+24+12+2	$6^1\ 3^{12}\ 1^9\ -2^{18}\ -3^5$	$\{6, 4, 2, 1; 1, 1, 4, 6\}$
!	v = 63 = 1+10+30+20+2	$10^1\ 5^{12}\ 1^{14}\ -2^{30}\ -4^6$	$\{10, 6, 4, 1; 1, 2, 6, 10\}$
!	v = 70 = 1+16+36+16+1	$16^1\ 8^7\ 2^{20}\ -2^{28}\ -4^{14}$	$\{16, 9, 4, 1; 1, 4, 9, 16\}$
?	v = 162 = 1+20+120+20+1	$20^1\ 5^{36}\ 2^{60}\ -4^{45}\ -7^{20}$	$\{20, 18, 3, 1; 1, 3, 18, 20\}$
	v = 243 = 1+20+180+40+2	$20^1\ 5^{72}\ 2^{60}\ -4^{90}\ -7^{20}$	$\{20, 18, 4, 1; 1, 2, 18, 20\}$
?	v = 486 = 1+20+360+100+5	$20^1\ 5^{180}\ 2^{60}\ -4^{225}\ -7^{20}$	$\{20, 18, 5, 1; 1, 1, 18, 20\}$
?	v = 486 = 1+21+420+42+2	$21^1\ 4.6^{162}\ 3^{105}\ -4.6^{162}\ -6^{56}$	$\{21, 20, 2, 1; 1, 1, 20, 21\}$
?	v = 200 = 1+22+154+22+1	$22^1\ 4.7^{50}\ 2^{77}\ -4.7^{50}\ -8^{22}$	$\{22, 21, 3, 1; 1, 3, 21, 22\}$
?	v = 300 = 1+22+231+44+2	$22^1\ 4.7^{100}\ 2^{77}\ -4.7^{100}\ -8^{22}$	$\{22, 21, 4, 1; 1, 2, 21, 22\}$
?	v = 600 = 1+22+462+110+5	$22^1\ 4.7^{250}\ 2^{77}\ -4.7^{250}\ -8^{22}$	$\{22, 21, 5, 1; 1, 1, 21, 22\}$
?	v = 352 = 1+25+300+25+1	$25^1\ 5^{88}\ 3^{120}\ -5^{88}\ -7^{55}$	$\{25, 24, 2, 1; 1, 2, 24, 25\}$
?	v = 704 = 1+25+600+75+3	$25^1\ 5^{264}\ 3^{120}\ -5^{264}\ -7^{55}$	$\{25, 24, 3, 1; 1, 1, 24, 25\}$
?	v = 704 = 1+26+650+26+1	$26^1\ 5.1^{176}\ 4^{208}\ -5.1^{176}\ -6^{143}$	$\{26, 25, 1, 1; 1, 1, 25, 26\}$
?	v = 264 = 1+27+180+54+2	$27^1\ 9^{44}\ 3^{55}\ -3^{132}\ -6^{32}$	$\{27, 20, 6, 1; 1, 3, 20, 27\}$
!	v = 128 = 1+28+70+28+1	$28^1\ 14^8\ 4^{28}\ -2^{56}\ -4^{35}$	$\{28, 15, 6, 1; 1, 6, 15, 28\}$
?	v = 210 = 1+32+144+32+1	$32^1\ 8^{35}\ 2^{84}\ -4^{70}\ -10^{20}$	$\{32, 27, 6, 1; 1, 6, 27, 32\}$
?	v = 315 = 1+32+216+64+2	$32^1\ 8^{70}\ 2^{84}\ -4^{140}\ -10^{20}$	$\{32, 27, 8, 1; 1, 4, 27, 32\}$
?	v = 420 = 1+32+288+96+3	$32^1\ 8^{105}\ 2^{84}\ -4^{210}\ -10^{20}$	$\{32, 27, 9, 1; 1, 3, 27, 32\}$
?	v = 630 = 1+32+432+160+5	$32^1\ 8^{175}\ 2^{84}\ -4^{350}\ -10^{20}$	$\{32, 27, 10, 1; 1, 2, 27, 32\}$
?	v = 420 = 1+33+352+33+1	$33^1\ 5.7^{105}\ 3^{154}\ -5.7^{105}\ -9^{55}$	$\{33, 32, 3, 1; 1, 3, 32, 33\}$
?	v = 630 = 1+33+528+66+2	$33^1\ 5.7^{210}\ 3^{154}\ -5.7^{210}\ -9^{55}$	$\{33, 32, 4, 1; 1, 2, 32, 33\}$
?	v = 1260 = 1+33+1056+165+5	$33^1\ 5.7^{525}\ 3^{154}\ -5.7^{525}\ -9^{55}$	$\{33, 32, 5, 1; 1, 1, 32, 33\}$
?	v = 704 = 1+36+630+36+1	$36^1\ 6^{176}\ 4^{231}\ -6^{176}\ -8^{120}$	$\{36, 35, 2, 1; 1, 2, 35, 36\}$
?	v = 1408 = 1+36+1260+108+3	$36^1\ 6^{528}\ 4^{231}\ -6^{528}\ -8^{120}$	$\{36, 35, 3, 1; 1, 1, 35, 36\}$
?	v = 1408 = 1+37+1332+37+1	$37^1\ 6.1^{352}\ 5^{407}\ -6.1^{352}\ -7^{296}$	$\{37, 36, 1, 1; 1, 1, 36, 37\}$
//	v = 252 = 1+45+160+45+1	$45^1\ 15^{21}\ 3^{90}\ -3^{105}\ -9^{35}$	$\{45, 32, 9, 1; 1, 9, 32, 45\}$
	v = 378 = 1+45+240+90+2	$45^1\ 15^{42}\ 3^{90}\ -3^{210}\ -9^{35}$	$\{45, 32, 12, 1; 1, 6, 32, 45\}$
//	v = 756 = 1+45+480+225+5	$45^1\ 15^{105}\ 3^{90}\ -3^{525}\ -9^{35}$	$\{45, 32, 15, 1; 1, 3, 32, 45\}$
?	v = 392 = 1+45+300+45+1	$45^1\ 9^{70}\ 3^{150}\ -5^{126}\ -11^{45}$	$\{45, 40, 6, 1; 1, 6, 40, 45\}$
?	v = 588 = 1+45+450+90+2	$45^1\ 9^{140}\ 3^{150}\ -5^{252}\ -11^{45}$	$\{45, 40, 8, 1; 1, 4, 40, 45\}$
?	v = 784 = 1+45+600+135+3	$45^1\ 9^{210}\ 3^{150}\ -5^{378}\ -11^{45}$	$\{45, 40, 9, 1; 1, 3, 40, 45\}$
?	v = 1176 = 1+45+900+225+5	$45^1\ 9^{350}\ 3^{150}\ -5^{630}\ -11^{45}$	$\{45, 40, 10, 1; 1, 2, 40, 45\}$
?	v = 2352 = 1+45+1800+495+11	$45^1\ 9^{770}\ 3^{150}\ -5^{1386}\ -11^{45}$	$\{45, 40, 11, 1; 1, 1, 40, 45\}$
?	v = 798 = 1+45+660+90+2	$45^1\ 6.7^{266}\ 3^{209}\ -6.7^{266}\ -12^{56}$	$\{45, 44, 6, 1; 1, 3, 44, 45\}$
?	v = 2394 = 1+45+1980+360+8	$45^1\ 6.7^{1064}\ 3^{209}\ -6.7^{1064}\ -12^{56}$	$\{45, 44, 8, 1; 1, 1, 44, 45\}$
?	v = 784 = 1+46+690+46+1	$46^1\ 6.8^{196}\ 4^{276}\ -6.8^{196}\ -10^{115}$	$\{46, 45, 3, 1; 1, 3, 45, 46\}$
?	v = 1176 = 1+46+1035+92+2	$46^1\ 6.8^{392}\ 4^{276}\ -6.8^{392}\ -10^{115}$	$\{46, 45, 4, 1; 1, 2, 45, 46\}$
?	v = 2352 = 1+46+2070+230+5	$46^1\ 6.8^{980}\ 4^{276}\ -6.8^{980}\ -10^{115}$	$\{46, 45, 5, 1; 1, 1, 45, 46\}$
?	v = 1276 = 1+49+1176+49+1	$49^1\ 7^{319}\ 5^{406}\ -7^{319}\ -9^{231}$	$\{49, 48, 2, 1; 1, 2, 48, 49\}$
?	v = 2552 = 1+49+2352+147+3	$49^1\ 7^{957}\ 5^{406}\ -7^{957}\ -9^{231}$	$\{49, 48, 3, 1; 1, 1, 48, 49\}$
?	v = 2552 = 1+50+2450+50+1	$50^1\ 7.1^{638}\ 6^{725}\ -7.1^{638}\ -8^{550}$	$\{50, 49, 1, 1; 1, 1, 49, 50\}$
?	v = 650 = 1+54+540+54+1	$54^1\ 9^{130}\ 4^{234}\ -6^{195}\ -11^{90}$	$\{54, 50, 5, 1; 1, 5, 50, 54\}$
?	v = 1625 = 1+54+1350+216+4	$54^1\ 9^{520}\ 4^{234}\ -6^{780}\ -11^{90}$	$\{54, 50, 8, 1; 1, 2, 50, 54\}$
?	v = 3250 = 1+55+2970+220+4	$55^1\ 7.4^{1300}\ 5^{429}\ -7.4^{1300}\ -10^{220}$	$\{55, 54, 4, 1; 1, 1, 54, 55\}$
//	v = 324 = 1+56+210+56+1	$56^1\ 14^{36}\ 2^{140}\ -4^{126}\ -16^{21}$	$\{56, 45, 12, 1; 1, 12, 45, 56\}$
!	v = 486 = 1+56+315+112+2	$56^1\ 14^{72}\ 2^{140}\ -4^{252}\ -16^{21}$	$\{56, 45, 16, 1; 1, 8, 45, 56\}$
//	v = 648 = 1+56+420+168+3	$56^1\ 14^{108}\ 2^{140}\ -4^{378}\ -16^{21}$	$\{56, 45, 18, 1; 1, 6, 45, 56\}$
//	v = 972 = 1+56+630+280+5	$56^1\ 14^{180}\ 2^{140}\ -4^{630}\ -16^{21}$	$\{56, 45, 20, 1; 1, 4, 45, 56\}$

//	$v = 1296 = 1+56+840+392+7$	$56^1\ 14^{252}\ 2^{140}\ -4^{882}\ -16^{21}$	$\{56, 45, 21, 1; 1, 3, 45, 56\}$
?	$v = 648 = 1+57+532+57+1$	$57^1\ 7.5^{162}\ 3^{266}\ -7.5^{162}\ -15^{57}$	$\{57, 56, 6, 1; 1, 6, 56, 57\}$
?	$v = 972 = 1+57+798+114+2$	$57^1\ 7.5^{324}\ 3^{266}\ -7.5^{324}\ -15^{57}$	$\{57, 56, 8, 1; 1, 4, 56, 57\}$
?	$v = 1296 = 1+57+1064+171+3$	$57^1\ 7.5^{486}\ 3^{266}\ -7.5^{486}\ -15^{57}$	$\{57, 56, 9, 1; 1, 3, 56, 57\}$
?	$v = 1944 = 1+57+1596+285+5$	$57^1\ 7.5^{810}\ 3^{266}\ -7.5^{810}\ -15^{57}$	$\{57, 56, 10, 1; 1, 2, 56, 57\}$
?	$v = 3888 = 1+57+3192+627+11$	$57^1\ 7.5^{1782}\ 3^{266}\ -7.5^{1782}\ -15^{57}$	$\{57, 56, 11, 1; 1, 1, 56, 57\}$
?	$v = 552 = 1+75+400+75+1$	$75^1\ 15^{69}\ 3^{230}\ -5^{207}\ -17^{45}$	$\{75, 64, 12, 1; 1, 12, 64, 75\}$
?	$v = 828 = 1+75+600+150+2$	$75^1\ 15^{138}\ 3^{230}\ -5^{414}\ -17^{45}$	$\{75, 64, 16, 1; 1, 8, 64, 75\}$
?	$v = 1104 = 1+75+800+225+3$	$75^1\ 15^{207}\ 3^{230}\ -5^{621}\ -17^{45}$	$\{75, 64, 18, 1; 1, 6, 64, 75\}$
?	$v = 1656 = 1+75+1200+375+5$	$75^1\ 15^{345}\ 3^{230}\ -5^{1035}\ -17^{45}$	$\{75, 64, 20, 1; 1, 4, 64, 75\}$
?	$v = 2208 = 1+75+1600+525+7$	$75^1\ 15^{483}\ 3^{230}\ -5^{1449}\ -17^{45}$	$\{75, 64, 21, 1; 1, 3, 64, 75\}$
?	$v = 3312 = 1+75+2400+825+11$	$75^1\ 15^{759}\ 3^{230}\ -5^{2277}\ -17^{45}$	$\{75, 64, 22, 1; 1, 2, 64, 75\}$
?	$v = 1104 = 1+76+950+76+1$	$76^1\ 8.7^{276}\ 4^{437}\ -8.7^{276}\ -16^{114}$	$\{76, 75, 6, 1; 1, 6, 75, 76\}$
?	$v = 1656 = 1+76+1425+152+2$	$76^1\ 8.7^{552}\ 4^{437}\ -8.7^{552}\ -16^{114}$	$\{76, 75, 8, 1; 1, 4, 75, 76\}$
?	$v = 2208 = 1+76+1900+228+3$	$76^1\ 8.7^{828}\ 4^{437}\ -8.7^{828}\ -16^{114}$	$\{76, 75, 9, 1; 1, 3, 75, 76\}$
?	$v = 3312 = 1+76+2850+380+5$	$76^1\ 8.7^{1380}\ 4^{437}\ -8.7^{1380}\ -16^{114}$	$\{76, 75, 10, 1; 1, 2, 75, 76\}$
?	$v = 6624 = 1+76+5700+836+11$	$76^1\ 8.7^{3036}\ 4^{437}\ -8.7^{3036}\ -16^{114}$	$\{76, 75, 11, 1; 1, 1, 75, 76\}$
?	$v = 1080 = 1+77+924+77+1$	$77^1\ 11^{210}\ 5^{385}\ -7^{330}\ -13^{154}$	$\{77, 72, 6, 1; 1, 6, 72, 77\}$
?	$v = 1620 = 1+77+1386+154+2$	$77^1\ 11^{420}\ 5^{385}\ -7^{660}\ -13^{154}$	$\{77, 72, 8, 1; 1, 4, 72, 77\}$
?	$v = 2160 = 1+77+1848+231+3$	$77^1\ 11^{630}\ 5^{385}\ -7^{990}\ -13^{154}$	$\{77, 72, 9, 1; 1, 3, 72, 77\}$
?	$v = 3240 = 1+77+2772+385+5$	$77^1\ 11^{1050}\ 5^{385}\ -7^{1650}\ -13^{154}$	$\{77, 72, 10, 1; 1, 2, 72, 77\}$
?	$v = 2160 = 1+78+2002+78+1$	$78^1\ 8.8^{540}\ 6^{715}\ -8.8^{540}\ -12^{364}$	$\{78, 77, 3, 1; 1, 3, 77, 78\}$
?	$v = 3240 = 1+78+3003+156+2$	$78^1\ 8.8^{1080}\ 6^{715}\ -8.8^{1080}\ -12^{364}$	$\{78, 77, 4, 1; 1, 2, 77, 78\}$
?	$v = 6480 = 1+78+6006+390+5$	$78^1\ 8.8^{2700}\ 6^{715}\ -8.8^{2700}\ -12^{364}$	$\{78, 77, 5, 1; 1, 1, 77, 78\}$
?	$v = 750 = 1+81+504+162+2$	$81^1\ 27^{50}\ 6^{144}\ -3^{450}\ -9^{105}$	$\{81, 56, 18, 1; 1, 9, 56, 81\}$
//	$v = 2250 = 1+81+1512+648+8$	$81^1\ 27^{200}\ 6^{144}\ -3^{1800}\ -9^{105}$	$\{81, 56, 24, 1; 1, 3, 56, 81\}$
?	$v = 3404 = 1+81+3240+81+1$	$81^1\ 9^{851}\ 7^{1035}\ -9^{851}\ -11^{666}$	$\{81, 80, 2, 1; 1, 2, 80, 81\}$
?	$v = 6808 = 1+81+6480+243+3$	$81^1\ 9^{2553}\ 7^{1035}\ -9^{2553}\ -11^{666}$	$\{81, 80, 3, 1; 1, 1, 80, 81\}$
?	$v = 6808 = 1+82+6642+82+1$	$82^1\ 9.1^{1702}\ 8^{1886}\ -9.1^{1702}\ -10^{1517}$	$\{82, 81, 1, 1; 1, 1, 81, 82\}$
?	$v = 800 = 1+84+630+84+1$	$84^1\ 14^{120}\ 4^{315}\ -6^{280}\ -16^{84}$	$\{84, 75, 10, 1; 1, 10, 75, 84\}$
?	$v = 1600 = 1+84+1260+252+3$	$84^1\ 14^{360}\ 4^{315}\ -6^{840}\ -16^{84}$	$\{84, 75, 15, 1; 1, 5, 75, 84\}$
?	$v = 2000 = 1+84+1575+336+4$	$84^1\ 14^{480}\ 4^{315}\ -6^{1120}\ -16^{84}$	$\{84, 75, 16, 1; 1, 4, 75, 84\}$
?	$v = 4000 = 1+84+3150+756+9$	$84^1\ 14^{1080}\ 4^{315}\ -6^{2520}\ -16^{84}$	$\{84, 75, 18, 1; 1, 2, 75, 84\}$
?	$v = 1600 = 1+85+1428+85+1$	$85^1\ 9.2^{400}\ 5^{595}\ -9.2^{400}\ -15^{204}$	$\{85, 84, 5, 1; 1, 5, 84, 85\}$
?	$v = 4000 = 1+85+3570+340+4$	$85^1\ 9.2^{1600}\ 5^{595}\ -9.2^{1600}\ -15^{204}$	$\{85, 84, 8, 1; 1, 2, 84, 85\}$
?	$v = 8000 = 1+85+7140+765+9$	$85^1\ 9.2^{3600}\ 5^{595}\ -9.2^{3600}\ -15^{204}$	$\{85, 84, 9, 1; 1, 1, 84, 85\}$
?	$v = 644 = 1+96+450+96+1$	$96^1\ 24^{46}\ 4^{252}\ -4^{276}\ -16^{69}$	$\{96, 75, 16, 1; 1, 16, 75, 96\}$
?	$v = 1288 = 1+96+900+288+3$	$96^1\ 24^{138}\ 4^{252}\ -4^{828}\ -16^{69}$	$\{96, 75, 24, 1; 1, 8, 75, 96\}$
//	$v = 2576 = 1+96+1800+672+7$	$96^1\ 24^{322}\ 4^{252}\ -4^{1932}\ -16^{69}$	$\{96, 75, 28, 1; 1, 4, 75, 96\}$
?	$v = 1650 = 1+96+1456+96+1$	$96^1\ 12^{330}\ 6^{572}\ -8^{495}\ -14^{252}$	$\{96, 91, 6, 1; 1, 6, 91, 96\}$
?	$v = 2475 = 1+96+2184+192+2$	$96^1\ 12^{660}\ 6^{572}\ -8^{990}\ -14^{252}$	$\{96, 91, 8, 1; 1, 4, 91, 96\}$
?	$v = 3300 = 1+96+2912+288+3$	$96^1\ 12^{990}\ 6^{572}\ -8^{1485}\ -14^{252}$	$\{96, 91, 9, 1; 1, 3, 91, 96\}$
?	$v = 4950 = 1+96+4368+480+5$	$96^1\ 12^{1650}\ 6^{572}\ -8^{2475}\ -14^{252}$	$\{96, 91, 10, 1; 1, 2, 91, 96\}$
?	$v = 3300 = 1+97+3104+97+1$	$97^1\ 9.8^{825}\ 7^{1067}\ -9.8^{825}\ -13^{582}$	$\{97, 96, 3, 1; 1, 3, 96, 97\}$
?	$v = 4950 = 1+97+4656+194+2$	$97^1\ 9.8^{1650}\ 7^{1067}\ -9.8^{1650}\ -13^{582}$	$\{97, 96, 4, 1; 1, 2, 96, 97\}$
?	$v = 9900 = 1+97+9312+485+5$	$97^1\ 9.8^{4125}\ 7^{1067}\ -9.8^{4125}\ -13^{582}$	$\{97, 96, 5, 1; 1, 1, 96, 97\}$
?	$v = 3534 = 1+99+3234+198+2$	$99^1\ 9.9^{1178}\ 6^{836}\ -9.9^{1178}\ -15^{341}$	$\{99, 98, 6, 1; 1, 3, 98, 99\}$
?	$v = 10602 = 1+99+9702+792+8$	$99^1\ 9.9^{4712}\ 6^{836}\ -9.9^{4712}\ -15^{341}$	$\{99, 98, 8, 1; 1, 1, 98, 99\}$
?	$v = 5152 = 1+100+4950+100+1$	$100^1\ 10^{1288}\ 8^{1540}\ -10^{1288}\ -12^{1035}$	$\{100, 99, 2, 1; 1, 2, 99, 100\}$
?	$v = 10304 = 1+100+9900+300+3$	$100^1\ 10^{3864}\ 8^{1540}\ -10^{3864}\ -12^{1035}$	$\{100, 99, 3, 1; 1, 1, 99, 100\}$

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CURRICULUM VITAE

Aleksandar Jurišić was born on December 12, 1963 in Ljubljana. After three years of the secondary school "Poljanska Gimnazija" he became an undergraduate student in Pure Mathematics at the University of Ljubljana and in 1987 he graduated with the diploma thesis "Use of Topology in Combinatorics" under the supervision of Jože Vrabec. He wrote a problem book for mathematics competitions in 1989.

From the fall of 1988 to the winter of 1995 he attended graduate studies at the Department of Combinatorics and Optimization, University of Waterloo, and studied algebraic combinatorics under supervision of Chris Godsil (M.Sc. in 1990, Ph.D. in 1995). He married in 1989, got his first child in 1991 and the second one in 1992.

Currently he has an industrial post doctorial position at Certicom Corp. and the University of Waterloo, working in cryptography and algorithmic number theory. In his free time he enjoys playing basketball and teaching recreational mathematics.

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